

## Nonexistence of global solutions for a fractional system of strongly coupled integro-differential equations

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**Abstract:** In this paper, we study the nonexistence of nontrivial global solutions for a system of two strongly coupled fractional differential equations. Each equation involves two fractional derivatives and a nonlinear source term. The fractional derivatives are of Caputo type of subfirst orders. The nonlinear sources are nonlocal in time. They have the form of a convolution of a polynomial of the state with a (possibly singular) kernel. The system under consideration is a generalization of many interesting special systems of equations whose solutions do not exist globally in time. We establish some criteria under which no nontrivial global solutions exist. Several integral inequalities and estimations are derived and the test function method is adopted. Special cases and examples are given to illustrate the results.

**Key words:** Nonexistence, global solution, fractional system, fractional integro-differential equation, Caputo fractional derivative, nonlocal source

### 1. Introduction

We consider the following system of nonlinear fractional integro-differential equations

$$\begin{cases} ({}^C D_{0+}^{\alpha_1} u)(t) + \mu_1 ({}^C D_{0+}^{\beta_1} u)(t) = \int_0^t k_1(t-s) |v(s)|^{q_1} ds, & t > 0, q_1 > 1, \\ ({}^C D_{0+}^{\alpha_2} v)(t) + \mu_2 ({}^C D_{0+}^{\beta_2} v)(t) = \int_0^t k_2(t-s) |u(s)|^{q_2} ds, & t > 0, q_2 > 1, \\ u(0) = u_0, v(0) = v_0, & u_0, v_0 \in \mathbb{R}. \end{cases} \quad (1.1)$$

Here  $0 \leq \beta_i < \alpha_i \leq 1$ ,  $\mu_i = 0, 1$ ,  $i = 1, 2$  and  ${}^C D_{0+}^{\alpha}$  is the Caputo fractional derivative of order  $\alpha$ .

The convolutions in the right-hand sides of (1.1) represent nonlinear history dependent power-type source terms. The kernels  $k_1 : (0, \infty) \rightarrow [0, \infty)$  and  $k_2 : (0, \infty) \rightarrow [0, \infty)$  are assumed to be locally integrable functions different from zero almost everywhere.

We prove nonexistence of nontrivial global solutions under some suitable assumptions on the kernels  $k_i$ , the parameters  $\alpha_i$ ,  $\beta_i$ ,  $q_i$ ,  $i = 1, 2$ , and the initial conditions. The proof is based on the weak formulation of the problem with the test function method used in [16], with some suitable estimation inequalities we proved.

Clearly,  $(u, v) \equiv (0, 0)$  is the trivial solution of the system (1.1) (with zero initial data). We will exclude this case and consider only nontrivial solutions.

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It is well known that lower order derivatives usually represent damping terms and therefore help stabilizing the system in addition to the existence of solutions for all time. On the contrary, polynomial sources destabilize the system and they can even force solutions to blow up in finite time. In fact, they are sometimes called blowing up terms. When they are both present in the system we will have a competition between these two terms.

There is a considerable number of results on the existence of solutions for several classes of fractional differential and integro-differential equations, we refer to [1, 5, 18] and the references therein. The nonexistence of global and local solutions is investigated for some fractional differential and integro-differential equations, (e.g., see [2–4, 6–9, 11–15]). Most of these works are about sources of power type, which are special cases of ours.

It has been shown in [6] that the system

$$\begin{cases} u'(t) + ({}^C D_{0+}^\alpha u)(t) = |v(t)|^p, & t > 0, 0 < \alpha < 1, p > 1, \\ v'(t) + ({}^C D_{0+}^\beta v)(t) = |u(t)|^q, & t > 0, 0 < \beta < 1, q > 1, \\ u(0) = u_0, \quad v(0) = v_0, \end{cases} \tag{1.2}$$

admits no global solutions when  $u_0 > 0$  and  $v_0 > 0$  with  $1 - \frac{1}{pq} \leq \beta + \frac{\alpha}{q}$  or  $1 - \frac{1}{pq} \leq \alpha + \frac{\beta}{p}$ .

In [13], the authors presented estimates for the blowing-up solutions of the system (1.2) by matching them to the solutions of the ordinary system

$$\begin{cases} u'(t) = a |v(t)|^p, & t > 0, p > 1, \\ v'(t) = a |u(t)|^q, & t > 0, q > 1, \end{cases} \tag{1.3}$$

and the fractional system

$$\begin{cases} ({}^C D_{0+}^\alpha u)(t) = a |v(t)|^p, & t > 0, 0 < \alpha < 1, p > 1, \\ ({}^C D_{0+}^\beta v)(t) = a |u(t)|^q, & t > 0, 0 < \beta < 1, q > 1, \end{cases} \tag{1.4}$$

with either  $a = \frac{1}{2}$  or  $a = 1$ .

In 2017, the authors of [9] proved that the problem

$$\begin{cases} (D_{0+}^\alpha u)(t) + (D_{0+}^\beta u)(t) \geq t^\gamma |u(t)|^p, & t > 0, 0 < \beta < \alpha \leq 1, \\ (I^{1-\alpha} u)(0^+) = u_0, \end{cases} \tag{1.5}$$

has no nontrivial global solution in the space  $C_{1-\alpha}^\alpha$  when  $\gamma > -\beta$ ,  $1 < p < \frac{1+\gamma}{1-\beta}$  and  $u_0 \geq 0$ . The space  $C_{1-\alpha}^\alpha$  is defined by

$$C_{1-\alpha}^\alpha [0, b] = \{u \in C_{1-\alpha} [0, b] : D_{0+}^\alpha u \in C_{1-\alpha} [0, b]\},$$

where

$$C_{1-\alpha} [0, b] = \{u : (0, b] \rightarrow \mathbb{R} : t^{1-\alpha} u(t) \in C [0, b]\}.$$

In 2017, it was proved in [3] that the positive solution  $(u, v)$  of the system

$$\begin{cases} u'(t) - ({}^C D_{0+}^\alpha u)(t) = u^p(t)v^q(t), & t > 0, 0 < \alpha < 1, \\ v'(t) - ({}^C D_{0+}^\beta v)(t) = u^r(t)v^s(t), & t > 0, 0 < \beta < 1, \\ u(0) = u_0 > 0, & v(0) = v_0 > 0, \end{cases}$$

with  $0 < p < 1$ ,  $0 < s < 1$ ,  $r > 1 - p$ ,  $q > 1 - s$ , blows up in finite time if  $1 - \frac{1}{p'q'} \leq \beta + \frac{\alpha}{q'}$  or  $1 - \frac{1}{p'q'} \leq \alpha + \frac{\beta}{p'}$  where  $p' = \frac{r}{1-p}$  and  $q' = \frac{q}{1-s}$ .

The present authors studied, in [2], the nonexistence of nontrivial global solutions for the fractional integro-differential problem

$$\begin{cases} (D_{0+}^\alpha u)(t) + \lambda (D_{0+}^\beta u)(t) \geq \int_0^t h(t-s)|u(s)|^p ds, & t > 0, p > 1, \\ (I^{1-\alpha}u)(0^+) = b, & b \in \mathbb{R}, \end{cases} \tag{1.6}$$

where  $D_{0+}^\alpha$  and  $D_{0+}^\beta$  are the Riemann–Liouville fractional derivatives of orders  $\alpha$  and  $\beta$ , respectively,  $0 \leq \beta < \alpha \leq 1$ ,  $\lambda = 0, 1$  and  $h$  is a nonnegative function different from zero almost everywhere. It has been shown that if  $(t^{-\alpha p'} + \lambda p' t^{-\beta p'}) h^{1-p'}(t) \in L^1_{loc}[0, \infty)$  and

$$\lim_{T \rightarrow \infty} T^{1-p'} \left( \int_0^T t^{-\alpha p'} h^{1-p'}(t) dt + \lambda p' \int_0^T t^{-\beta p'} h^{1-p'}(t) dt \right) = 0,$$

where  $p' = \frac{p}{p-1}$ , then, the problem (1.6) does not have any nontrivial global solution when  $b \geq 0$ .

System (1.1) in this paper is a generalization of many interesting special systems of equations. Our results will extend the existing related results from the integer order case to the fractional case and from local sources to nonlocal sources. For example, when  $\mu_1 = \mu_2 = 1 = \alpha_1 = \beta_1$ ,  $\alpha_2 = \alpha$ ,  $\beta_2 = \beta$ , and  $k_1(t) = k_2(t) = \delta(t)$  (the Dirac delta function), we recover the system (1.2). Moreover, when  $\alpha_1 = \beta_1 = 1$ ,  $\mu_1 = \mu_2 = 0$  and  $k_1(t) = k_2(t) = \delta(t)$ , we obtain (1.3) with  $a = 1$ . The system (1.4) is a special case of (1.1) with  $a = 1$ ,  $\alpha_1 = \alpha$ ,  $\beta_1 = \beta$ ,  $\mu_1 = \mu_2 = 0$ , and  $k_1(t) = k_2(t) = \delta(t)$ .

The structure of this paper is as follows. In Section 2, we recall some definitions and lemmas from fractional calculus that will be used throughout the paper. Moreover, we introduce our selected test function, state and prove some of its properties. Section 3 is devoted to the statements and proofs of our results illustrated by some special cases and examples.

## 2. Preliminaries

In this section, we introduce some notation, definitions, and properties of fractional-order operators and the test function that will be used in the next section.

Let  $[a, b]$  be a finite interval of the real line  $\mathbb{R}$ . The Riemann–Liouville left-sided and right-sided fractional

derivatives of order  $\alpha \geq 0$ , are defined by

$$(D_{a+}^\alpha f)(t) = D^n (I_{a+}^{n-\alpha} f)(t), \tag{2.1}$$

$$(D_{b-}^\alpha f)(t) = (-1)^n D^n (I_{b-}^{n-\alpha} f)(t), \tag{2.2}$$

respectively, where  $D^n = \frac{d^n}{dt^n}$ ,  $n = [\alpha] + 1$  and  $[\alpha]$  is the integral part of  $\alpha$ .  $I_{a+}^\alpha$  and  $I_{b-}^\alpha$  are the Riemann–Liouville left-sided and right-sided fractional integrals of order  $\alpha > 0$  defined by

$$(I_{a+}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a,$$

$$(I_{b-}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, \quad t < b,$$

respectively, provided the right-hand sides exist. We define  $I_{a+}^0 f = I_{b-}^0 f = f$ . The function  $\Gamma(\alpha)$  is the Euler Gamma function. In particular, when  $\alpha = m \in \mathbb{N}_0$ , it follows from the definitions that

$$D_{a+}^m f = D^m f, \quad D_{b-}^\alpha f = (-1)^m D^m f.$$

The Caputo left-sided and right-sided fractional derivatives of order  $\alpha \geq 0$ , are defined by

$$({}^C D_{a+}^\alpha f)(t) = \left( D_{a+}^\alpha \left( f(s) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!} (s-a)^i \right) \right) (t),$$

$$({}^C D_{b-}^\alpha f)(t) = \left( D_{b-}^\alpha \left( f(s) - \sum_{i=0}^{n-1} \frac{f^{(i)}(b)}{i!} (b-s)^i \right) \right) (t),$$

respectively, where  $n = [\alpha] + 1$  for  $\alpha \notin \mathbb{N}_0$  and  $n = \alpha$  for  $\alpha \in \mathbb{N}_0$ .

In particular, when  $\alpha = n \in \mathbb{N}_0$ , it follows from the definitions that

$${}^C D_{a+}^0 f = {}^C D_{b-}^0 f = f, \quad {}^C D_{a+}^n f = D^n f, \quad {}^C D_{b-}^n f = (-1)^n D^n f.$$

Notice that if  $f^{(i)}(a) = 0$  for all  $i = 0, 1, \dots, n - 1$ , then  ${}^C D_{a+}^\alpha f = D_{a+}^\alpha f$ , and if  $f^{(i)}(b) = 0$  for all  $i = 0, 1, \dots, n - 1$ , then  ${}^C D_{b-}^\alpha f = D_{b-}^\alpha f$ . For more on fractional derivatives and integrals, we refer the reader to the books [10, 17].

The space of absolutely continuous functions on  $[a, b]$  is denote by  $AC[a, b]$ . In general, for  $n \in \mathbb{N}$ ,

$$AC^n[a, b] = \{g : [a, b] \rightarrow \mathbb{R} \text{ such that } D^{n-1}g \in AC[a, b]\}.$$

If  $g \in AC^n[a, b]$ , then  ${}^C D_{a+}^\alpha g$  and  ${}^C D_{b-}^\alpha g$  exist almost everywhere on  $[a, b]$  and are represented by

$$({}^C D_{a+}^\alpha g)(t) = (I_{a+}^{n-\alpha} D^n g)(t), \tag{2.3}$$

$$({}^C D_{b-}^\alpha g)(t) = (-1)^n (I_{b-}^{n-\alpha} D^n g)(t). \tag{2.4}$$

**Lemma 2.1** [10] *If  $\alpha \geq 0, \beta > 0$ , then*

$$\left(I_{b^-}^\alpha (b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (b-t)^{\beta+\alpha-1},$$

$$\left(D_{b^-}^\alpha (b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{\beta-\alpha-1}.$$

**Lemma 2.2** [17] *Let  $\alpha \geq 0, p \geq 1, q \geq 1$  and  $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$  ( $p \neq 1$  and  $q \neq 1$  in the case when  $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$ ). If  $\psi_1 \in L^p(a, b)$  and  $\psi_2 \in L^q(a, b)$ , then*

$$\int_a^b \psi_1(t) (I_{a^+}^\alpha \psi_2)(t) dt = \int_a^b \psi_2(t) (I_{b^-}^\alpha \psi_1)(t) dt.$$

**Lemma 2.3** *Let  $\alpha \geq 0$  and  $n = [\alpha] + 1$  for  $\alpha \notin \mathbb{N}_0$  and  $n = \alpha$  for  $\alpha \in \mathbb{N}_0$ . For  $f \in C[a, b]$  and  $g, I_{b^-}^{n-\alpha} f \in AC^n[a, b]$ , we have*

$$\int_a^b f(t) ({}^C D_{a^+}^\alpha g)(t) dt = \int_a^b g(t) (D_{b^-}^\alpha f)(t) dt + \sum_{i=0}^{n-1} [(D_{b^-}^{\alpha+i-n} f)(t) (D^{n-1-i} g)(t)]_a^b.$$

**Proof** Since  $g \in AC^n[a, b]$ , then we have from the definition (2.3),

$$\int_a^b f(t) ({}^C D_{a^+}^\alpha g)(t) dt = \int_a^b f(t) (I_{a^+}^{n-\alpha} D^n g)(t) dt.$$

Now, as  $f \in L^{m_1}(a, b)$  for any  $m_1 \geq 1$  and  $D^n g \in L^1(a, b)$ , we deduce from Lemma 2.2,

$$\int_a^b f(t) (I_{a^+}^{n-\alpha} D^n g)(t) dt = \int_a^b D^n g(t) (I_{b^-}^{n-\alpha} f)(t) dt.$$

Taking into account the facts  $I_{b^-}^{n-\alpha} f \in AC^n[a, b]$  and  $D^{n-1} g \in AC[a, b]$ , then integrating by parts  $n$  times yields

$$\int_a^b f(t) ({}^C D_{a^+}^\alpha g)(t) dt = \sum_{i=0}^{n-1} [(D_{b^-}^{\alpha+i-n} f)(t) (D^{n-1-i} g)(t)]_a^b + (-1)^n \int_a^b g(t) D^n (I_{b^-}^{n-\alpha} f)(t) dt.$$

The conclusion follows from (2.2). □

In this paper, we use the test function

$$\psi(t) := \begin{cases} T^{-\sigma} (T-t)^\sigma, & 0 \leq t \leq T, \\ 0, & t > T. \end{cases} \tag{2.5}$$

This test function enjoys the following properties.

**Lemma 2.4** Let  $\psi$  be as in (2.5) and  $p > 1$ , then for  $\sigma > np - 1$ ,  $n = 0, 1, 2, \dots$ , we have

$$\int_0^T \psi^{1-p}(t) |D^n \psi(t)|^p dt = C_{n,p} T^{1-np}, \quad T > 0,$$

where

$$C_{n,p} = \frac{\Gamma^p(\sigma + 1)}{(\sigma - np + 1) \Gamma^p(\sigma - n + 1)}.$$

**Proof** Since

$$\begin{aligned} D^n \psi(t) &= (-1)^n \sigma(\sigma - 1)(\sigma - 2) \dots (\sigma - n + 1) T^{-\sigma} (T - t)^{\sigma-n} \\ &= \frac{(-1)^n \Gamma(\sigma + 1)}{\Gamma(\sigma - n + 1)} T^{-\sigma} (T - t)^{\sigma-n}, \end{aligned}$$

it follows that

$$\begin{aligned} \int_0^T \psi^{1-p}(t) |D^n \psi(t)|^p dt &= \frac{\Gamma^p(\sigma + 1)}{\Gamma^p(\sigma - n + 1)} T^{-\sigma} \int_0^T (T - t)^{\sigma-np} dt \\ &= C_{n,p} T^{1-np}. \end{aligned}$$

□

**Lemma 2.5** Let  $\alpha \geq 0$  and  $\psi$  be as in (2.5) with  $\sigma > \alpha - 1$ , then we have for all  $0 \leq t \leq T$ ,

$$(D_{T-}^\alpha \psi)(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha + 1)} T^{-\sigma} (T - t)^{\sigma-\alpha}, \tag{2.6}$$

$$\int_0^T t^m (D_{T-}^\alpha \psi)(t) dt = G_{m,\sigma} T^{m+1-\alpha}, \quad m = 0, 1, 2, \dots, n - 1, \quad n = [\alpha] + 1, \tag{2.7}$$

where  $G_{m,\sigma} = \frac{(-1)^m m! \Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha + m + 2)}$ .

**Proof** We have from Lemma 2.1,

$$(D_{T-}^\alpha \psi)(t) = (D_{T-}^\alpha T^{-\sigma} (T - s)^\sigma)(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha + 1)} T^{-\sigma} (T - t)^{\sigma-\alpha}.$$

An integration by parts  $m$  times gives

$$\begin{aligned} \int_0^T t^m (D_{T-}^\alpha \psi)(t) dt &= \sum_{i=0}^{m-1} \left[ (-1)^i \frac{m!}{(m-i)!} t^{m-i} (I_{T-}^{i+1} D_{T-}^\alpha \psi)(t) \right]_0^T \\ &\quad + (-1)^m m! \int_0^T (I_{T-}^m D_{T-}^\alpha \psi)(t) dt. \end{aligned} \tag{2.8}$$

Using (2.6) and Lemma 2.1, we find

$$(I_{T^-}^{i+1} D_{T^-}^\alpha \psi)(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha + i + 2)} T^{-\sigma} (T - t)^{\sigma - \alpha + i + 1},$$

$$(I_{T^-}^m D_{T^-}^\alpha \psi)(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha + m + 1)} T^{-\sigma} (T - t)^{\sigma - \alpha + m}.$$

Therefore,

$$[t^{m-i} (I_{T^-}^{i+1} D_{T^-}^\alpha \psi)(t)]_0^T = 0 \text{ for all } i = 0, 1, 2, \dots, m - 1, \tag{2.9}$$

and

$$\int_0^T (I_{T^-}^m D_{T^-}^\alpha \psi)(t) dt = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha + m + 2)} T^{m - \alpha + 1}. \tag{2.10}$$

Now, substituting (2.9) and (2.10) in (2.8) gives the desired result in (2.7). □

**Lemma 2.6** Let  $\alpha \geq 0$ ,  $n = [\alpha] + 1$  and  $\psi$  be as in (2.5) with  $\sigma > \alpha - 1$ , then

$$(I_{T^-}^{n-\alpha} \psi)(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + n - \alpha + 1)} T^{-\sigma} (T - t)^{\sigma + n - \alpha},$$

for all  $0 \leq t \leq T$ . Moreover,  $I_{T^-}^{n-\alpha} \psi \in AC^n[0, T]$ .

**Proof** It follows from Lemma 2.1 that

$$(I_{T^-}^{n-\alpha} \psi)(t) = (I_{T^-}^{n-\alpha} (T^{-\sigma} (T - s)^\sigma))(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + n - \alpha + 1)} T^{-\sigma} (T - t)^{\sigma + n - \alpha},$$

which is clearly in the space  $AC^n[0, T]$  for  $\sigma > \alpha - 1$ . □

**Lemma 2.7** Let  $\alpha \geq 0$  and  $\psi$  be as in (2.5) with  $\sigma > \max\{0, \alpha - 1\}$ . Suppose that  $f \in AC^n[0, T]$ ,  $n = [\alpha] + 1$  for  $\alpha \notin \mathbb{N}_0$  and  $n = \alpha$  for  $\alpha \in \mathbb{N}_0$ . Then

$$\int_0^T \psi(t) ({}^C D_{0^+}^\alpha f)(t) dt = \int_0^T f(t) (D_{T^-}^\alpha \psi)(t) dt - \sum_{i=0}^{n-1} \hat{G}_{i,\alpha} T^{n-\alpha-i} (D^{n-1-i} f)(0),$$

where  $\hat{G}_{i,\alpha} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-\alpha-i+n)}$ .

**Proof** From (2.6) in Lemma 2.5, we have for  $i = 0, 1, 2, \dots, n - 1$ ,

$$(D_{T^-}^{\alpha+i-n} \psi)(0) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + 1 - \alpha - i + n)} T^{n-\alpha-i},$$

$$(D_{T^-}^{\alpha+i-n} \psi)(T) = 0.$$

As  $\psi \in C[0, T]$  for  $\sigma > 0$  and  $I_{T^-}^{n-\alpha} \psi \in AC^n[0, T]$  by Lemma 2.6, then the result follows in force of Lemma 2.3. □

**3. Main results**

In this section we study the nonexistence of nontrivial global solutions for the fractional system (1.1). To prove our results, we start with the following lemma.

**Lemma 3.1** *Let  $\beta \geq 0$ ,  $n = [\beta] + 1$  and  $p > 1$ . Let  $\psi$  be as in (2.5) with  $\sigma > np - 1$ . Suppose that  $k$  is a nonnegative function which is different from zero almost everywhere and  $t^{p(n-\beta-1)}k^{1-p}(t) \in L^1_{loc}[0, +\infty)$ . Then, for any  $T > 0$*

$$\int_0^T \left( D_{T-}^\beta \psi \right)^p(t) \left( \int_t^T k(s-t)\psi(s) ds \right)^{1-p} dt \leq \hat{C}_{\beta,p} T^{1-np} \int_0^T t^{p(n-\beta-1)}k^{1-p}(t)dt,$$

where  $\hat{C}_{\beta,p} = \frac{C_{n,p}}{\Gamma^p(n-\beta)}$ ,  $C_{n,p}$  is given in Lemma 2.4.

**Proof** Since  $\psi^{(i)}(T) = 0$  for all  $i = 0, 1, \dots, n - 1$ , then  $D_{T-}^\beta \psi = {}^C D_{T-}^\beta \psi$ . Moreover, because  $\psi \in AC^n[0, T]$  for  $\sigma > n - 1$ , we may write  ${}^C D_{T-}^\beta \psi = (-1)^n I_{T-}^{n-\beta} D^n \psi$  and

$$\begin{aligned} \left( D_{T-}^\beta \psi \right)(t) &\leq \left( I_{T-}^{n-\beta} |D^n \psi| \right)(t) = \frac{1}{\Gamma(n-\beta)} \int_t^T (s-t)^{n-\beta-1} |(D^n \psi)(s)| ds \\ &= \frac{1}{\Gamma(n-\beta)} \int_t^T (s-t)^{n-\beta-1} k^{\frac{1}{p'}}(s-t)\psi^{\frac{1}{p'}}(s) k^{-\frac{1}{p'}}(s-t)\psi^{-\frac{1}{p'}}(s) |(D^n \psi)(s)| ds. \end{aligned}$$

Using Hölder inequality with  $\frac{1}{p} + \frac{1}{p'} = 1$ , we find

$$\begin{aligned} \left( D_{T-}^\beta \psi \right)(t) &\leq \frac{1}{\Gamma(n-\beta)} \left( \int_t^T k(s-t)\psi(s) ds \right)^{\frac{1}{p'}} \\ &\quad \times \left( \int_t^T (s-t)^{(n-\beta-1)p} k^{-\frac{p}{p'}}(s-t)\psi^{-\frac{p}{p'}}(s) |(D^n \psi)(s)|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^T \left( D_{T-}^\beta \psi \right)^p(t) \left( \int_t^T k(s-t)\psi(s) ds \right)^{1-p} dt \\ &\leq b_1 \int_0^T \int_t^T (s-t)^{p(n-\beta-1)} k^{-\frac{p}{p'}}(s-t)\psi^{-\frac{p}{p'}}(s) |(D^n \psi)(s)|^p ds dt, \quad b_1 = \frac{1}{\Gamma^p(n-\beta)} \\ &= b_1 \int_0^T \int_0^s (s-t)^{p(n-\beta-1)} k^{1-p}(s-t)\psi^{1-p}(s) |(D^n \psi)(s)|^p dt ds \\ &= b_1 \int_0^T \psi^{1-p}(s) |(D^n \psi)(s)|^p \left( \int_0^s (s-t)^{p(n-\beta-1)} k^{1-p}(s-t) dt \right) ds. \end{aligned}$$



Letting  $\tau = s - t$  in the inner integral, we obtain the uniform bound

$$\int_0^s \tau^{p(n-\beta-1)} k^{1-p}(\tau) d\tau \leq \int_0^T \tau^{p(n-\beta-1)} k^{1-p}(\tau) d\tau,$$

and the result follows from Lemma 2.4. □

Concerning the kernels  $k_i$  and the parameters  $\alpha_i, \beta_i, q_i$  and  $q'_i = \frac{q_i}{q_i-1}, i = 1, 2$ , we introduce the following conditions:

$$(H_1) \quad \lim_{T \rightarrow \infty} T^{\frac{-q_1-1}{q_1 q_2-1}} \left( \int_0^T t^{-\alpha_1 q'_2} k_2^{1-q'_2}(t) dt \right)^{\frac{q_1(q_2-1)}{q_1 q_2-1}} \left( \int_0^T t^{-\alpha_2 q'_1} k_1^{1-q'_1}(t) dt \right)^{\frac{q_1-1}{q_1 q_2-1}} = 0.$$

$$(H_2) \quad \lim_{T \rightarrow \infty} T^{\frac{-q_2-1}{q_1 q_2-1}} \left( \int_0^T t^{-\alpha_1 q'_2} k_2^{1-q'_2}(t) dt \right)^{\frac{q_2-1}{q_1 q_2-1}} \left( \int_0^T t^{-\alpha_2 q'_1} k_1^{1-q'_1}(t) dt \right)^{\frac{q_2(q_1-1)}{q_1 q_2-1}} = 0.$$

$$(H_3) \quad \lim_{T \rightarrow \infty} T^{\frac{-q_1-1}{q_1 q_2-1}} \left( \left( \int_0^T t^{-\alpha_2 q'_1} k_1^{1-q'_1}(t) dt \right)^{\frac{q_1-1}{q_1 q_2-1}} + \left( \int_0^T t^{-\beta_2 q'_1} k_1^{1-q'_1}(t) dt \right)^{\frac{q_1-1}{q_1 q_2-1}} \right) \\ \left( \left( \int_0^T t^{-\alpha_1 q'_2} k_2^{1-q'_2}(t) dt \right)^{\frac{q_1(q_2-1)}{q_1 q_2-1}} + \left( \int_0^T t^{-\beta_1 q'_2} k_2^{1-q'_2}(t) dt \right)^{\frac{q_1(q_2-1)}{q_1 q_2-1}} \right) = 0.$$

$$(H_4) \quad \lim_{T \rightarrow \infty} T^{\frac{-q_2-1}{q_1 q_2-1}} \left( \left( \int_0^T t^{-\alpha_1 q'_2} k_2^{1-q'_2}(t) dt \right)^{\frac{q_2-1}{q_1 q_2-1}} + \left( \int_0^T t^{-\beta_1 q'_2} k_2^{1-q'_2}(t) dt \right)^{\frac{q_2-1}{q_1 q_2-1}} \right) \\ \left( \left( \int_0^T t^{-\alpha_2 q'_1} k_1^{1-q'_1}(t) dt \right)^{\frac{q_2(q_1-1)}{q_1 q_2-1}} + \left( \int_0^T t^{-\beta_2 q'_1} k_1^{1-q'_1}(t) dt \right)^{\frac{q_2(q_1-1)}{q_1 q_2-1}} \right) = 0.$$

**Definition 3.2** By a global nontrivial solution to Problem (1.1), we mean nonzero functions  $u$  and  $v$  defined on  $[0, \infty)$  such that  $(u, v) \in AC[0, T] \times AC[0, T]$  for all  $T > 0$ , for which the equations and initial conditions in (1.1) holds for all  $t > 0$ .

We start with the case of having only one fractional derivative in the left hand sides of (1.1). That is, when  $\mu_1 = 0 = \mu_2$ .

**Theorem 3.3** Let  $k_1$  and  $k_2$  be nonnegative functions which are different from zero almost everywhere with  $t^{-\alpha_1 q'_2} k_2^{1-q'_2}(t), t^{-\alpha_2 q'_1} k_1^{1-q'_1}(t) \in L^1_{loc}[0, +\infty), 0 \leq \alpha_1, \alpha_2 < 1$ . Suppose that  $k_1$  and  $k_2$  satisfy either  $(H_1)$  or  $(H_2)$ , then Problem (1.1), with  $\mu_1 = \mu_2 = 0$ , does not admit any global nontrivial solution when  $u_0 \geq 0$  and  $v_0 \geq 0$ .

**Proof** Assume, on the contrary, that a solution  $(u, v)$  exists for all  $T > 0$ . Multiplying both sides of each equation in (1.1) by the test function  $\psi$  defined in (2.5) with  $\sigma > 2q'_i - 1, i = 1, 2$  and integrating, we obtain

$$\int_0^T \psi(t) ({}^C D_{0^+}^{\alpha_1} u)(t) dt = \int_0^T \psi(t) \left( \int_0^t k_1(t-s) |v(s)|^{q_1} ds \right) dt \tag{3.1}$$

and

$$\int_0^T \psi(t) ({}^C D_{0^+}^{\alpha_2} v)(t) dt = \int_0^T \psi(t) \left( \int_0^t k_2(t-s) |u(s)|^{q_2} ds \right) dt. \tag{3.2}$$

By Lemma 2.7, the relations (3.1) and (3.2) become,

$$\int_0^T u(t) (D_{T^-}^{\alpha_1} \psi)(t) dt = u_0 \hat{G}_{\alpha_1} T^{1-\alpha_1} + \int_0^T \psi(t) \left( \int_0^t k_1(t-s) |v(s)|^{q_1} ds \right) dt \tag{3.3}$$

and

$$\int_0^T v(t) (D_{T^-}^{\alpha_2} \psi)(t) dt = v_0 \hat{G}_{\alpha_2} T^{1-\alpha_2} + \int_0^T \psi(t) \left( \int_0^t k_2(t-s) |u(s)|^{q_2} ds \right) dt, \tag{3.4}$$

where  $\hat{G}_\alpha = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\alpha+2)}$ .

Let

$$\begin{aligned} J_1 &:= \int_0^T \psi(t) \left( \int_0^t k_1(t-s) |v(s)|^{q_1} ds \right) dt \text{ and} \\ J_2 &:= \int_0^T \psi(t) \left( \int_0^t k_2(t-s) |u(s)|^{q_2} ds \right) dt. \end{aligned} \tag{3.5}$$

To obtain bounds for the integrals  $J_1$  and  $J_2$ , we rewrite them in the forms

$$J_1 = \int_0^T |v(s)|^{q_1} \left( \int_s^T k_1(t-s) \psi(t) dt \right) ds = \int_0^T |v(s)|^{q_1} K_1(s) ds,$$

and

$$J_2 = \int_0^T |u(s)|^{q_2} \left( \int_s^T k_2(t-s) \psi(t) dt \right) ds = \int_0^T |u(s)|^{q_2} K_2(s) ds,$$

where

$$K_i(s) := \int_s^T k_i(t-s) \psi(t) dt, \quad 0 \leq s < t \leq T, \quad i = 1, 2.$$

Applying the Hölder inequality with  $\frac{1}{q_i} + \frac{1}{q'_i} = 1, \quad i = 1, 2$  on the integrals on the left hand sides of (3.3) and (3.4) gives

$$\int_0^T u(t) (D_{T^-}^{\alpha_1} \psi)(t) dt \leq \left( \int_0^T |u(t)|^{q_2} K_2(t) dt \right)^{\frac{1}{q_2}} \left( \int_0^T K_2^{-\frac{q'_2}{q_2}}(t) (D_{T^-}^{\alpha_1} \psi)^{q'_2}(t) dt \right)^{\frac{1}{q'_2}},$$

and

$$\int_0^T v(t) (D_{T^-}^{\alpha_2} \psi)(t) dt \leq \left( \int_0^T |v(t)|^{q_1} K_1(t) dt \right)^{\frac{1}{q_1}} \left( \int_0^T K_1^{-\frac{q'_1}{q_1}}(t) (D_{T^-}^{\alpha_2} \psi)^{q'_1}(t) dt \right)^{\frac{1}{q'_1}}.$$

Therefore, (3.3) and (3.4) can be rewritten as

$$\begin{aligned} J_1 + u_0 \hat{G}_{\alpha_1} T^{1-\alpha_1} &\leq J_2^{\frac{1}{q_2}} A_1^{\frac{1}{q'_2}} \text{ and} \\ J_2 + v_0 \hat{G}_{\alpha_2} T^{1-\alpha_2} &\leq J_1^{\frac{1}{q_1}} A_2^{\frac{1}{q'_1}}, \end{aligned} \tag{3.6}$$

where

$$A_1 := \int_0^T K_2^{-\frac{q_2'}{q_2}}(t) (D_{T^-}^{\alpha_1} \psi)^{q_2'}(t) dt \text{ and} \tag{3.7}$$

$$A_2 := \int_0^T K_1^{-\frac{q_1'}{q_1}}(t) (D_{T^-}^{\alpha_2} \psi)^{q_1'}(t) dt.$$

From the nonnegativity of  $u_0, v_0$ , and  $\hat{G}_{\alpha_i}$ , the relations (28) imply (3.6) that

$$J_1 \leq J_2^{\frac{1}{q_2'}} A_1^{\frac{1}{q_2'}} \text{ and } J_2 \leq J_1^{\frac{1}{q_1'}} A_2^{\frac{1}{q_1'}},$$

and consequently

$$J_1 \leq \left( A_2^{\frac{1}{q_1' q_2}} A_1^{\frac{1}{q_2'}} \right)^{\frac{q_1 q_2}{q_1 q_2 - 1}} = A_1^{\frac{q_1(q_2-1)}{q_1 q_2 - 1}} A_2^{\frac{q_1-1}{q_1 q_2 - 1}} \text{ and} \tag{3.8}$$

$$J_2 \leq \left( A_1^{\frac{1}{q_1' q_2}} A_2^{\frac{1}{q_1'}} \right)^{\frac{q_1 q_2}{q_1 q_2 - 1}} = A_1^{\frac{q_2-1}{q_1 q_2 - 1}} A_2^{\frac{q_2(q_1-1)}{q_1 q_2 - 1}}.$$

Now,  $A_1$  defined in (3.7) can be estimated using Lemma 3.1, with  $p = q_2'$ ,  $0 \leq \beta = \alpha_1 < 1$ , as follows

$$A_1 = \int_0^T \left( \int_t^T k_2(s-t) \psi(s) ds \right)^{1-q_2'} (D_{T^-}^{\alpha_1} \psi(t))^{q_2'} dt \leq \hat{C}_{\alpha_1, q_2'} T^{1-q_2'} \int_0^T t^{-\alpha_1 q_2'} k_2^{1-q_2'}(t) dt. \tag{3.9}$$

Similarly, we see that

$$A_2 \leq \hat{C}_{\alpha_2, q_1'} T^{1-q_1'} \int_0^T t^{-\alpha_2 q_1'} k_1^{1-q_1'}(t) dt. \tag{3.10}$$

Substituting (3.9) and (3.10) in (3.8) we end up with

$$\begin{aligned} J_1 &\leq \left( \hat{C}_{\alpha_1, q_2'} T^{1-q_2'} \int_0^T t^{-\alpha_1 q_2'} k_2^{1-q_2'}(t) dt \right)^{\frac{q_1(q_2-1)}{q_1 q_2 - 1}} \left( \hat{C}_{\alpha_2, q_1'} T^{1-q_1'} \int_0^T t^{-\alpha_2 q_1'} k_1^{1-q_1'}(t) dt \right)^{\frac{q_1-1}{q_1 q_2 - 1}} \\ &= \Omega_1 T^{\frac{-q_1-1}{q_1 q_2 - 1}} \left( \int_0^T t^{-\alpha_1 q_2'} k_2^{1-q_2'}(t) dt \right)^{\frac{q_1(q_2-1)}{q_1 q_2 - 1}} \left( \int_0^T t^{-\alpha_2 q_1'} k_1^{1-q_1'}(t) dt \right)^{\frac{q_1-1}{q_1 q_2 - 1}} \end{aligned}$$

and

$$\begin{aligned} J_2 &\leq \left( \hat{C}_{\alpha_1, q_2'} T^{1-q_2'} \int_0^T t^{-\alpha_1 q_2'} k_2^{1-q_2'}(t) dt \right)^{\frac{q_2-1}{q_1 q_2 - 1}} \left( \hat{C}_{\alpha_2, q_1'} T^{1-q_1'} \int_0^T t^{-\alpha_2 q_1'} k_1^{1-q_1'}(t) dt \right)^{\frac{q_2(q_1-1)}{q_1 q_2 - 1}} \\ &= \Omega_2 T^{\frac{-q_2-1}{q_1 q_2 - 1}} \left( \int_0^T t^{-\alpha_1 q_2'} k_2^{1-q_2'}(t) dt \right)^{\frac{q_2-1}{q_1 q_2 - 1}} \left( \int_0^T t^{-\alpha_2 q_1'} k_1^{1-q_1'}(t) dt \right)^{\frac{q_2(q_1-1)}{q_1 q_2 - 1}}, \end{aligned}$$

where  $\Omega_1 = \hat{C}_{\alpha_1, q_2'}^{\frac{q_1(q_2-1)}{q_1 q_2 - 1}} \hat{C}_{\alpha_2, q_1'}^{\frac{q_1-1}{q_1 q_2 - 1}}$  and  $\Omega_2 = \hat{C}_{\alpha_1, q_2'}^{\frac{q_2-1}{q_1 q_2 - 1}} \hat{C}_{\alpha_2, q_1'}^{\frac{q_2(q_1-1)}{q_1 q_2 - 1}}$ .

It follows from the assumptions (H<sub>1</sub>) and (H<sub>2</sub>) that  $(u, v) \equiv (0, 0)$  and this completes the proof. □

Now, we pass to prove the nonexistence result for System (1.1) when  $\mu_1 = \mu_2 = 1$ .

**Theorem 3.4** *Let  $k_1$  and  $k_2$  be nonnegative functions which are different from zero almost everywhere with  $t^{-\alpha_1 q_2'} k_2^{1-q_2'}(t)$ ,  $t^{-\alpha_2 q_1'} k_1^{1-q_1'}(t)$ ,  $t^{-\beta_1 q_2'} k_2^{1-q_2'}(t)$  and  $t^{-\beta_2 q_1'} k_1^{1-q_1'}(t) \in L_{loc}^1[0, +\infty)$ ,  $0 \leq \beta_i < \alpha_i \leq 1$ ,  $i = 1, 2$ .*

Suppose that  $k_1$  and  $k_2$  satisfy either  $(H_3)$  or  $(H_4)$ , then Problem (1.1) does not admit any global nontrivial solution when  $u_0 \geq 0$  and  $v_0 \geq 0$ .

**Proof** Assume, on the contrary, that there exists a nontrivial solution  $(u, v)$  for all  $T > 0$ . Multiplying both sides of each equation in (1.1) by  $\psi$  and integrating, we obtain

$$\int_0^T \psi(t) ({}^C D_{0+}^{\alpha_1} u)(t) dt + \int_0^T \psi(t) ({}^C D_{0+}^{\beta_1} u)(t) dt = J_1, \tag{3.11}$$

$$\int_0^T \psi(t) ({}^C D_{0+}^{\alpha_2} v)(t) dt + \int_0^T \psi(t) ({}^C D_{0+}^{\beta_2} v)(t) dt = J_2, \tag{3.12}$$

where  $J_1$  and  $J_2$  are as in (3.5). Using Lemma 2.7, we have

$$J_1 + u_0 (\hat{G}_{\alpha_1} T^{1-\alpha_1} + \hat{G}_{\beta_1} T^{1-\beta_1}) = \int_0^T u(t) (D_{T-}^{\alpha_1} \psi)(t) dt + \int_0^T u(t) (D_{T-}^{\beta_1} \psi)(t) dt, \tag{3.13}$$

and

$$J_2 + v_0 (\hat{G}_{\alpha_2} T^{1-\alpha_2} + \hat{G}_{\beta_2} T^{1-\beta_2}) = \int_0^T v(t) (D_{T-}^{\alpha_2} \psi)(t) dt + \int_0^T v(t) (D_{T-}^{\beta_2} \psi)(t) dt. \tag{3.14}$$

Proceeding as in the proof of Theorem 3.3, (3.13) and (3.14) become

$$\begin{aligned} J_1 + u_0 (\hat{G}_{\alpha_1} T^{1-\alpha_1} + \hat{G}_{\beta_1} T^{1-\beta_1}) &\leq J_2^{\frac{1}{q_2}} \left( A_1^{\frac{1}{q_2}} + B_1^{\frac{1}{q_2}} \right), \\ J_2 + v_0 (\hat{G}_{\alpha_2} T^{1-\alpha_2} + \hat{G}_{\beta_2} T^{1-\beta_2}) &\leq J_1^{\frac{1}{q_1}} \left( A_2^{\frac{1}{q_1}} + B_2^{\frac{1}{q_1}} \right), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} A_1 &:= \int_0^T K_2^{-\frac{q_2'}{q_2}}(t) (D_{T-}^{\alpha_1} \psi)^{q_2'}(t) dt, \quad B_1 := \int_0^T K_2^{-\frac{q_2'}{q_2}}(t) (D_{T-}^{\beta_1} \psi)^{q_2'}(t) dt, \\ A_2 &:= \int_0^T K_1^{-\frac{q_1'}{q_1}}(t) (D_{T-}^{\alpha_2} \psi)^{q_1'}(t) dt, \quad B_2 := \int_0^T K_1^{-\frac{q_1'}{q_1}}(t) (D_{T-}^{\beta_2} \psi)^{q_1'}(t) dt. \end{aligned} \tag{3.16}$$

As  $u_0, v_0, \hat{G}_{\alpha_i}$  and  $\hat{G}_{\beta_i}$  are nonnegative, (3.15), we deduce

$$J_1 \leq J_2^{\frac{1}{q_2}} \left( A_1^{\frac{1}{q_2}} + B_1^{\frac{1}{q_2}} \right) \quad \text{and} \quad J_2 \leq J_1^{\frac{1}{q_1}} \left( A_2^{\frac{1}{q_1}} + B_2^{\frac{1}{q_1}} \right),$$

which imply that

$$\begin{aligned} J_1^{1-\frac{1}{q_1 q_2}} &\leq \left( A_2^{\frac{1}{q_1}} + B_2^{\frac{1}{q_1}} \right)^{\frac{1}{q_2}} \left( A_1^{\frac{1}{q_2}} + B_1^{\frac{1}{q_2}} \right) \leq \left( A_2^{\frac{1}{q_1 q_2}} + B_2^{\frac{1}{q_1 q_2}} \right) \left( A_1^{\frac{1}{q_2}} + B_1^{\frac{1}{q_2}} \right), \\ J_2^{1-\frac{1}{q_1 q_2}} &\leq \left( A_1^{\frac{1}{q_2}} + B_1^{\frac{1}{q_2}} \right)^{\frac{1}{q_1}} \left( A_2^{\frac{1}{q_1}} + B_2^{\frac{1}{q_1}} \right) \leq \left( A_1^{\frac{1}{q_1 q_2}} + B_1^{\frac{1}{q_1 q_2}} \right) \left( A_2^{\frac{1}{q_1}} + B_2^{\frac{1}{q_1}} \right), \end{aligned}$$

where the basic inequality  $(a + b)^r \leq a^r + b^r$ ,  $a, b \geq 0$ ,  $0 \leq r \leq 1$ , is used. Moreover, with the help of the inequality  $(a + b)^r \leq 2^{r-1}(a^r + b^r)$ ,  $a, b \geq 0$ ,  $r \geq 1$ , we obtain

$$\begin{aligned}
 J_1 &\leq \left( A_2^{\frac{1}{q_1^{q_2}}} + B_2^{\frac{1}{q_1^{q_2}}} \right)^{\frac{q_1 q_2}{q_1 q_2 - 1}} \left( A_1^{\frac{1}{q_2}} + B_1^{\frac{1}{q_2}} \right)^{\frac{q_1 q_2}{q_1 q_2 - 1}} \\
 &\leq 4^{\frac{1}{q_1 q_2 - 1}} \left( A_2^{\frac{q_1 - 1}{q_1 q_2 - 1}} + B_2^{\frac{q_1 - 1}{q_1 q_2 - 1}} \right) \left( A_1^{\frac{q_1(q_2 - 1)}{q_1 q_2 - 1}} + B_1^{\frac{q_1(q_2 - 1)}{q_1 q_2 - 1}} \right) \quad \text{and} \\
 J_2 &\leq \left( A_1^{\frac{1}{q_1^{q_2}}} + B_1^{\frac{1}{q_1^{q_2}}} \right)^{\frac{q_1 q_2}{q_1 q_2 - 1}} \left( A_2^{\frac{1}{q_2}} + B_2^{\frac{1}{q_2}} \right)^{\frac{q_1 q_2}{q_1 q_2 - 1}} \\
 &\leq 4^{\frac{1}{q_1 q_2 - 1}} \left( A_1^{\frac{q_2 - 1}{q_1 q_2 - 1}} + B_1^{\frac{q_2 - 1}{q_1 q_2 - 1}} \right) \left( A_2^{\frac{q_2(q_1 - 1)}{q_1 q_2 - 1}} + B_2^{\frac{q_2(q_1 - 1)}{q_1 q_2 - 1}} \right).
 \end{aligned}
 \tag{3.17}$$

Estimating the integrals  $A_1, A_2, B_1$ , and  $B_2$  as in the proof of Theorem 3.3 and employing Lemma 3.1, we find

$$A_1 \leq \hat{C}_{\alpha_1, q_2'} T^{1 - q_2'} \int_0^T t^{-\alpha_1 q_2'} k_2^{1 - q_2'}(t) dt, \quad B_1 \leq \hat{C}_{\beta_1, q_2'} T^{1 - q_2'} \int_0^T t^{-\beta_1 q_2'} k_2^{1 - q_2'}(t) dt,$$

(3.18)

$$A_2 \leq \hat{C}_{\alpha_2, q_1'} T^{1 - q_1'} \int_0^T t^{-\alpha_2 q_1'} k_1^{1 - q_1'}(t) dt \quad \text{and} \quad B_2 \leq \hat{C}_{\beta_2, q_1'} T^{1 - q_1'} \int_0^T t^{-\beta_2 q_1'} k_1^{1 - q_1'}(t) dt.$$

Using (3.18) in (3.17), we reach a contradiction, in the light of the assumptions  $(H_3)$  and  $(H_4)$ , as the solution is supposed to be nontrivial. □

We notice from Theorem 3.4 that bringing another fractional derivative of sub-first order into the left-hand side of each equation in System (1.1) does not provide global solutions. The following results can be considered as corollaries of Theorem 3.3. Similar results can be obtained for Theorem 3.4.

**Theorem 3.5** *Let  $k_1$  and  $k_2$  be nonnegative functions which are different from zero almost everywhere. Suppose that, for any  $T > 0$ , there are some positive constants  $c_1, c_2, \eta_1$  and  $\eta_2$  with*

$$\eta_1 q_1 (q_2 - 1) + \eta_2 (q_1 - 1) < q_1 + 1 \quad \text{or} \quad \eta_1 (q_2 - 1) + \eta_2 q_2 (q_1 - 1) < q_2 + 1$$

(3.19)

such that

$$\int_0^T t^{-\alpha_1 q_2'} k_2^{1 - q_2'}(t) dt \leq c_1 T^{\eta_1} \quad \text{and} \quad \int_0^T t^{-\alpha_2 q_1'} k_1^{1 - q_1'}(t) dt \leq c_2 T^{\eta_2},$$

(3.20)

where  $0 \leq \alpha_i < 1$  and  $q_i' = \frac{q_i}{q_i - 1}, i = 1, 2$ . Then Problem (1.1) does not admit any global nontrivial solution when  $u_0 \geq 0$  and  $v_0 \geq 0$ .

**Proof** It suffices to observe that the assumptions (3.19) and (3.20) imply the hypothesis  $(H_1)$  and  $(H_2)$  of Theorem 3.3. In fact, from (3.20), we obtain

$$\begin{aligned}
 0 &\leq T^{\frac{-q_1 - 1}{q_1 q_2 - 1}} \left( \int_0^T t^{-\alpha_1 q_2'} k_2^{1 - q_2'}(t) dt \right)^{\frac{q_1(q_2 - 1)}{q_1 q_2 - 1}} \left( \int_0^T t^{-\alpha_2 q_1'} k_1^{1 - q_1'}(t) dt \right)^{\frac{q_1 - 1}{q_1 q_2 - 1}} \\
 &\leq c_1^{\frac{q_1(q_2 - 1)}{q_1 q_2 - 1}} c_2^{\frac{q_1 - 1}{q_1 q_2 - 1}} T^{\frac{-q_1 - 1 + \eta_1 q_1(q_2 - 1) + \eta_2(q_1 - 1)}{q_1 q_2 - 1}},
 \end{aligned}$$

or

$$\begin{aligned}
 0 &\leq T^{\frac{-q_2-1}{q_1q_2-1}} \left( \int_0^T t^{-\alpha_1q'_2} k_2^{1-q'_2}(t) dt \right)^{\frac{q_2-1}{q_1q_2-1}} \left( \int_0^T t^{-\alpha_2q'_1} k_1^{1-q'_1}(t) dt \right)^{\frac{q_2(q_1-1)}{q_1q_2-1}} \\
 &\leq c_1^{\frac{q_2-1}{q_1q_2-1}} c_2^{\frac{q_2(q_1-1)}{q_1q_2-1}} T^{\frac{-q_2-1+\eta_1(q_2-1)+\eta_2q_2(q_1-1)}{q_1q_2-1}}.
 \end{aligned}$$

The conditions (H<sub>1</sub>) and (H<sub>2</sub>) follows from (3.19). □

The next result deals with a large class of kernels that arise in applications.

**Theorem 3.6** *Let  $\gamma_1 > 1 - q_1(1 - \alpha_2)$ ,  $\gamma_2 > 1 - q_2(1 - \alpha_1)$ ,  $0 \leq \alpha_i < 1$  and  $q_i > 1, i = 1, 2$ . Suppose that  $k_i(t) \geq a_i t^{-\gamma_i}$ ,  $t > 0$ , for some constants  $a_i > 0$ , and*

$$\gamma_1 + q_1\gamma_2 < q_1(\alpha_2 + (\alpha_1 - 1)q_2) + q_1 + 2, \quad \text{or} \quad \gamma_2 + q_2\gamma_1 < q_2(\alpha_1 + (\alpha_2 - 1)q_1) + q_2 + 2, \tag{3.21}$$

then Problem (1.1) does not admit a global nontrivial solution when  $u_0 \geq 0$  and  $v_0 \geq 0$ .

**Proof** It is enough to verify the conditions (H<sub>1</sub>) and (H<sub>2</sub>). As  $k_i^{1-q'_i}(t) \leq a_i^{1-q'_i} t^{\gamma_i(q'_i-1)}$ , then

$$T^{\frac{-q_1-1}{q_1q_2-1}} \left( \int_0^T t^{-\alpha_1q'_2} k_2^{1-q'_2}(t) dt \right)^{\frac{q_1(q_2-1)}{q_1q_2-1}} \left( \int_0^T t^{-\alpha_2q'_1} k_1^{1-q'_1}(t) dt \right)^{\frac{q_1-1}{q_1q_2-1}} \leq M_1 T^{\omega_1}$$

and

$$T^{\frac{-q_2-1}{q_1q_2-1}} \left( \int_0^T t^{-\alpha_1q'_2} k_2^{1-q'_2}(t) dt \right)^{\frac{q_2-1}{q_1q_2-1}} \left( \int_0^T t^{-\alpha_2q'_1} k_1^{1-q'_1}(t) dt \right)^{\frac{q_2(q_1-1)}{q_1q_2-1}} \leq M_2 T^{\omega_2},$$

where

$$\begin{aligned}
 M_1 &= \left( \frac{a_2^{1-q'_2}}{\gamma_2(q'_2-1) - \alpha_1q'_2 + 1} \right)^{\frac{q_1(q_2-1)}{q_1q_2-1}} \left( \frac{a_1^{1-q'_1}}{\gamma_1(q'_1-1) - \alpha_2q'_1 + 1} \right)^{\frac{q_1-1}{q_1q_2-1}}, \\
 M_2 &= \left( \frac{a_2^{1-q'_2}}{\gamma_2(q'_2-1) - \alpha_1q'_2 + 1} \right)^{\frac{q_2-1}{q_1q_2-1}} \left( \frac{a_1^{1-q'_1}}{\gamma_1(q'_1-1) - \alpha_2q'_1 + 1} \right)^{\frac{q_2(q_1-1)}{q_1q_2-1}}, \\
 \omega_1 &= \frac{-q_1-1 + (\gamma_2(q'_2-1) - \alpha_1q'_2 + 1)q_1(q_2-1) + (\gamma_1(q'_1-1) - \alpha_2q'_1 + 1)(q_1-1)}{q_1q_2-1} \\
 &= \frac{\gamma_1 - \alpha_2q_1 - \gamma_2q_1 - \alpha_1q_1q_2 + q_1q_2 - q_1 - 2}{q_1q_2-1}, \\
 \omega_2 &= \frac{-q_2-1 + (\gamma_2(q'_2-1) - \alpha_1q'_2 + 1)(q_2-1) + (\gamma_1(q'_1-1) - \alpha_2q'_1 + 1)q_2(q_1-1)}{q_1q_2-1} \\
 &= \frac{\gamma_2 - \alpha_1q_2 + \gamma_1q_2 - \alpha_2q_1q_2 + q_1q_2 - q_2 - 2}{q_1q_2-1}.
 \end{aligned}$$

It follows from (3.21) that both  $\omega_1$  and  $\omega_2$  are negative and so  $(H_1)$  and  $(H_2)$  are satisfied. □

**Remark 3.7** *Theorem 3.6 can be considered also as a consequence of Theorem 3.5 with*

$$\begin{aligned} c_1 &= \frac{a_2^{1-q'_2}}{\gamma_2(q'_2 - 1) - \alpha_1 q'_2 + 1}, \quad c_2 = \frac{a_1^{1-q'_1}}{\gamma_1(q'_1 - 1) - \alpha_2 q'_1 + 1}, \\ \eta_1 &= \gamma_2(q'_2 - 1) - \alpha_1 q'_2 + 1 = \frac{q_2(1 - \alpha_1) + \gamma_2 - 1}{q_2 - 1} \\ \eta_2 &= \gamma_1(q'_1 - 1) - \alpha_2 q'_1 + 1 = \frac{q_1(1 - \alpha_2) + \gamma_1 - 1}{q_1 - 1}. \end{aligned}$$

In the following example, the source term is the Riemann–Liouville fractional integral of a power of the state. This provides a special type of the kernels in System (1.1).

**Example 3.8** *The fractional integro-differential system*

$$\begin{cases} ({}^C D_{0+}^{\alpha_1} u)(t) = (I_{0+}^{\rho_1} |v(s)|^{q_1})(t), \quad t > 0, \rho_1 > 0, \quad q_1 > 1, \\ ({}^C D_{0+}^{\alpha_2} v)(t) = (I_{0+}^{\rho_2} |u(s)|^{q_2})(t), \quad t > 0, \rho_2 > 0, \quad q_2 > 1, \\ u(0) = u_0, \quad v(0) = v_0, \quad u_0, v_0 \in \mathbb{R} \end{cases} \quad (3.22)$$

is a special case of (1.1) when  $0 < \alpha_i < 1, k_i(t) = t^{\rho_i-1}$  and  $\mu_i = 0, i = 1, 2$ . Therefore, as a direct result of Theorem 3.6 with  $\gamma_i = 1 - \rho_i, \rho_1 < q_1(1 - \alpha_2)$  and  $\rho_2 < q_2(1 - \alpha_1)$ , the problem (3.22) does not admit a global nontrivial solution when  $u_0, v_0 \geq 0$ .

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