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Unbounded absolutely weak Dunford–Pettis operators

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Abstract: In the present article, we expose various properties of unbounded absolutely weak Dunford–Pettis and unbounded absolutely weak compact operators on a Banach lattice E. In addition to their topological and lattice properties, we investigate relationships between M-weakly compact operators, L-weakly compact operators, and order weakly compact operators with unbounded absolutely weak Dunford–Pettis operators. We show that the square of any positive uaw-Dunford–Pettis (M-weakly compact) operator on an order continuous Banach lattice is compact. Many examples are given to illustrate the essential conditions.

Key words: uaw-Convergence, uaw-Dunford–Pettis operator, Banach lattice

1. Introduction and preliminaries

The concept of unbounded order convergence under the name of individual convergence was first considered in [13] and "uo-convergence" was initially proposed in [6]. Recently, several papers about uo-convergence in Banach lattices have been published; see [3–5, 8–10, 16] for more details on these results. Unbounded norm convergence was introduced by Troitsky in [15] and further considered in [7, 11]. Unbounded absolutely weak convergence, or uaw-convergence, was presented by Zabeti and investigated in [17].

Let E be a Banach lattice. For a net x_α in E, if there is a net u_γ, possibly over a different index set, with $u_\gamma \downarrow 0$ and for every γ there exists α_0 such that $|x_\alpha - x| \leq u_\gamma$ whenever $\alpha \geq \alpha_0$, we say that x_α converges to x in order, in notation $x_\alpha \xrightarrow{o} x$. A net x_α in E is said to be unbounded order convergent (uo-convergent) to $x \in E$ if for each $u \in E_+$, the net $(|x_\alpha - x| \wedge u)$ converges to zero in order. It is called unbounded norm convergent (un-convergent) if $\||x_\alpha - x| \wedge u\| \to 0$. A net x_α in a Banach lattice E is said to be unbounded absolutely weakly convergent to $x \in E$ ($x_\alpha \xrightarrow{uaw} x$) if for each positive $u \in E$, one has $|x_\alpha - x| \wedge u \xrightarrow{w} 0$.

Suppose that E is a Banach lattice and that X is a Banach space. We say that an operator $T\colon E \to X$ is an unbounded absolutely weak Dunford–Pettis operator, abbreviated as uaw-Dunford–Pettis, if for every norm bounded sequence x_n in E, $x_n \xrightarrow{uaw} 0$ implies $||Tx_n|| \to 0$. We remark that uaw-Dunford–Pettis operators are continuous. We remark further that an example of a uaw-null sequence that is not norm bounded can be found in [17]. We denote by $B_{UDP}(E)$ the space of all uaw-Dunford–Pettis operators on a Banach lattice E.

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In the present paper, we reveal the relationships between uaw-Dunford–Pettis operators, unbounded absolutely weak compact operators (definition given below), M-weakly compact operators, L-weakly compact operators, and o-weakly compact operators. As one of main consequences, we deduce that the square of a positive uaw-Dunford–Pettis (M-weakly compact) operator on an order continuous Banach lattice is compact. In addition, various examples are given to make the concepts and hypotheses more understandable. For the general theory of Dunford–Pettis operators, the reader is referred to [2, 12, 14]. For other necessary terminology on vectors and Banach lattices, we refer the reader to [1, 2].

2. Main results

Proposition 2.1 *Suppose that E is a Banach lattice whose dual space is order continuous and X is a Banach space. In this case, every Dunford–Pettis operator $T\colon E \to X$ is uaw-Dunford–Pettis.*

Proof Suppose T is Dunford–Pettis and x_n is a norm bounded sequence in E, which is uaw-convergent to zero. By [17, Theorem 7], it is weakly convergent. By the assumption, $\|Tx_n\| \to 0$, as desired. □

Note that order continuity of E' is essential in Proposition 2.1 and it cannot be dropped. To see this, consider the identity operator I on ℓ_1. It follows from the Schur property of ℓ_1 that I is Dunford–Pettis. However, it can not be uaw-Dunford–Pettis as the uaw-null sequence $(e_i)_i$ formed by the standard basis of ℓ_1 is not norm convergent to zero. In addition, it can be easily seen that every uaw-Dunford–Pettis operator is continuous, but the converse is not true. Indeed, the identity operator on ℓ_1 is not uaw-Dunford–Pettis.

Remark 2.2 *Suppose that E is an AM-space and X is a Banach space. Since the lattice operations in E are weakly sequentially continuous [2, Theorem 4.31] and in view of Proposition 2.1, it can be seen that an operator $T : E \to X$ is uaw-Dunford–Pettis if and only if it is Dunford–Pettis. Suppose further that E is an atomic order continuous Banach lattice. It follows from [12, Proposition 2.5.23] that if an operator $T\colon E \to X$ is uaw-Dunford–Pettis, then it is a Dunford–Pettis operator.*

It is known that every compact operator between Banach lattices is Dunford–Pettis. In the following example, we show that in the case of a uaw-Dunford–Pettis operator, the situation is different.

Example 2.3 *Let $T\colon \ell_1 \to \mathbb{R}$ be the compact operator defined by $T((x_n)) = \sum_{n=1}^{\infty} x_n$ for every $(x_n) \in \ell_1$. It follows by considering the standard basis of ℓ_1 that T is not a uaw-Dunford–Pettis operator.*

A typical example of a Dunford–Pettis operator that is not compact is the identity operator on ℓ_1 because of the Schur property. However, this operator does not do the job for the uaw-case since it is not also uaw-Dunford–Pettis. Nevertheless, there is good news if one considers a version of Lozanovsky's example as described in [2, page 289, Exercise 10].

Example 2.4 *Consider the operator $T\colon C[0,1] \to c_0$ given by*

$$T(f) = \left(\int_0^1 f(t)\sin t\, dt, \int_0^1 f(t)\sin 2t\, dt, \ldots\right)$$

for every $f \in C[0,1]$. It follows that that T is not order bounded. Hence, by [2, Theorem 5.7], T is not compact. Denote by $(f_n) \subseteq C[0,1]$ a norm bounded sequence for which $f_n \xrightarrow{uaw} 0$ holds. It follows from [17, Theorem

7] that $f_n \xrightarrow{w} 0$ *and that* $||T(f_n)|| = \sup_{m\geq 1} |\int_0^1 f_n(t)\sin mt\, dt| \leq \int_0^1 |f_n(t)|dt \to 0$. *Hence, the noncompact operator* T *is a* uaw*-Dunford–Pettis operator.*

It follows that post- and precompositions of finitely many uaw-Dunford–Pettis operators are again uaw-Dunford–Pettis operators.

Proposition 2.5 *Suppose that* E *is a Banach lattice. Then* $B_{UDP}(E)$ *is a subalgebra of the algebra* $B(E)$ *of continuous operators on* E*.*

Proof If T and S are two uaw-Dunford–Pettis operators and x_n is a norm bounded sequence satisfying $x_n \xrightarrow{uaw} 0$ then $||TS(x_n)|| \to 0$ and $||(T+S)x_n|| \to 0$. □

Recall (see [2] for details) that an operator $T\colon E \to F$ is said to be M-weakly compact if for every norm bounded disjoint sequence x_n in E one has $||Tx_n|| \to 0$. The operator $T\colon E \to F$ is said to be L-weakly compact if every disjoint sequence y_n in the solid hull of $T(B_E)$ is norm null.

Proposition 2.6 *If* $T\colon E \to F$ *is a* uaw*-Dunford–Pettis operator then* T *is* M*-weakly compact. In particular,* $T\colon E \to F$ *is weakly compact.*

Proof If x_n is a norm bounded disjoint sequence in E, by [17, Lemma 2], $x_n \xrightarrow{uaw} 0$. Hence, $||Tx_n|| \to 0$. □

For the converse, we have the following result.

Theorem 2.7 *Suppose* E *and* F *are Banach lattices such that either* E *or* F *is order continuous. Then every positive* M*-weakly compact operator from* E *into* F *is* uaw*-Dunford–Pettis.*

Proof Suppose x_n is a bounded positive uaw-null sequence in E. Let $\varepsilon > 0$ be arbitrary. By [2, Theorem 5.60], due to Meyer-Nieberg, there is a positive $u \in E$ with $\|T(x_n) - T(x_n \wedge u)\| < \frac{\varepsilon}{2}$. First, suppose E is order continuous; since $x_n \wedge u \xrightarrow{w} 0$ and the sequence is order bounded, by [2, Theorem 4.17], we conclude that $\|x_n \wedge u\| \to 0$ so that $\|T(x_n \wedge u)\| \to 0$. Now, assume F is order continuous; $x_n \wedge u \xrightarrow{w} 0$ results in $T(x_n \wedge u) \xrightarrow{w} 0$. Note that this sequence is order bounded so that, by [2, Theorem 4.17], $\|T(x_n \wedge u)\| \to 0$. In any case, we see that $\|Tx_n\| < \varepsilon$ for sufficiently large n, as claimed. □

Corollary 2.8 *Suppose that either* E *or* F *is order continuous. Then every* L*-weakly compact lattice homomorphism from* E *to* F *is* uaw*-Dunford–Pettis.*

Proof It can be verified that T is M-weakly compact (for example, see [2, page 337, Exercise 4]). The conclusion follows from Theorem 2.7. □

Remark 2.9 *Suppose that* E *and* F *are Banach lattices. An operator* $T : E \to F$ *is said to be* uaw*-continuous if it maps bounded* uaw*-null sequences to* uaw*-null ones. It can be verified that every* uaw*-Dunford–Pettis operator is* uaw*-continuous but the converse is not true. The identity operator on* ℓ_1 *is* uaw*-continuous but not* uaw*-Dunford–Pettis.*

We remark that L-weakly compact operators are fruitful tools because of the following result.

Theorem 2.10 *Suppose that E is a Banach lattice and F is an order continuous Banach lattice. Then every L-weakly compact uaw-continuous operator from E into F is uaw-Dunford–Pettis.*

Proof Suppose that x_n is a bounded positive uaw-null sequence in E. Let $\varepsilon > 0$ be arbitrary. By [2, Theorem 5.60], there is a positive $u \in F$ with $\||T(x_n)| - |T(x_n)| \wedge u\| < \frac{\varepsilon}{2}$. Since $Tx_n \xrightarrow{uaw} 0$, we see that $|Tx_n| \wedge u \xrightarrow{w} 0$. Note that this sequence is order bounded so that by [2, Theorem 4.17], $\||Tx_n| \wedge u\| \to 0$. Therefore, $\|Tx_n\| < \varepsilon$ for sufficiently large n, as claimed. □

In the following example, we show that adjoint of a uaw-Dunford–Pettis operator need not be uaw-Dunford–Pettis.

Example 2.11 *Consider the operator T given in Example 2.4. We claim that its adjoint is not uaw-Dunford–Pettis. The adjoint $T' : \ell_1 \to M[0,1]$ is defined via*

$$T'(x_n)(f) = \sum_{n=1}^{\infty} x_n\left(\int_0^1 f(t)\sin nt dt\right),$$

where $M[0,1]$ is the space of all regular Borel measures on $[0,1]$. Note that the standard basis $(e_n)_n$ of ℓ_1 is uaw-null. For each $n \in \mathbb{N}$, put $f_n(t) = \sin nt$. Hence, we have

$$\|T'(e_n)\| \geq \|T'(e_n)(f_n)\| = \int_0^1 (\sin nt)^2 dt \nrightarrow 0.$$

Remark 2.12 *Observe that Example 2.11 can be employed to show that the positivity assumption in Theorem 2.7 and uaw-continuity hypothesis in Theorem 2.10 are essential and cannot be removed. The operator T' is not positive. Since T is uaw-Dunford–Pettis, it is M-weakly-compact. By [2, Theorem 5.67], T' is also M-weakly compact. However, as we see from Example 2.11, it is not uaw-Dunford–Pettis. Furthermore, [2, Theorem 5.67] convinces us that T' is also L-weakly compact. We claim that T' is not uaw-continuous. Note that $e_n \xrightarrow{uaw} 0$. For every $n \in \mathbb{N}$, consider $f_n(t) = \sin nt$. Also, since the sequence $(\sin n)_n$ is dense in $[-1,1]$, we can choose sufficiently large $n \in \mathbb{N}$ with $\sin n > \frac{1}{4}$. Suppose that δ_1 is the Dirac measure at point $x_0 = 1$. Then $(T'(e_n) \wedge \delta_1)(\sin nt) > \frac{1}{4}$.*

Recall that an operator $T : E \to X$ from a Banach lattice E into a Banach space X is o-weakly compact if the image of an order interval of E under T is a weakly relatively compact set in X. Compatible with [2, Theorem 5.91 and Corollary 5.92] and [17, Lemma 2], one may verify the following.

Proposition 2.13 *Every uaw-Dunford–Pettis operator $T : E \to X$ from a Banach lattice E into a Banach space X is o-weakly compact.*

Proposition 2.14 *The square of a uaw-Dunford–Pettis operator carries order intervals into norm totally bounded sets.*

Now we have the following.

Theorem 2.15 *Suppose that E is a Banach lattice and T is a positive uaw-Dunford–Pettis operator on E. Let S be a positive operator on E dominated by T^2. Then the operator S^2 is compact.*

Proof By Proposition 2.6 and Proposition 2.13, T is both o-weakly compact and M-weakly compact. Moreover, by Proposition 2.14, T^2 maps order intervals into norm totally bounded sets. The conclusion follows from [2, page 338, Exercise 13]. □

Observe that since the identity operator on ℓ_1 is Dunford–Pettis, we can not expect compactness of any power of T. However, the following result is surprising.

Corollary 2.16 *Suppose that E is a Banach lattice. Then, for every positive uaw-Dunford–Pettis operator T on E, the operator T^4 is compact.*

Proof The positive operator T^2 is dominated by itself. It follows from Theorem 2.15 that T^4 is compact. □

Corollary 2.17 *Suppose that E is a Banach lattice. The identity operator on E is uaw-Dunford–Pettis if and only if E is finite dimensional.*

Proof Suppose that the identity operator on E is uaw-Dunford–Pettis. By Corollary 2.16, it is compact. This yields that E is finite dimensional. Suppose E is a finite dimensional. Hence, it is atomic and reflexive. Therefore, every uaw-null sequence is weakly null and so norm null. This means that the identity operator on E is uaw-Dunford–Pettis. □

Proposition 2.18 *Suppose that E is an order continuous Banach lattice. Let T be a positive uaw-Dunford–Pettis operator on E. If an operator S satisfies $0 \leq S \leq T$, then the operator S^2 is compact. In particular, the square of a positive uaw-Dunford–Pettis operator is compact.*

Proof By Proposition 2.13, T is o-weakly compact. This means that the order bounded set $T[0,x]$ is relatively weakly compact. By [2, Theorem 4.17], the set $T[0,x]$ is relatively compact in E. By using [2, page 338, Exercise 13], we conclude that if a positive operator S is dominated by T, then the square of S is a compact operator. □

Furthermore, considering Theorem 2.7, we get the following important result.

Corollary 2.19 *The square of a positive M-weakly compact operator on an order continuous Banach lattice E is compact.*

For the uaw-convergence, we have $x_\alpha \xrightarrow{uaw} x$ in Banach lattice E if and only if $|x_\alpha - x| \xrightarrow{uaw} 0$; see [17, Lemma 1]. It allows one to reduce uaw-convergence to the uaw-convergence of positive nets to zero.

Proposition 2.20 *Let $T\colon E \to F$ be a positive uaw-Dunford–Pettis operator between Banach lattices with F Dedekind complete. Then the Kantorovich-like extension $S\colon E \to F$ defined via*

$$S(y) = \sup\left\{T(y \wedge y_n)\colon (y_n) \subseteq E_+, y_n \xrightarrow{uaw} 0\right\}$$

for $y \in E_+$ is again uaw-Dunford–Pettis.

Proof Suppose $y, z \in E_+$. Then

$$S(y+z) = \sup_n\{T((y+z) \wedge \gamma_n)\} \leq \sup_n\{T(y \wedge \gamma_n)\} + \sup_n\{T(z \wedge \gamma_n)\} \leq S(y) + S(z),$$

in which γ_n is a positive sequence that is uaw-null. On the other hand,

$$T(y\wedge\alpha_n)+T(z\wedge\beta_n)=T(y\wedge\alpha_n+z\wedge\beta_n)\leq T((y+z)\wedge(\alpha_n+\beta_n))\leq S(y+z),$$

provided that two positive sequences α_n,β_n are uaw-null so that $S(y)+S(z)\leq S(y+z)$. Therefore, by the Kantorovich extension theorem [2, Theorem 1.10], S extends to a positive operator. Denote by S the extended operator $S\colon E\to F$.

We show that S is also uaw-Dunford–Pettis. Let y_n be a be a norm bounded sequence in E, which is uaw-null. By [17, Lemma 1], $y_n\xrightarrow{uaw}0$ implies $|y_n|\xrightarrow{uaw}0$. We write $y_n=y_n^+-y_n^-$ for each n. Therefore, we have

$$\|S(y_n^+)\|\leq\|S(|y_n|)\|=\|\sup_m T(|y_n|\wedge\alpha_m)\|\leq\|T(|y_n|)\|\to 0,$$

in which α_m is a positive sequence in E, which is convergent to zero in the uaw-topology. Similarly, $\|S(y_n^-)\|\to 0$. Hence, $\|Sy_n\|=\|Sy_n^+-Sy_n^-\|\leq\|Sy_n^+\|+\|Sy_n^-\|\to 0$.

□

In the next example, we show that adjoint of a non-uaw-Dunford–Pettis operator can be uaw-Dunford–Pettis.

Example 2.21 *Consider the operator $T\colon\ell_1\to L^2[0,1]$ defined by $T(x_n)=(\sum_{i=1}^{\infty}x_n)\chi_{[0,1]}$ for all $x_n\in\ell_2$ where $\chi_{[0,1]}$ denotes the characteristic function of $[0,1]$. The operator T is compact but it is not uaw-Dunford–Pettis. Its adjoint $T'\colon L^2[0,1]\to\ell_\infty$ is compact, and hence it is Dunford–Pettis. By Proposition 2.1, we conclude that it is uaw-Dunford–Pettis.*

Remark 2.22 *One may verify that every positive operator dominated by a positive uaw-Dunford–Pettis operator is again uaw-Dunford–Pettis. Therefore, if T is an operator whose modulus is uaw-Dunford–Pettis, it can be easily seen that T is also uaw-Dunford–Pettis. Furthermore, the remarkable theorem of Kalton and Saab ([2, Theorem 5.90]) asserts that if the range space is order continuous, then we can deduce the former statement in the case of Dunford–Pettis operators. Hence, this point can be considered as an advantage for uaw-Dunford–Pettis operators.*

In this step, we investigate closedness properties of $B_{UDP}(E)$.

Proposition 2.23 *$B_{UDP}(E)$ is a closed subalgebra of $B(E)$.*

Proof Suppose that T_m is a sequence of uaw-Dunford–Pettis operators, which is convergent to the operator T. We show that T is also uaw-Dunford–Pettis. Assume that x_n is a bounded uaw-null sequence in E. Given any $\varepsilon>0$, there is an m_0 such that $\|T_m-T\|<\frac{\varepsilon}{2}$ for each $m>m_0$. Fix an $m>m_0$. For sufficiently large n, we have $\|T_m(x_n)\|<\frac{\varepsilon}{2}$. Therefore,

$$\|T(x_n)\|<\|T_m-T\|+\|T_m(x_n)\|<\varepsilon.$$

□

As the following example shows, the closed algebra of all uaw-Dunford–Pettis operators is not order closed.

Example 2.24 *Put $E = c_0$. Suppose that P_n is the projection onto the nth first components. Each P_n is a finite rank operator and so Dunford–Pettis. By Proposition 2.1, P_n is uaw-Dunford–Pettis for all n. Also, $P_n \uparrow I$, where I denotes the identity operator on E. However, I is not uaw-Dunford–Pettis as the standard basis $(e_i)_{i=1}^{\infty}$ is uaw-null but not norm convergent to zero.*

Remark 2.25 *It is a natural question to ask whether the algebra $B_{UDP}(E)$ has a lattice structure or not. This can be reduced as follows. When does the modulus of a uaw-Dunford–Pettis operator exist, and is it again uaw-Dunford–Pettis? In general, the answer to this question is not affirmative. Consider [2, Example 5.6], which is due to Krengel. Observe that the space E mentioned there is a Dedekind complete order continuous Banach lattice whose dual is again order continuous. The operator T is compact and so Dunford–Pettis. By Proposition 2.1, it is uaw-Dunford–Pettis. The sequence $\hat{x}_n$ is disjoint so that by [17, Lemma 2] it is uaw-null. However, as we see in the example, $|T|(\hat{x}_n)$ is not norm null.*

Recall that an operator T between vector lattices E and F is said to preserve disjointness if $x \perp y$ in E implies $Tx \perp Ty$ in F.

Theorem 2.26 *Suppose that E is a Banach lattice. Let T be an order bounded uaw-Dunford–Pettis operator. If T preserves disjointness then T possesses a modulus $|T|$, which is uaw-Dunford–Pettis.*

Proof By [2, Theorem 2.40], the modulus of T exists, and it satisfies the identity $|T|(x) = |T(x)|$ for each positive element $x \in E$. Suppose that x_n is a bounded positive sequence, which is uaw-null. By the hypothesis, $\|Tx_n\| \to 0$. Hence, $|T|(x_n)$ is also norm null. □

Remark 2.27 *Observe that there is no inclusion relation between the algebra of uaw-Dunford–Pettis operators and the class of disjointness preserving operators. The identity operator on ℓ_1 preserves disjointness but it is not uaw-Dunford–Pettis. Furthermore, consider the operator T on $C[0,1]$ defined via $T(f) = (f(0) + f(1))\mathbf{1}$. One may verify that T is a compact operator and so Dunford–Pettis. By Proposition 2.1, it is uaw-Dunford–Pettis but it is not disjoint preserving, as mentioned in [2, Page 117].*

An operator $T\colon X \to E$, where X is a Banach space and E is a Banach lattice, is said to be (sequentially) uaw-compact if $T(B_X)$ is relatively (sequentially) uaw-compact where B_X denotes the closed unit ball of the Banach space X. Equivalently, for every bounded net x_α (respectively, every bounded sequence x_n), its image has a subnet (respectively, subsequence), which is uaw-convergent.

We further say that the operator T is un-compact if $T(B_X)$ is relatively un-compact in E. In [11], some properties of un-compact operators are studied. A more general treatment can be found in [3, 4].

Recall that an element $0 \neq e \in X^+$ of a normed latice X is called a quasi-interior point if the principal ideal I_e generated by e is norm dense in X. The element $0 < e \in X$ is a quasi-interior point if and only if for every $x \in X^+$ we have $||x - x \wedge ne|| \to 0$ as $n \to \infty$.

As in [11, Proposition 9.1] and using [17, Theorem 4 and Proposition 14], we have the same conditions for uaw-compactness and sequentially uaw-compactness of an operator.

Proposition 2.28 *Let $T\colon E \to F$ be an operator between Banach lattices.*

(i) If F is order continuous and has a quasi-interior point then T is uaw-compact if and only if it is sequentially uaw-compact;

(ii) If F is order continuous and T is uaw-compact then T is sequentially uaw-compact;

(iii) If F is an atomic KB-space then T is uaw-compact if and only if T is sequentially uaw-compact.

Remark 2.29 *One of the facts used in the proof of [11, Proposition 9.1, (i)] is that un-topology on a Banach lattice E is metrizable if and only if E has a quasi-interior point. This result can be restated in terms of uaw-topology provided that E is order continuous. Note that order continuity is essential and cannot be dropped; for instance, consider $E = \ell_\infty$. It is easy to see that uaw-topology and absolute weak topology agree on the unit ball B_E of E. However, B_E is not weakly metrizable since E' is not separable. This implies that E cannot be metrizable with respect to the uaw-topology.*

Similar to the case of usual compact and Dunford–Pettis operators, it might seem at first glance that every *uaw*-compact operator is *uaw*-Dunford–Pettis; the following example is surprising.

Example 2.30 *The inclusion $\ell_2 \hookrightarrow \ell_\infty$ is weakly compact by [2, Theorem 5.24]. This operator is sequentially uaw-compact. However, it is not uaw-Dunford–Pettis. For the standard basis $(e_n)_n$ is uaw-null but it is not norm convergent to zero.*

Also, the other implication may fail, as well.

Example 2.31 *Consider the inclusion map $J\colon L^\infty[0,1] \to L^1[0,1]$. It follows from [2, page 313, Exercise 7] that J is weakly compact. In fact, J is uaw-Dunford–Pettis. To see this, suppose f_n is a norm bounded sequence, which converges to zero in the uaw-topology. By [17, Theorem 7], it follows that it is weakly convergent. Since $L^1[0,1] \subseteq (L^\infty[0,1])'$ and the constant function one lies in $L^1[0,1]$, we conclude that $\|f_n\|_1 \to 0$, as claimed. However, J is not uaw-compact, since the norm bounded sequence r_n of the Rademacher functions does not have any uaw-convergent subsequence.*

Let us continue with several ideal properties.

Proposition 2.32 *Let $S\colon E \to F$ and $T\colon F \to G$ be two operators between Banach lattices E, F, and G.*

(i) *If T is (sequentially) uaw-compact and S is continuous then TS is (sequentially) uaw-compact.*

(ii) *If T is a uaw-Dunford–Pettis operator and S is either (sequentially) un-compact or uaw-compact then TS is compact.*

(iii) *If T is uaw-Dunford–Pettis and S is Dunford–Pettis then TS is Dunford–Pettis.*

(iv) *If T is continuous and S is uaw-Dunford–Pettis, then TS is uaw-Dunford–Pettis.*

Proof (i) We prove the results for the sequence case. For nets, the proof is similar. Suppose $(x_n) \subseteq E$ is a bounded sequence. By the assumption, the sequence Sx_n is also norm bounded. Therefore, there is a subsequence $TS(x_{n_k})$ that is *uaw*-convergent.

(ii) Suppose that x_n is a bounded sequence in E. There is a subsequence x_{n_k} such that $S(x_{n_k}) \xrightarrow{uaw} x$ for some $x \in F$. Thus, by the hypothesis, $\|TS(x_{n_k}) - TS(x)\| \to 0$, as desired.

(iii) Suppose that x_n is a sequence in E, which is weakly null. By the assumption, $\|Sx_n\| \to 0$. It follows that $Sx_n \xrightarrow{uaw} 0$. Again, this implies that $\|TS(x_n)\| \to 0$.

(iv) Suppose that x_n is a norm bounded sequence in E, which is uaw-null. By the hypothesis, $\|Sx_n\| \to 0$ so that $\|TS(x_n)\| \to 0$, as desired. □

Denote by $K_{uaw}(E), K_{un}(E)$ the spaces of all uaw-compact and un-compact operators on the Banach lattice E, respectively. In general, we have $K(E) \subseteq K_{un}(E) \subseteq K_{uaw}(E)$. In the next discussion, we show that not every uaw-compact operator is un-compact.

Example 2.33 *The inclusion $\ell_2 \hookrightarrow \ell_\infty$ is weakly compact by [2, Theorem 5.24]. Hence, it is sequentially uaw-compact because the range of the operator is an AM-space. However it is not sequentially un-compact since by [11, Theorem 2.3], it should be compact, which is not possible.*

Remark 2.34 *$K_{un}(E)$ and $K_{uaw}(E)$ are not order closed in the usual order of the space of all continuous operators on E, as shown by [11, Example 9.3]; see also [17, Theorem 4].*

The following results are motivated by the Krengel's theorem; see [2, Theorem 5.9].

Theorem 2.35 *If E is an AL-space and F is a Banach lattice whose dual space is order continuous, then every sequentially uaw-compact operator T from E into F has a sequentially uaw-compact adjoint.*

Proof Let $T\colon E \to F$ be a sequentially uaw-compact operator. For every norm bounded sequence x_n in E, the sequence Tx_n has a subsequence Tx_{n_k}, which is convergent in the uaw-topology. By [17, Theorem 7], the subsequence is weakly convergent. This implies that the operator T is weakly compact. By Gantmacher's theorem [2, Theorem 5.23], it follows that T' is weakly compact. Since the range of T' is an AM-space, it is sequentially uaw-compact. □

Remark 2.36 *Note that order continuity of F' is essential and cannot be removed. Consider the identity operator on ℓ_1. One may verify that it is uaw-compact; ℓ_1 is an atomic KB-space and therefore using [11, Theorem 7.5] and [17, Theorem 4] yields the desired result. However, its adjoint is the identity operator on ℓ_∞, which is not sequentially uaw-compact.*

Theorem 2.37 *If E is an AL-space and F is a reflexive Banach lattice, then every order bounded sequentially uaw-compact operator T from E into F has a weakly compact modulus.*

Proof By Theorem 2.35, if T is sequentially uaw-compact then T' is a sequentially uaw-compact operator. Note that E' is an AM-space. Hence, the operator T' is weakly compact and the result follows from [2, Theorem 5.35]. □

Proposition 2.38 *Let E be a Banach lattice whose dual space is atomic and order continuous. Also, let F be a Banach lattice whose dual is order continuous. Then every (sequentially) un-compact operator $T\colon E \to F$ has a (sequentially) un-compact adjoint operator $T'\colon F' \to E'$.*

Proof For any norm bounded sequence x_n in E, the sequence Tx_n has a subsequence that is un-convergent to zero by un-compactness. By [7, Theorem 6.4], it is weakly convergent. Hence, the operator T is weakly compact. It follows from Gantmacher's theorem that T' is weakly compact. By [11, Proposition 4.16], the operator T' is un-compact. □

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References

[1] Abramovich Y, Aliprantis CD. An Invitation to Operator Theory, Vol. 50. Providence, RI, USA: American Mathematical Society, 2002.

[2] Aliprantis CD, Burkinshaw O. Positive Operators. Berlin, Germany: Springer, 2006.

[3] Aydin A, Emelyanov EY, Erkurşun-Özcan E, Marabeh MAA. Compact-like operators in lattice-normed spaces. Indagationes Mathematicae 2018; 29 (2): 633-656. doi: 10.1016/j.indag.2017.11.002

[4] Aydin A, Emelyanov EY, Erkurşun-Özcan E, Marabeh MAA. Unbounded p-convergence in lattice-normed vector lattices. Siberian Advances in Mathematics 2019; 29 (3): 164-182. doi: 10.3103/S1055134419030027

[5] Dabboorasad YA, Emelyanov EY, Marabeh MAA. Uτ-convergence in locally solid vector lattices. Positivity 2018; 22 (4): 1065-1080. doi: 10.1007/s11117-018-0559-4

[6] DeMarr R. Partially ordered linear spaces and locally convex linear topological spaces. Illinois Journal of Mathematics 1964; 8: 601-606. doi: 10.1215/ijm/1256059459

[7] Deng Y, O'Brien M, Troitsky VG. Unbounded norm convergence in Banach lattices. Positivity 2017; 21 (3): 963-974. doi: 10.1007/s11117-016-0446-9

[8] Gao N. Unbounded order convergence in dual spaces. Journal of Mathematical Analysis and Applications 2014; 419: 347-354. doi: 10.1016/j.jmaa.2014.04.067

[9] Gao N, Troitsky VG, Xanthos F. Uo-convergence and its applications to Cesàro means in Banach lattices. Israel Journal of Mathematics 2017; 220: 649-689. doi: 10.1007/s11856-017-1530-y

[10] Gao N, Xanthos F. Unbounded order convergence and application to martingales without probability. Journal of Mathematical Analysis and Applications 2014; 415: 931-947. doi: 10.1016/j.jmaa.2014.01.078

[11] Kandić M, Marabeh MAA, Troitsky VG. Unbounded norm topology in Banach lattices. Journal of Mathematical Analysis and Applications 2017; 451 (1): 259-279. doi: 10.1016/j.jmaa.2017.01.041

[12] Meyer-Nieberg P. Banach Lattices. Berlin, Germany: Springer-Verlag, 1991.

[13] Nakano H. Ergodic theorems in semi-ordered linear spaces. Annals of Mathematics 1948; 49 (2): 538–556. doi: 10.2307/1969044

[14] Schaefer HH. Banach Lattices and Positive Operators. Berlin, Germany: Springer-Verlag, 1974.

[15] Troitsky VG. Measures of non-compactness of operators on Banach lattices. Positivity 2004; 8 (2): 165–178. doi: 10.1023/B:POST.0000042833.31340.6b

[16] Wickstead AW. Weak and unbounded order convergence in Banach lattices. Journal of Australian Mathematical Society A 1977; 24 (3): 312-319. doi: 10.1017/S1446788700020346

[17] Zabeti O. Unbounded absolute weak convergence in Banach lattices. Positivity 2018; 22 (1): 501-505. doi: 10.1007/s11117-017-0524-7