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Classifying semisymmetric cubic graphs of order $20p$

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Abstract: A simple graph is called semisymmetric if it is regular and edge-transitive but not vertex-transitive. In this paper we classify all connected cubic semisymmetric graphs of order $20p$, p prime.

Key words: Edge-transitive graph, vertex-transitive graph, semisymmetric graph, order of a graph, classification of cubic semisymmetric graphs

1. Introduction

In this paper all graphs are finite, undirected, and simple, i.e. without loops and multiple edges. A graph is called semisymmetric if it is regular and edge-transitive but not vertex-transitive.

The class of semisymmetric graphs was first studied by Folkman [9], who found several infinite families of such graphs and posed eight open problems.

An interesting research problem is to classify connected cubic semisymmetric graphs of various types of orders. In [9], Folkman proved that there are no semisymmetric graphs of order $2p$ or $2p^2$ for any prime p. The classification of semisymmetric graphs of order $2pq$, where p and q are distinct primes, was given in [7].

For prime p, cubic semisymmetric graphs of order $2p^3$ were investigated in [17], in which the authors proved that there is no connected cubic semisymmetric graph of order $2p^3$ for any prime $p \neq 3$ and that for $p = 3$ the only such graph is the Gray graph.

Connected cubic semisymmetric graphs of orders $4p^3$, $6p^2$, $6p^3$, $8p^2$, $8p^3$, $10p^3$, and $18p^n\,(n \geq 1)$ were also classified in [1, 2, 8, 11, 13, 21].

In this paper we investigate connected cubic semisymmetric graphs of order $20p$ for all primes p. Note that for orders like $4p$, $6p$, $10p$, and $14p$, which are of the form $2qp$ for some fixed prime q, the problem of classifying such graphs follows from the general result of [7].

We prove that if Γ is a connected cubic semisymmetric graph of order $20p$, p prime, then $p = 11$ and Γ is isomorphic to a known graph. We go beyond this, however, and prove that there is no connected cubic G-semisymmetric graph of order $20p$ for any prime $p \neq 2, 11$. This will put us near the classification of all connected cubic G-semisymmetric graphs of order $20p$: if there is any such graph, then its order must be either 40 or 220.

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2. Preliminaries

In this paper the symmetric and alternating groups of degree n, the dihedral group of order $2n$, and the cyclic group of order n are respectively denoted by $\mathbb{S}_n$, $\mathbb{A}_n$, D_{2n}, and $\mathbb{Z}_n$. If G is a group and $H \leq G$, then $\mathrm{Aut}(G)$, G', $Z(G)$, $C_G(H)$, and $N_G(H)$ respectively denote the group of automorphisms of G, the commutator subgroup of G, the center of G, the centralizer, and the normalizer of H in G. We also write $H \trianglelefteq^c G$ to denote that H is a characteristic subgroup of G. If $H \trianglelefteq^c K \trianglelefteq G$, then $H \trianglelefteq G$. For a prime p dividing the order of a finite group G, $O_p(G)$ will denote the largest normal p-subgroup of G. It is easy to verify that $O_p(G) \trianglelefteq^c G$.

For a group G and a nonempty set Ω, an action of G on Ω is a function $(g,\omega) \to g.\omega$ from $G \times \Omega$ to Ω, where $1.\omega = \omega$ and $g.(h.\omega) = (gh).\omega$, for every $g,h \in G$ and every $\omega \in \Omega$. We write $g\omega$ instead of $g.\omega$ if there is no fear of ambiguity. For $\omega \in \Omega$, the stabilizer of ω in G is defined as $G_\omega = \{g \in G : g\omega = \omega\}$. The action is called semiregular if the stabilizer of each element in Ω is trivial; it is called regular if it is semiregular and transitive.

For any two groups G and H and any homomorphism $\varphi : H \to \mathrm{Aut}(G)$ the external semidirect product $G \rtimes_\varphi H$ is defined as the group whose underlying set is the Cartesian product $G \times H$ and whose binary operation is defined as $(g_1,h_1)(g_2,h_2) = (g_1\varphi(h_1)(g_2), h_1h_2)$. If $\varphi(h) = 1$ for each $h \in H$, then the semidirect product will coincide with the usual direct product. If $G = NK$ where $N \trianglelefteq G$, $K \leq G$, and $N \cap K = 1$, then G is said to be the internal semidirect product of N and K. These two concepts are in fact equivalent in the sense that there is some homomorphism $\varphi : K \to \mathrm{Aut}(N)$ where $G \simeq N \rtimes_\varphi K$.

The dihedral group D_{2n} is defined as

$$D_{2n} = \left\langle a,b | a^n = b^2 = 1, b^{-1}ab = a^{-1} \right\rangle,$$

so $D_{2n} = \{a^i | i = 0,\ldots,n-1\} \cup \{ba^i | i = 0,\ldots,n-1\}$. All the elements of the form ba^i are of order 2.

Let Γ be a graph. For two vertices u and v, we write $u \sim v$ to denote that u is adjacent to v. If $u \sim v$, then each of the ordered pairs (u,v) and (v,u) is called an arc. The set of all vertices adjacent to a vertex u is denoted by $\Gamma(u)$. The degree or valency of u is $|\Gamma(u)|$. We call Γ regular if all of its vertices have the same valency. The vertex set, the edge set, the arc set, and the set of all automorphisms of Γ are respectively denoted by $V(\Gamma)$, $E(\Gamma)$, $\mathrm{Arc}(\Gamma)$, and $\mathrm{Aut}(\Gamma)$. If Γ is a graph and $N \trianglelefteq \mathrm{Aut}(\Gamma)$, then Γ_N will denote a simple undirected graph whose vertices are the orbits of N in its action on $V(\Gamma)$, and where two vertices Nu and Nv are adjacent if and only if $u \sim nv$ in Γ, for some $n \in N$.

Let Γ_c and Γ be two graphs. Then Γ_c is said to be a covering graph for Γ if there is a surjection $f : V(\Gamma_c) \to V(\Gamma)$ that preserves adjacency, and for each $u \in V(\Gamma_c)$, the restricted function $f|_{\Gamma_c(u)} : \Gamma_c(u) \to \Gamma(f(u))$ is a one to one correspondence. f is called a covering projection. Clearly, if Γ is bipartite, then so is Γ_c. For each $u \in V(\Gamma)$, the fiber on u is defined as $fib_u = f^{-1}(u)$. The following important set is a subgroup of $\mathrm{Aut}(\Gamma_c)$ and is called the group of covering transformations for f:

$$CT(f) = \{\sigma \in \mathrm{Aut}(\Gamma_c) | \forall u \in V(\Gamma), \sigma(fib_u) = fib_u\}.$$

It is known that $K = CT(f)$ acts semiregularly on each fiber [14]. If this action is regular, then Γ_c is said to be a regular K-cover of Γ.

Let $X \leq \mathrm{Aut}(\Gamma)$. Then Γ is said to be X-vertex transitive, X-edge transitive, or X-arc transitive if X acts transitively on $V(\Gamma)$, $E(\Gamma)$, or $\mathrm{Arc}(\Gamma)$, respectively. The graph Γ is called X-semisymmetric if it is regular and X-edge transitive but not X-vertex transitive. Also, Γ is called X-symmetric if it is X-vertex transitive and X-arc transitive. For $X = \mathrm{Aut}(\Gamma)$, we omit X and simply talk about Γ being edge-transitive, vertex-transitive, symmetric, or semisymmetric. As an example, $\Gamma = K_{3,3}$, the complete bipartite graph on 6 vertices, is not semisymmetric but it is X-semisymmetric for some $X \leq \mathrm{Aut}(\Gamma)$.

An X-edge transitive but not X-vertex transitive graph is necessarily bipartite, where the two partites are the orbits of the action of X on $V(\Gamma)$. If Γ is regular, then the two partite sets have equal cardinality, so an X-semisymmetric graph is bipartite such that X is transitive on each partite but X carries no vertex from one partite set to the other.

According to [5], if there is a unique known cubic semisymmetric graph of order n, then it is denoted by **Sn**. The symmetric counterpart of **Sn** is denoted by **Fn** [6]. There are only two symmetric cubic graphs of order 20, which are denoted by **F20A** and **F20B**. Only **F20B** is bipartite [6].

Any minimal normal subgroup of a finite group is the internal direct product of isomorphic copies of a simple group.

A finite group G is called a K_n-group if its order has exactly n distinct prime divisors, where $n \in \mathbb{N}$. The following two results determine all simple K_3-groups and K_4-groups [3, 12, 19, 24].

Theorem 2.1 *(i) If G is a simple K_3-group, then G is isomorphic to one of the following groups:* $\mathbb{A}_5$, $\mathbb{A}_6$, $L_2(7)$, $L_2(2^3)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.
(ii) If G is a simple K_4-group, then G is isomorphic to one of the following groups:

(1) $\mathbb{A}_7$, $\mathbb{A}_8$, $\mathbb{A}_9$, $\mathbb{A}_{10}$, M_{11}, M_{12}, J_2, $L_2(2^4)$, $L_2(5^2)$, $L_2(7^2)$, $L_2(3^4)$, $L_2(97)$, $L_2(3^5)$, $L_2(577)$, $L_3(2^2)$, $L_3(5)$, $L_3(7)$, $L_3(2^3)$, $L_3(17)$, $L_4(3)$, $U_3(2^2)$, $U_3(5)$, $U_3(7)$, $U_3(2^3)$, $U_3(3^2)$, $U_4(3)$, $U_5(2)$, $S_4(2^2)$, $S_4(5)$, $S_4(7)$, $S_4(3^2)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $Sz(2^3)$, $Sz(2^5)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;

(2) $L_2(r)$ *where r is a prime,* $r^2-1 = 2^a \cdot 3^b \cdot s$, $s > 3$ *is a prime,* $a, b \in \mathbb{N}$;

(3) $L_2(2^m)$ *where* m, $2^m - 1$, $\frac{2^m+1}{3}$ *are primes greater than* 3;

(4) $L_2(3^m)$ *where* m, $\frac{3^m+1}{4}$ *and* $\frac{3^m-1}{2}$ *are odd primes.*

Proposition 2.2 *([18], Theorem 9.1.2) Let G be a finite group and $N \trianglelefteq G$. If $|N|$ and $|\frac{G}{N}|$ are relatively prime, then G has a subgroup H such that $G = NH$ and $N \cap H = 1$ (therefore, G is the internal semidirect product of N and H).*

An immediate consequence of the following theorem of Burnside is that the order of every nonabelian simple group is divisible by at least 3 distinct primes.

Theorem 2.3 *[20] For any two distinct primes p and q and any two nonnegative integers a and b, every finite group of order $p^a q^b$ is solvable.*

In the following theorem, the inverse of a pair (a, b) is meant to be (b, a). Also, for each i, A_i, B_i, C_i, and D_i are certain groups of order i with known structures. We will not need their structures.

Theorem 2.4 *[10] If Γ is a connected cubic X-semisymmetric graph, then the order of the stabilizer of any vertex is of the form $2^r \cdot 3$ for some $0 \le r \le 7$. More precisely, if $\{u,v\}$ is any edge of Γ, then the pair (X_u, X_v) can only be one of the following fifteen pairs or their inverses:*
$(\mathbb{Z}_3,\mathbb{Z}_3)$, $(\mathbb{S}_3,\mathbb{S}_3)$, $(\mathbb{S}_3,\mathbb{Z}_6)$, (D_{12},D_{12}), $(D_{12},\mathbb{A}_4)$, $(\mathbb{S}_4,D_{24})$, $(\mathbb{S}_4,\mathbb{Z}_3 \rtimes D_8)$, $(\mathbb{A}_4 \times \mathbb{Z}_2, D_{12} \times \mathbb{Z}_2)$, $(\mathbb{S}_4 \times \mathbb{Z}_2, D_8 \times \mathbb{S}_3)$, $(\mathbb{S}_4,\mathbb{S}_4)$, $(\mathbb{S}_4 \times \mathbb{Z}_2, \mathbb{S}_4 \times \mathbb{Z}_2)$, (A_{96},B_{96}), (A_{192},B_{192}), (C_{192},D_{192}), (A_{384},B_{384}).

Proposition 2.5 *[17] Let Γ be a connected cubic X-semisymmetric graph for some $X \le Aut(\Gamma)$; then either $\Gamma \simeq K_{3,3}$, the complete bipartite graph on 6 vertices, or X acts faithfully on each of the bipartition sets of Γ.*

Theorem 2.6 *[15] Let Γ be a connected cubic X-semisymmetric graph. Let $\{U,W\}$ be a bipartition for Γ and assume $N \trianglelefteq X$. If the actions of N on both U and W are intransitive, then N acts semiregularly on both U and W, Γ_N is $\frac{X}{N}$-semisymmetric, and Γ is a regular N-covering of Γ_N.*

This theorem has a nice result. For every normal subgroup $N \trianglelefteq X$ either N is transitive on at least one partite set or it is intransitive on both partite sets. In the former case, the order of N is divisible by $|U| = |W|$. In the latter case, according to Theorem 2.6, the induced action of N on both U and W is semiregular and hence the order of N divides $|U| = |W|$. Thus, we have the following handy corollary.

Corollary 2.7 *If Γ is a connected cubic X-semisymmetric graph with $\{U,W\}$ as a bipartition and $N \trianglelefteq X$, then either $|N|$ divides $|U|$ or $|U|$ divides $|N|$.*

Following [10] (see also [16]), the coset graph $C(G;H_0,H_1)$ of a group G with respect to finite subgroups H_0 and H_1 is a bipartite graph with $\{H_0g|g \in G\}$ and $\{H_1g|g \in G\}$ as its bipartition sets of vertices where H_0g is adjacent to H_1g' whenever $H_0g \cap H_1g' \neq \emptyset$. The following proposition may be extracted from [10]:

Proposition 2.8 *Let G be a finite group and $H_0, H_1 \le G$. The coset graph $C(G;H_0,H_1)$ has the following properties:*

(i) $C(G;H_0,H_1)$ *is regular of valency d if and only if $H_0 \cap H_1$ has index d in both H_0 and H_1.*

(ii) $C(G;H_0,H_1)$ *is connected if and only if $G = \langle H_0, H_1 \rangle$.*

(iii) *G acts on $C(G;H_0,H_1)$ by right multiplication. Moreover, this action is faithful if and only if $Core_G(H_0 \cap H_1) = 1$.*

(iv) *In the case when the action of G is faithful, the coset graph $C(G;H_0,H_1)$ is G-semisymmetric.*

Proposition 2.9 *[16] Let Γ be a regular graph and $G \le Aut(\Gamma)$. If Γ is G-semisymmetric, then Γ is isomorphic to the coset graph $C(G;G_u,G_v)$ where u and v are adjacent vertices.*

3. Main results

Our goal in this paper is to fully classify connected cubic semisymmetric graphs of order $20p$. We also derive a very restrictive necessary condition for the existence of connected cubic G-semisymmetric graphs of order $20p$. We prove the following important result. Part (i) is a full classification whereas part (ii) is only a necessary condition.

Theorem 3.1 *Let p be a prime.*

(i) *If Γ is a connected cubic semisymmetric graph of order $20p$, then $p = 11$ and $\Gamma \simeq$ **S220**.*

(ii) *If Γ is a connected cubic G-semisymmetric graph of order $20p$ for some $G \leq Aut(\Gamma)$, then $p = 2$ or 11.*

To prove the main theorem, we need some lemmas.

Lemma 3.2 *The only simple K_4-groups whose orders are of the form $2^i \cdot 3 \cdot 5 \cdot p$ for some prime $p > 5$ and some $1 \leq i \leq 8$ are the following three projective special linear groups: $L_2(2^4)$, $L_2(11)$, and $L_2(31)$.*

Proof Considering the powers of primes, there is no possibility for such a group in subitem (4) of item (ii) of Theorem 2.1. By inspecting orders of groups in subitem (1), the only group of the desired form is $L_2(2^4)$. As for subitem (3), let $L_2(2^m)$ be a group of order $2^i \cdot 3 \cdot 5 \cdot p$, and then

$$2^m \cdot 3 \cdot (2^m - 1) \cdot (\tfrac{2^m+1}{3}) = 2^i \cdot 3 \cdot 5 \cdot p,$$

where m, $2^m - 1$, and $\frac{2^m+1}{3}$ are all primes according to Theorem 2.1. This equation has no answer as neither $2^m - 1$ nor $\frac{2^m+1}{3}$ could be equal to 5. Finally, consider groups $L_2(r)$ in subitem (2). If for odd prime r and for prime $s > 3$, we have $r^2 - 1 = 2^a \cdot 3^b \cdot s$ and

$$2^{a-1} \cdot 3^b \cdot s \cdot r = 2^i \cdot 3 \cdot 5 \cdot p,$$

then $b = 1$, $a - 1 = i$, and either $s = 5$ or $r = 5$. The equality $r = 5$ is not possible, since $L_2(5)$ is not a K_4-group. Also, if $s = 5$, then the equation $r^2 - 1 = 2^a \cdot 3 \cdot 5$ gives us only two solutions, $r = 11, 31$, when a spans integers $2, 3, \ldots, 9$. □

Lemma 3.3 *Let $p > 11$ be a prime and $p \neq 17, 31$. If Γ is a connected cubic G-semisymmetric graph of order $20p$, then G has a normal Sylow p-subgroup.*

Proof Take $\{U, W\}$ to be a bipartition for Γ. Then $|U| = |W| = 10p$. For $u \in U$ according to Theorem 2.4, $|G_u| = 2^r \cdot 3$ for some $0 \leq r \leq 7$. Due to transitivity of G on U, the equality $[G : G_u] = |U|$ holds, which yields $|G| = 2^{r+1} \cdot 3 \cdot 5 \cdot p$. If G does not have a normal Sylow p-subgroup, then $O_p(G) = 1$. We derive a contradiction out of this.

Suppose G has a normal subgroup M of order 10. Due to its order, M is intransitive on the partite sets, and according to Theorem 2.6, the quotient graph Γ_M is $\frac{G}{M}$-semisymmetric with a bipartition $\{U_M, W_M\}$ where $|U_M| = |W_M| = p$ and $|\frac{G}{M}| = 2^r \cdot 3 \cdot p$.

Let $\frac{K}{M}$ be a minimal normal subgroup of $\frac{G}{M}$. If $\frac{K}{M}$ is unsolvable, it must be simple of order $2^i \cdot 3 \cdot p$ for some i, so $\frac{K}{M} \simeq \mathbb{A}_5$ or $L_2(7)$. However, these are not possible since $p > 11$. Now if $\frac{K}{M}$ is solvable and hence elementary abelian, then by Corollary 2.7, its order must be p, implying $|K| = 10p$. The Sylow p-subgroup of K is normal and hence characteristic in K. Therefore, it is normal in G, contradicting the assumption that $O_p(G) = 1$. Thus, $O_p(G) = 1$ implies that G does not have a normal subgroup of order 10.

Next, let $N \simeq T^k$ be a minimal normal subgroup of G, where T is simple. If T is nonabelian, then $k = 1$ and $N = T$ since the powers of 3 and 5 in $|G|$ equal 1. According to Corollary 2.7, either $|N|$ divides $|U| = 10p$ or $10p$ divides $|N|$.

If $|N|$ divides $10p$, then $|N| = 2 \cdot 5 \cdot p$, since $|N|$ should be divisible by at least three distinct primes (Theorem 2.3). However, there is no simple K_3-group of order $2 \cdot 5 \cdot p$ according to part (i) of Theorem 2.1, so $10p$ divides $|N|$. Since the order of every simple K_3-group is divisible by 3, N must be a simple K_4-group whose order is of the form $2^i \cdot 3 \cdot 5 \cdot p$. According to Lemma 3.2, $N \simeq L_2(2^4)$, $L_2(11)$, or $L_2(31)$ corresponding to $p = 17$, 11, and 31 respectively. However, these cases are ruled out in the statement of the lemma.

Now suppose that T is abelian and hence N would be elementary abelian. It follows from Corollary 2.7 that $|N|$ divides $10p$ and so $|N| = 2$, 5, or p. Certainly $|N| = p$ contradicts the assumption on $O_p(G)$. In the remaining two cases Γ_N would itself be a connected cubic $\frac{G}{N}$-semisymmetric graph of order $\frac{20p}{|N|}$. Take $\{U_N, W_N\}$ to be the bipartition for Γ_N. Also let $\frac{M}{N}$ be a minimal normal subgroup of $\frac{G}{N}$.

If $N \simeq \mathbb{Z}_2$, then $|\frac{G}{N}| = 2^r \cdot 3 \cdot 5 \cdot p$ and $|U_N| = |W_N| = 5p$. If $\frac{M}{N}$ is unsolvable, then it must be a simple K_4-group whose order is of the form $2^i \cdot 3 \cdot 5 \cdot p$. It follows from Lemma 3.2 that $p = 17$, 11, or 31, which are ruled out by our assumption on p. On the other hand, if $\frac{M}{N}$ is solvable, then its order should divide $|U_N| = 5p$ and hence $|\frac{M}{N}| = 5$ or p. If $|\frac{M}{N}| = 5$, then $|M| = 10$, which is not possible (as we showed at the beginning of the proof), and if $|\frac{M}{N}| = p$, then $|M| = 2p$, which contradicts our assumption on $O_p(G)$ since a Sylow p-subgroup of M would be characteristic in M and so would be normal in G.

Now if $N \simeq \mathbb{Z}_5$, then $|\frac{G}{N}| = 2^{r+1} \cdot 3 \cdot p$ and $|U_N| = |W_N| = 2p$. In this case, if $\frac{M}{N}$ is unsolvable, it would be a simple group of order $2^i \cdot 3 \cdot p$ for some i, and hence, according to Theorem 2.1, $\frac{M}{N} \simeq \mathbb{A}_5$ or $L_2(7)$ implying $p = 5$ or 7. This is in contradiction to our assumption on p. On the other hand, if $\frac{M}{N}$ is solvable, then like before, we conclude that $|\frac{M}{N}| = 2$ or p, which again leads to contradictions as in the previous case. □

Lemma 3.4 *Let $p > 11$ be a prime and $p \neq 17, 31$. Suppose Γ is a connected cubic G-semisymmetric graph of order $20p$. Let M be the Sylow p-subgroup of G. If $\frac{G}{M} \simeq H$, then:*

(1) For each vertex u the stabilizer G_u is isomorphic to a subgroup of H.

(2) $G \simeq M \rtimes_\varphi H$ for some homomorphism $\varphi : H \to Aut(M)$.

Proof For each vertex u of Γ, $MG_u \leq G$. Therefore, $G_u \simeq \frac{G_u}{M \cap G_u} \simeq \frac{MG_u}{M} \leq \frac{G}{M}$. This proves (1). Now since obviously the orders of M and $\frac{G}{M}$ are coprime, it follows from Proposition 2.2 that $G = MK$ for some subgroup $K \leq G$ where $M \cap K = 1$. Thus, G is the internal semidirect product of M and K and hence it is isomorphic to the external semidirect product of M and K, i.e. $G \simeq M \rtimes_\psi K$ for some $\psi : K \to \mathrm{Aut}(M)$. Since $H \simeq \frac{G}{M} = \frac{MK}{M} \simeq \frac{K}{M \cap K} \simeq K$, we can write $G \simeq M \rtimes_\varphi H$ for some $\varphi : H \to \mathrm{Aut}(M)$. □

Lemma 3.5 *Let $p > 11$ be a prime and $p \neq 17, 31$. If Γ is a connected cubic G-semisymmetric graph of order $20p$ and if M is the Sylow p-subgroup of G, then $\frac{G}{M}$ cannot be isomorphic to $\mathbb{A}_5$.*

Proof Suppose, on the contrary, that $\frac{G}{M} \simeq \mathbb{A}_5$. Then for any vertex u from $[G : G_u] = 10p$ we obtain $|G_u| = 6$ and hence $G_u \simeq \mathbb{Z}_6$ or $\mathbb{S}_3$. By Lemma 3.4, $G_u \leq \mathbb{A}_5$. Since $\mathbb{A}_5$ does not have elements of order 6, we conclude that $G_u \simeq \mathbb{S}_3$. Also, according to Lemma 3.4, $G \simeq M \rtimes_\varphi \mathbb{A}_5$. There are only two possibilities for the kernel of $\varphi : \mathbb{A}_5 \to \mathrm{Aut}(M)$:

(a) If $ker(\varphi) = 1$, then $\mathbb{A}_5$ is isomorphic to a subgroup of $\mathrm{Aut}(M) \simeq \mathrm{Aut}(\mathbb{Z}_p) \simeq \mathbb{Z}_{p-1}$, which is obviously not the case.

(b) If $ker(\varphi) = \mathbb{A}_5$, then φ is the trivial homomorphism and so $G \simeq M \times \mathbb{A}_5$. Since Γ is G-semisymmetric, according to Proposition 2.9, Γ is isomorphic to $C(G; G_u, G_v)$ where u and v are two adjacent vertices in Γ. As Γ is connected, according to Proposition 2.8, we must have $G = \langle G_u, G_v \rangle$. In view of $G_u \simeq G_v \simeq \mathbb{S}_3$, this means that $M \times \mathbb{A}_5$ is generated by two of its subgroups, say H and K, both isomorphic to $\mathbb{S}_3$. Now for each element $(m, a) \in H$ we have $(m, a)^6 = 1$, which means $m^6 = 1$ in M. As $|M| = p > 31$, we conclude $m = 1$. Therefore, the first component of each element of H (and similarly for K) equals 1. Consequently, the first component of each element in $M \times \mathbb{A}_5 = \langle H, K \rangle$ equals 1, which is a contradiction. $\square$

Consider a semidirect product $\mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$ where $\varphi : \mathbb{S}_5 \to \mathrm{Aut}(\mathbb{Z}_p)$ is a homomorphism. Let $H \leq \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$ and $H \simeq D_{12}$ or $\mathbb{A}_4$. We call H **type A** if all the elements of H have their second component in $\mathbb{A}_5$. We also call H **type D** if there is at least one element in H whose second component is not in $\mathbb{A}_5$. Also, for any $x \in \mathbb{Z}_p$ and any $g, h \in \mathbb{S}_5$, we define two subsets $R_{x,g,h}, S_{x,g,h} \subset \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$ as follows:

$$R_{x,g,h} = \{(1,1),(x,g),(1,g^2),(x,g^3),(1,g^4),(x,g^5),\ (1,h),(x,hg),(1,hg^2),(x,hg^3),(1,hg^4),(x,hg^5)\}$$

and

$$S_{x,g,h} = \{(1,1),(x,g),(1,g^2),(x,g^3),(1,g^4),(x,g^5),\ (x,h),(1,hg),(x,hg^2),(1,hg^3),(x,hg^4),(1,hg^5)\}.$$

As we will see later, these two subsets are sometimes subgroups of $\mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$.

The group $D_{12} = \left\langle a, b | a^6 = b^2 = 1, b^{-1}ab = a^{-1} \right\rangle$ has exactly three Sylow 2-subgroups, all isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, which are listed below:

$$P_1 = \{1, a^3, b, ba^3\},\ P_2 = \{1, a^3, ba, ba^4\},\ P_3 = \{1, a^3, ba^2, ba^5\}.$$

Lemma 3.6 *Let $p > 3$ be a prime and let $\varphi : \mathbb{S}_5 \to Aut(\mathbb{Z}_p)$ be a homomorphism where $ker(\varphi) = \mathbb{A}_5$. Let $H \leq \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$.*

(i) If $H \simeq \mathbb{A}_4$, then H is of type A, and if $H \simeq D_{12}$, then H is of type D.

(ii) Moreover, if $H \simeq D_{12}$, then there are some $x \in \mathbb{Z}_p$, some $g, g' \notin \mathbb{A}_5$, and some $h \in \mathbb{A}_5$ where $H = R_{x,g,h}$ or $H = S_{x,g,g'}$.

Proof The image of φ is isomorphic to $\frac{\mathbb{S}_5}{\mathbb{A}_5} \simeq \mathbb{Z}_2$, so there is some $F \in \mathrm{Aut}(\mathbb{Z}_p)$ of order 2 for which $\varphi(x) = 1$ for all $x \in \mathbb{A}_5$ and $\varphi(x) = F$ for any $x \notin \mathbb{A}_5$. For any two elements (x, g) and (y, h) from $\mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$ the multiplication $(x, g)(y, h)$ equals (xy, gh) if $g \in \mathbb{A}_5$ and equals $(xF(y), gh)$ if $g \notin \mathbb{A}_5$. It is easy to see that for any positive integer n, if $g \in \mathbb{A}_5$, then $(x, g)^n = (x^n, g^n)$ for all $x \in \mathbb{Z}_p$, and if $g \notin \mathbb{A}_5$, then $(x, g)^{2n} = (x^n F(x^n), g^{2n})$ and $(x, g)^{2n+1} = (x^{n+1} F(x^n), g^{2n+1})$ for all $x \in \mathbb{Z}_p$.

Now, to prove part (i), it suffices to prove that for a subgroup $H \leq \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$ that is isomorphic to either D_{12} or $\mathbb{A}_4$, both of the following statements are true:

- if H is of type A, then $H \simeq \mathbb{A}_4$.
- if H is of type D, then $H \simeq D_{12}$.

Let $H \leq \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$ and $H \simeq D_{12}$ or $\mathbb{A}_4$. If $K := \{(x,g) \in H | g \in \mathbb{A}_5\}$ then $K \leq H$. For the homomorphism $f : K \to \mathbb{Z}_p$ defined by $f(x,g) = x$, the isomorphism $\frac{K}{ker(f)} \simeq Im(f)$ implies that $|\frac{K}{ker(f)}|$ divides both 12 and p and hence $K = ker(f)$. Therefore, for each $(x,g) \in H$, if $g \in \mathbb{A}_5$, then $x = 1$.

It follows immediately that if H is of type A, then the first component of each element of H equals 1 and hence H is isomorphic to a subgroup of $\mathbb{A}_5$. As $\mathbb{A}_5$ has no element of order 6, H cannot be isomorphic to D_{12} and so $H \simeq \mathbb{A}_4$.

Now suppose H is of type D. For two arbitrary elements $(x,g),(y,h) \in H$ with $g,h \notin \mathbb{A}_5$, we have $(xF(x), g^2) = (x,g)^2 \in H$ and $(yF(x), hg) = (y,h)(x,g) \in H$. Since g^2 and hg are in $\mathbb{A}_5$, the first components must equal 1, i.e. $xF(x) = 1$ and $yF(x) = 1$, which imply $x = y$. In other words, for any pair of elements $(x,g) \in H$ and $(y,h) \in H$ with $g,h \notin \mathbb{A}_5$ we must have $x = y$. There are always elements in H whose second component lies in $\mathbb{A}_5$ and hence their first component is 1. Therefore, we can write

$$H = \{(x,g_1),(x,g_2),\ldots,(x,g_n),(1,h_1),\ldots,(1,h_m)\}, \tag{3.1}$$

where $n + m = 12$ and where $g_1,\ldots,g_n \notin \mathbb{A}_5$ and $h_1,\ldots,h_m \in \mathbb{A}_5$. It also follows that for this specific x, $F(x) = x^{-1}$. Let

$$\overline{H} = \{(1,h_1),\ldots,(1,h_m)\}, H_1 = \{h_1,\ldots,h_m\};$$

then $H_1 \simeq \overline{H}$, $\overline{H} \leq H$ and $H_1 \leq \mathbb{A}_5$. Multiplying all the elements of H from equation 3.1, by (x,g_t) for an arbitrary t, we again obtain H. Therefore,

$$H = \{(1,g_tg_1),(1,g_tg_2),\ldots,(1,g_tg_n),(x,g_th_1),\ldots,(x,g_th_m)\}. \tag{3.2}$$

Comparing equalities 3.1 and 3.2, and by taking into account that $g_tg_i \in \mathbb{A}_5$ for $i = 1,\ldots,n$ and $g_th_j \notin \mathbb{A}_5$ for $j = 1,\ldots,m$, it follows that

$$\{g_th_1,\ldots,g_th_m\} = \{g_1,\ldots,g_n\}.$$

Therefore, $m = n = 6$ and so $|\overline{H}| = 6$. Since $\mathbb{A}_4$ does not have a subgroup of order 6, we conclude that $H \simeq D_{12}$.

We now proceed to prove part (ii). Let $D_{12} \simeq H \leq \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$. According to part (i) H is of type D. We continue to use the notations invented in the proof of part (i). Group D_{12} has only two subgroups of order 6, namely $\mathbb{Z}_6$ and $\mathbb{S}_3$. Since $\overline{H} \leq H$ and $|\overline{H}| = 6$, we have $\overline{H} \simeq \mathbb{Z}_6$ or $\mathbb{S}_3$. Since $\overline{H} \simeq H_1 \leq \mathbb{A}_5$ and $\mathbb{A}_5$ does not have elements of order 6, it follows that $\overline{H}$ cannot be isomorphic to $\mathbb{Z}_6$ and hence $\overline{H} \simeq \mathbb{S}_3$. Also, as $H \simeq D_{12}$, we can write $H = \{a^i | i = 0,\ldots,5\} \cup \{ba^i | i = 0,\ldots,5\}$. As $\overline{H} \simeq \mathbb{S}_3$ does not have any element of order 6, we must have $a \in H - \overline{H}$, i.e. $a = (x,g)$ for some $g \notin \mathbb{A}_5$ (see equation 3.1). As for b, there are two possible cases: either $b = (1,h) \in \overline{H}$ or $b = (x,g') \in H - \overline{H}$.

If $b = (1,h)$, $h \in \mathbb{A}_5$, then

$$\begin{aligned} H = \{(x,g)^i | i = 0,\ldots,5\} \cup \{(1,h)(x,g)^i | i = 0,\ldots,5\} = \ &\{(1,1),(x,g),(1,g^2),(x,g^3),(1,g^4),(x,g^5), \\ &(1,h),(x,hg),(1,hg^2),(x,hg^3),(1,hg^4),(x,hg^5)\} = R_{x,g,h}. \end{aligned}$$

Also, if $b = (x,g')$, $g' \notin \mathbb{A}_5$, then

$$H = \{(x,g)^i | i = 0,\ldots,5\} \cup \{(x,g')(x,g)^i | i = 0,\ldots,5\} = \{(1,1),(x,g),(1,g^2),(x,g^3),(1,g^4),(x,g^5),\\ (x,g'),(1,g'g),(x,g'g^2),(1,g'g^3),(x,g'g^4),(1,g'g^5)\} = S_{x,g,g'}.$$

□

Lemma 3.7 *Let $p > 3$ be a prime. A semidirect product $\mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$ does not have two subgroups U and V with all the following properties:*

1) *$(U,V) \simeq (D_{12}, D_{12})$ or $(D_{12}, \mathbb{A}_4)$; and*
2) *$\mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5 = \langle U, V\rangle$; and*
3) *$U \cap V$ is a common Sylow 2-subgroup of both U and V.*

Proof Let $\varphi : \mathbb{S}_5 \to \mathrm{Aut}(\mathbb{Z}_p)$ be a homomorphism. The kernel of φ could not be identity since otherwise $\mathbb{S}_5$ would be isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{Z}_p) \simeq \mathbb{Z}_{p-1}$, which is impossible. On the other hand, if $ker(\varphi) = \mathbb{S}_5$, then φ is the trivial homomorphism and so $\mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5 = \mathbb{Z}_p \times \mathbb{S}_5$. For any two subgroups $U, V \le \mathbb{Z}_p \times \mathbb{S}_5$ both of order 12, the equality $(x,a)^{12} = 1$ holds for each $(x,a) \in U \cup V$. This implies $x^{12} = 1$ and hence $x = 1$, since $x \in \mathbb{Z}_p$. Consequently, the equality $\mathbb{Z}_p \times \mathbb{S}_5 = \langle U, V\rangle$ cannot hold.

The only remaining possibility is to have $ker(\varphi) = \mathbb{A}_5$. We assume there are subgroups U, V with the desired properties and reach a contradiction, so $U \simeq D_{12}$, and hence according to Lemma 3.6, there are some $x \in \mathbb{Z}_p$, some $g, k \notin \mathbb{A}_5$, and some $h \in \mathbb{A}_5$ where $U = R_{x,g,h}$ or $U = S_{x,g,k}$. If $U = R_{x,g,h}$, then all the Sylow 2-subgroups of U are as follows:

$$RP^1_{x,g,h} = \{(1,1),(x,g^3),(1,h),(x,hg^3)\},$$
$$RP^2_{x,g,h} = \{(1,1),(x,g^3),(x,hg),(1,hg^4)\},$$
$$RP^3_{x,g,h} = \{(1,1),(x,g^3),(1,hg^2),(x,hg^5)\},$$

and if $U = S_{x,g,k}$, then all the Sylow 2-subgroups of U are as follows:

$$SP^1_{x,g,k} = \{(1,1),(x,g^3),(x,k),(1,kg^3)\},$$
$$SP^2_{x,g,k} = \{(1,1),(x,g^3),(1,kg),(x,kg^4)\},$$
$$SP^3_{x,g,k} = \{(1,1),(x,g^3),(x,kg^2),(1,kg^5)\}.$$

For some i either $RP^i_{x,g,h}$ or $SP^i_{x,g,k}$ must also be a Sylow 2-subgroup of V. If $V \simeq \mathbb{A}_4$, then according to Lemma 3.6, it is of type A and hence the first components of all the elements of each of its Sylow 2-subgroups equal 1. However, there are elements in $RP^i_{x,g,h}$ and in $SP^i_{x,g,k}$ whose first components are equal to x, so if $V \simeq \mathbb{A}_4$, then $x = 1$. Every element of $\langle U, V\rangle$ is an alternating product of elements from U and V. Since in the semidirect product we have $(1,t)(1,s) = (1,ts)$ for any $t, s \in \mathbb{S}_5$, it follows that the first component of every element from $\langle U, V\rangle$ is 1 and hence $\langle U, V\rangle \ne \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$.

On the other hand, if $V \simeq D_{12}$, then according to Lemma 3.6, either $V = R_{y,g',h'}$ or $V = S_{y,g',k'}$ for some $y \in \mathbb{Z}_p$, some $g', k' \notin \mathbb{A}_5$, and some $h' \in \mathbb{A}_5$. Again, all the Sylow 2-subgroups of V are known. The first component of each element from any Sylow 2-subgroup of U is 1 or x and the first component of each element from any Sylow 2-subgroup of V is 1 or y. Since U and V have at least one common Sylow 2-subgroup (namely $U \cap V$), we must have $x = y$.

Now define $W = (\{1\} \times \mathbb{A}_5) \cup (\{x\} \times (\mathbb{S}_5 - \mathbb{A}_5))$. It is easy to check that $W \leq \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$. Obviously $U \cup V \subset W$, and so $\langle U, V\rangle \leq W$. Therefore, $\langle U, V\rangle \neq \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$. □

Proof of Theorem 3.1. We first provide a general discussion on G-semisymmetric graphs. Let Γ be a connected cubic G-semisymmetric graph of order n. Then Γ is regular and bipartite. Moreover, it is G-edge-transitive and hence edge-transitive. Now, if Γ is not vertex-transitive, then by definition it is semisymmetric cubic of order n. On the other hand, if Γ is vertex-transitive, then it is symmetric cubic of order n, since according to [23] a cubic vertex- and edge-transitive graph is necessarily symmetric. Therefore, Γ is either a bipartite cubic symmetric graph of order n or it is a cubic semisymmetric graph of order n.

We now set off to prove part (ii) of Theorem 3.1. For $p = 3, 5, 7, 17, 31$ there is no connected cubic semisymmetric graph of order $20p$ according to [5]. Also, for $p = 5, 7, 17$, no connected cubic symmetric graph of order $20p$ exists according to [6]. As for $p = 3, 31$, according to [6] there exists only one connected cubic symmetric graph of order $20p$, which is not bipartite. Therefore, we conclude that for $p = 3, 5, 7, 17, 31$ there is no connected cubic G-semisymmetric graph of order $20p$.

Now let $p > 11$ be a prime such that $p \neq 17, 31$. Suppose, on the contrary, that Γ is a connected cubic G-semisymmetric graph of order $20p$ for some $G \leq \mathrm{Aut}(\Gamma)$. Let $\{U, W\}$ be the bipartition for Γ. Then $|U| = |W| = 10p$ and $|G| = 2^{r+1} \cdot 3 \cdot 5 \cdot p$ for some $0 \leq r \leq 7$. If M is a Sylow p-subgroup of G, then according to Lemma 3.3, $M \trianglelefteq G$. Due to its order, M is intransitive on both U and W and so, according to Theorem 2.6, Γ_M is a connected cubic G_M-semisymmetric graph of order 20 with the bipartition $\{U_M, W_M\}$, where $G_M \simeq \frac{G}{M}$ and $|U_M| = |W_M| = 10$. According to the general discussion we just had, Γ_M is either a bipartite cubic symmetric graph or a cubic semisymmetric graph of order 20. By [5] there is no semisymmetric cubic graph of order 20 and by [6] there is only one bipartite symmetric cubic graph of order 20, namely **F20B**. Therefore, $\Gamma_M \simeq$ **F20B**.

The automorphism group of **F20B** has 240 elements [6] and G_M is isomorphic to a subgroup of Aut(**F20B**) of order $|G_M| = 2^{r+1} \cdot 3 \cdot 5$. The equality is not possible since G_M is not transitive on V(**F20B**) whereas Aut(**F20B**) is. Thus, $|G_M| < 240$ and hence $1 \leq r+1 \leq 3$. Also, G_M is transitive on both U_M and W_M, and according to Proposition 2.5, the action of G_M on each of U_M and W_M is faithful. Therefore, G_M is a transitive permutation group of degree 10. Transitive permutation groups of degree 10 were completely classified in [4]. There are 45 such groups up to isomorphism, which are denoted $T1, T2, \cdots, T45$ in [4] and the only ones whose orders are of the form $2^i \cdot 3 \cdot 5$ for $1 \leq i \leq 3$ are $T7 \simeq \mathbb{A}_5$ of order 60, and $T11$, $T12$, and $T13 \simeq \mathbb{S}_5$ of order 120.

First note that $G_M \simeq T7$ is not possible according to Lemma 3.5. Next, we argue that G_M could not be isomorphic to $T11$ or $T12$.

In [4] all the transitive groups of degree 10 are defined with a set of generating permutations on ten points. If

$$a = (1,2,3,4,5),\ b = (6,7,8,9,10),\ e = (1,5)(2,3),\ f = (6,10)(7,8),$$
$$g = (1,2),\ h = (6,7),\ \text{and}\ i = (1,6)(2,7)(3,8)(4,9)(5,10),$$

then $T11 = \langle ab, ef, i\rangle$ and $T12 = \langle ab, ef, ghi\rangle$. Using the GAP software [22], it is easy to verify that $H = \langle i\rangle$ of order 2 is a normal subgroup of $T11$.

If $G_M \simeq T11$, then according to Theorem 2.6, the quotient graph of Γ_M with respect to H, which we denote by $(\Gamma_M)_H$, would be R-semisymmetric of order 10, where $R \simeq \frac{T11}{H}$. This implies that R is transitive on each partite set, and by Proposition 2.5, R would be a transitive permutation group of degree 5. Again according to [4], the only transitive permutation group of degree 5 and of order 60 is $\mathbb{A}_5$, so we should have $R \simeq \mathbb{A}_5$. Now the stabilizer of any vertex of $(\Gamma_M)_H$ under the action of R has $\frac{|R|}{5} = 12$ points and the only subgroup of $\mathbb{A}_5$ of order 12 is isomorphic to $\mathbb{A}_4$. For an edge $\{u, w\}$ of the cubic R-semisymmetric graph $(\Gamma_M)_H$, we have $(R_u, R_w) = (\mathbb{A}_4, \mathbb{A}_4)$, which is not possible according to Theorem 2.4. Therefore, the assumption that $G_M \simeq T11$ leads to a contradiction.

Now suppose $G_M \simeq T12$. Calculated by GAP, the stabilizer of 1 under $T12$ is

$$(T12)_1 = \langle (2,4)(3,5)(7,9)(8,10), (3,5,4)(8,10,9) \rangle .$$

Again, using GAP, one finds out that this group is nonabelian of order 12, which has the following group as a normal subgroup:

$$\langle (2,3)(4,5)(7,8)(9,10), (2,4)(3,5)(7,9)(8,10) \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 .$$

There are only 3 nonabelian groups of order 12 up to isomorphism: $\mathbb{A}_4$, D_{12} and the dicyclic group of order 12. Among these, only $\mathbb{A}_4$ has a normal subgroup of order 4. Thus, $(T12)_i \simeq (T12)_1 \simeq \mathbb{A}_4$ for any $i = 1, 2, \ldots 10$. However, this is impossible by Theorem 2.4.

Finally, suppose $G_M \simeq \frac{G}{M} \simeq T13$. Since $M \simeq \mathbb{Z}_p$, by Lemma 3.4, $G \simeq \mathbb{Z}_p \rtimes_\varphi \mathbb{S}_5$ for some homomorphism $\varphi : \mathbb{S}_5 \to \mathrm{Aut}(\mathbb{Z}_p)$. From $[G : G_u] = 10p$ we have $|G_u| = 12$ for any vertex u, so if $\{u, v\}$ is a fixed edge of Γ, then it follows from Theorem 2.4, that $(G_u, G_v) \simeq (D_{12}, D_{12})$ or $(D_{12}, \mathbb{A}_4)$. Of course, $(G_u, G_v) \simeq (\mathbb{A}_4, D_{12})$ is nothing new, since then we can change the roles of u and v.

Also, $\Gamma \simeq C(G; G_u, G_v)$ by Proposition 2.9. Now it follows from part (ii) of Proposition 2.8 that $G = \langle G_u, G_v \rangle$ and from part (i) of the same Proposition that $|G_u \cap G_v| = 4$, i.e. $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v. However, the existence of G_u and G_v with all these properties contradicts Lemma 3.7.

Since every case for G_M is contradictory, part (ii) follows.

Next we turn to part (i) of Theorem 3.1. For $p \neq 2, 11$ there is no connected cubic semisymmetric graph of order $20p$ according to part (ii). Also, there is no such graph of order 20×2 according to [5], and by the same reference, there is only one connected cubic semisymmetric graph of order 20×11, namely **S220**.

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