


## On the regularity of the solution map of the Euler–Poisson system

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**Abstract:** In this paper we consider the Euler–Poisson system (describing a plasma consisting of positive ions with a negligible temperature and massless electrons in thermodynamical equilibrium) on the Sobolev spaces  $H^s(\mathbb{R}^3)$ ,  $s > 5/2$ . Using a geometric approach we show that for any time  $T > 0$  the corresponding solution map,  $(\rho_0, u_0) \mapsto (\rho(T), u(T))$ , is nowhere locally uniformly continuous. On the other hand it turns out that the trajectories of the ions are analytic curves in  $\mathbb{R}^3$ .

**Key words:** Euler–Poisson system, solution map

### 1. Introduction

The initial value problem for the Euler–Poisson system in  $\mathbb{R}^3$  is given by

$$\begin{aligned}\rho_t + \operatorname{div}(\rho u) &= 0, \\ u_t + (u \cdot \nabla)u &= -\nabla\theta, \\ e^\theta - \Delta\theta &= \rho, \\ \lim_{|x| \rightarrow \infty} \rho &= 1, \\ u(0) = u_0, \rho(0) &= \rho_0,\end{aligned}\tag{1.1}$$

where  $\theta : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is the electric potential,  $\rho : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  the ion density and  $u = (u_1, u_2, u_3) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  the ion velocity. The equations (1.1) describe the dynamics of a plasma consisting of positive ions with negligible temperature and massless electrons in thermodynamical equilibrium. The ions are described by conservation of mass (first equation) and conservation of momentum for charged particles (second equation). The third equation is the Poisson equation for the electric potential, where the Boltzmann relation “electron density =  $e^\theta$ ” is used. The fourth equation is the assumption that the plasma is uniform at infinity.

The system (1.1) is well posed in the Sobolev spaces  $H^s$ ,  $s > 5/2$ , see [14] and see also [6] for local well-posedness of a modified version of (1.1) in 1D. More precisely, given  $(\rho_0, u_0) \in (1 + H^{s-1}(\mathbb{R}^3)) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$  with  $\rho_0(x) > 0$  for all  $x \in \mathbb{R}^3$ , there is  $T > 0$  such that we have a unique solution

$$(\rho, u) \in C([0, T]; (1 + H^{s-1}(\mathbb{R}^3)) \times H^s(\mathbb{R}^3; \mathbb{R}^3))$$

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to (1.1). Moreover, the solution map

$$(\rho_0, u_0) \mapsto (\rho(T), u(T))$$

is continuous. To state our main result we denote the time  $T$  solution map by  $\Phi_T$ . Its domain of definition around the equilibrium point  $(1, 0) \in (1 + H^{s-1}(\mathbb{R}^3)) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$  by  $U_T \subseteq (1 + H^{s-1}(\mathbb{R}^3)) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$ . We then have

**Theorem 1.1** *Let  $s > 5/2$  and  $T > 0$ . Then the solution map*

$$\Phi_T : U_T \rightarrow (1 + H^{s-1}(\mathbb{R}^3)) \times H^s(\mathbb{R}^3; \mathbb{R}^3), \quad (\rho_0, u_0) \mapsto (\rho(T), u(T))$$

*is nowhere locally uniformly continuous.*

Theorem 1.1 tells us that the dependence of system (1.1) on the initial conditions is very “rough”, i.e. that it is not more than continuous. That the solution map of the modified version of (1.1) in 1D does not have the property of being uniformly continuous on bounded sets was established in [8]. They used an approach similar to the one in [7] developed for the incompressible Euler equations. We think that with the improvements made in the recent work [2], one could use the methods in [7] to prove Theorem 1.1. However, we proceed in a different way. Our approach is more along the lines of [10, 11, 13]. In [10] the analog of Theorem 1.1 was proved for the incompressible Euler equations, in [11] for the Holm-Staley  $b$ -family of equations, and in [13] for the inviscid SQG equation.

We also show

**Theorem 1.2** *The trajectories of the ions are analytic curves in  $\mathbb{R}^3$ .*

The particle trajectories are given by integrating

$$\dot{\gamma} = u \circ \gamma, \quad \gamma(0) = x \in \mathbb{R}^3.$$

From ODE theory we know that the maximal regularity guaranteed for  $\gamma$  is determined by the regularity of  $u$ . However, Theorem 1.2 tells us that the trajectories are always analytic, even if  $u$  has just Sobolev regularity  $H^s$ , which is very astonishing. This phenomenon occurs also in other hydrodynamical models, see [4].

To establish our results we give in a first step a geometric formulation of (1.1). The strategy is to express (1.1) in Lagrangian coordinates, i.e. in terms of the flow map

$$\varphi_t = u \circ \varphi, \quad \varphi(0) = \text{id},$$

where  $\text{id} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is the identity map. This approach was popularized by the work cited in [1, 5] for the incompressible Euler equations. In a second step we use the fact that a modified vorticity is “transported” by the flow to produce perturbations which are irregular in some sense.

**2. Geometric formulation**

To motivate the geometric formulation let us assume that  $(\rho, u)$  is a solution to (1.1). Consider the flow map of  $u$ , i.e.

$$\varphi_t = u \circ \varphi, \quad \varphi(0) = \text{id}.$$

We take the  $t$  derivative in

$$\frac{d}{dt}(\det(d\varphi)\rho \circ \varphi) = (\text{div } u) \circ \varphi \det(d\varphi)\rho \circ \varphi + \det(d\varphi)(\rho_t + (u \cdot \nabla)\rho) \circ \varphi = 0,$$

where  $d\varphi$  denotes the jacobian of  $\varphi$ . Reexpressing gives

$$\rho = \left( \frac{\rho_0}{\det(d\varphi)} \right) \circ \varphi^{-1}.$$

Denote by  $\theta = F^{-1}(\rho)$  the solution for  $\theta$  in (1.1). We can thus write

$$u_t + (u \cdot \nabla)u = -\nabla F^{-1}\left(\left(\frac{\rho_0}{\det(d\varphi)}\right) \circ \varphi^{-1}\right).$$

Using  $\varphi_{tt} = (u_t + (u \cdot \nabla)u) \circ \varphi$

$$\varphi_{tt} = -\left(\nabla F^{-1}\left(\left(\frac{\rho_0}{\det(d\varphi)}\right) \circ \varphi^{-1}\right)\right) \circ \varphi.$$

Our goal is to prove that the right hand side above is an analytic expression in  $\varphi$ . However, first we have to introduce the appropriate functional space for  $\varphi$ . Consider for  $s > 5/2$

$$\mathcal{D}^s(\mathbb{R}^3) = \{\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \mid \varphi - \text{id} \in H^s(\mathbb{R}^3; \mathbb{R}^3) \text{ and } \det(d_x\varphi) > 0 \forall x \in \mathbb{R}^3\}.$$

By the Sobolev imbedding theorem this space consists of  $C^1$  diffeomorphisms. As a subset of  $\text{id} + H^s(\mathbb{R}^3; \mathbb{R}^3)$  it gets a differential structure and is a connected topological group under composition, see [9]. We know that every  $u \in C([0, T]; H^s(\mathbb{R}^3; \mathbb{R}^3))$  generates a unique flow  $\varphi \in C^1([0, T]; \mathcal{D}^s(\mathbb{R}^3))$  with  $\varphi(0) = \text{id}$ , see [12]. To handle the elliptic equation in (1.1) we use the lemma from [14].

**Lemma 2.1** *Let  $\rho \in 1 + H^{s-1}(\mathbb{R}^3)$  with  $\rho > 0$  in  $\mathbb{R}^3$ . Then there is a unique  $\theta \in H^{s+1}(\mathbb{R}^3)$  with*

$$e^\theta - \Delta\theta = \rho.$$

Denote this  $\theta$  by  $F^{-1}(\rho)$ . We also introduce the following open subset of  $H^{s-1}(\mathbb{R}^3)$

$$U_{\bar{\rho}} = \{\bar{\rho} \in H^{s-1}(\mathbb{R}^3) \mid \bar{\rho}(x) > -1 \quad \forall x \in \mathbb{R}^3\}.$$

In particular our  $\rho$  in (1.1) lies in  $1 + U_{\bar{\rho}}$ .

**Lemma 2.2** *Let  $\bar{\rho}_0 = \rho_0 - 1 \in U_{\bar{\rho}}$ . Then there is an open set  $W \subseteq H^{s-2}(\mathbb{R}^3)$  with  $\bar{\rho}_0 \in W$  and an open set  $V \subseteq H^s(\mathbb{R}^3)$  such that*

$$V \subseteq H^s(\mathbb{R}^3) \rightarrow W \subseteq H^{s-2}(\mathbb{R}^3), \quad \theta \mapsto e^\theta - 1 - \Delta\theta$$

*is an analytic diffeomorphism.*

**Proof** From the Banach algebra property of  $H^s$  we see that

$$H^s(\mathbb{R}^3) \rightarrow H^s(\mathbb{R}^3), \quad \theta \mapsto e^\theta - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} \theta^k$$

is analytic. Therefore

$$\Gamma : H^s(\mathbb{R}^3) \rightarrow H^{s-2}(\mathbb{R}^3), \quad \theta \mapsto e^\theta - 1 - \Delta\theta$$

is analytic. Let  $\theta_0 = F^{-1}(\rho_0)$ . The differential of  $\Gamma$  at  $\theta_0$  is given by

$$d_{\theta_0}\Gamma : H^s(\mathbb{R}^3) \rightarrow H^{s-2}(\mathbb{R}^3), \quad h \mapsto e^{\theta_0} \cdot h - \Delta h,$$

which by linear elliptic theory is known to be an isomorphism. By the inverse function theorem we get open neighborhoods of  $\theta_0$  in  $H^s$  resp.  $\bar{\rho}_0$  in  $H^{s-2}$  on which  $\Gamma$  is an analytic diffeomorphism.  $\square$

For  $\varphi \in \mathcal{D}^s(\mathbb{R}^3)$  we denote by  $R_\varphi$  the linear map  $f \mapsto f \circ \varphi$ . This map is continuous on  $H^{s'}$  for  $0 \leq s' \leq s$ . Note also that  $R_{\varphi^{-1}} = R_\varphi^{-1}$ . For  $\Gamma$  introduced above, i.e.

$$\Gamma : H^s(\mathbb{R}^3) \rightarrow H^{s-2}(\mathbb{R}^3), \quad \theta \mapsto e^\theta - 1 - \Delta\theta,$$

one sees that it is injective, see [14]. Therefore, we can use  $\Gamma^{-1}$  in the following. Using the notation of Lemma 2.2 we have:

**Lemma 2.3** *Let  $\bar{\rho}_0 = \rho_0 - 1 \in U_{\bar{\rho}}$  and  $\varphi_0 \in \mathcal{D}^s(\mathbb{R}^3)$ . Then there is an open neighborhood  $W \times V \subseteq \mathcal{D}^s(\mathbb{R}^3) \times H^{s-2}(\mathbb{R}^3)$  of  $(\varphi_0, \bar{\rho}_0)$  such that*

$$W \times V \rightarrow H^s(\mathbb{R}^3), \quad (\varphi, \bar{\rho}) \mapsto R_\varphi \Gamma^{-1}(\bar{\rho} \circ \varphi^{-1})$$

is analytic.

Note that by Lemma 2.2 the expression  $R_\varphi \Gamma^{-1}(\bar{\rho} \circ \varphi^{-1})$  is defined for  $W$  and  $V$  small enough.

**Proof** Consider the map

$$\Pi : (\varphi, \theta) \mapsto (\varphi, R_\varphi \Gamma(\theta \circ \varphi^{-1})). \tag{2.1}$$

Note that

$$\Pi(\varphi, \theta) = (\varphi, e^\theta - 1 - R_\varphi \Delta(\theta \circ \varphi^{-1})).$$

We have

$$R_\varphi \Delta(\theta \circ \varphi^{-1}) = \sum_{k=1}^3 R_\varphi \partial_k R_\varphi^{-1} R_\varphi \partial_k R_\varphi^{-1} \theta.$$

By the chain rule

$$R_\varphi \nabla(\theta \circ \varphi^{-1}) = [d\varphi^\top]^{-1} \nabla \theta,$$

which shows that for  $1 \leq s' \leq s$  and  $k = 1, 2, 3$

$$\mathcal{D}^s(\mathbb{R}^3) \times H^{s'}(\mathbb{R}^3) \rightarrow H^{s'-1}(\mathbb{R}^3), \quad (\varphi, \theta) \mapsto R_\varphi \partial_k R_\varphi^{-1} \theta$$

is analytic, see also [12]. Thus,  $\Pi$  is analytic. Let  $\theta_0 = F^{-1}(\rho_0)$ . The differential of  $\Pi$  at  $(\varphi_0, \theta_0)$  is of the form

$$d_{(\varphi_0, \theta_0)}\Pi(g, h) = \begin{pmatrix} g & 0 \\ * & R_{\varphi_0} \left( e^{\theta_0 \circ \varphi_0^{-1}} \cdot (h \circ \varphi_0^{-1}) - \Delta(h \circ \varphi_0^{-1}) \right) \end{pmatrix},$$

showing that it is an isomorphism. By the inverse function theorem we conclude that  $\Pi$  is an analytic diffeomorphism in a neighborhood of  $(\varphi_0, \theta_0)$  in  $\mathcal{D}^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ . Since

$$R_\varphi \Gamma^{-1}(\bar{\rho} \circ \varphi^{-1})$$

is the second component of  $\Pi^{-1}$  the claim follows. □

Note that  $\det(d\varphi) \in 1 + H^{s-1}$ . The map

$$\mathcal{D}^s(\mathbb{R}^3) \times (1 + U_{\bar{\rho}}) \rightarrow H^{s-1}(\mathbb{R}^3), \quad (\varphi, \rho_0) \mapsto \frac{\rho_0}{\det(d\varphi)} - 1$$

is analytic. Thus, we get from Lemma 2.3 that

$$\mathcal{D}^s(\mathbb{R}^3) \times (1 + U_{\bar{\rho}}) \rightarrow H^s(\mathbb{R}^3), \quad (\varphi, \rho_0) \mapsto \theta \circ \varphi = R_\varphi \Gamma^{-1} \left( \left( \frac{\rho_0}{\det(d\varphi)} - 1 \right) \circ \varphi^{-1} \right)$$

is analytic. However, we need more regularity for  $\nabla\theta$ .

**Lemma 2.4** *The map*

$$\begin{aligned} \mathcal{D}^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3) \times H^{s-2}(\mathbb{R}^3) &\rightarrow H^s(\mathbb{R}^3) \\ (\varphi, \theta, \bar{\rho}) &\mapsto R_\varphi (e^{\theta \circ \varphi^{-1}} - \Delta)^{-1}(\bar{\rho} \circ \varphi^{-1}) \end{aligned}$$

*is analytic.*

**Proof** We proceed as in Lemma 2.2 and look at the inverse expression, so consider

$$\begin{aligned} \Theta : \mathcal{D}^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3) &\rightarrow \mathcal{D}^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3) \times H^{s-2}(\mathbb{R}^3) \\ (\varphi, \theta, \xi) &\mapsto (\varphi, \theta, R_\varphi (e^{\theta \circ \varphi^{-1}} - \Delta)(\xi \circ \varphi^{-1})), \end{aligned} \tag{2.2}$$

which is analytic. Its differential is of the form

$$d_{(\varphi, \theta, \xi)}\Theta(g, h, f) = \begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ * & * & R_\varphi (e^{\theta \circ \varphi^{-1}} - \Delta)(f \circ \varphi^{-1}) \end{pmatrix}.$$

By the inverse function theorem we get the conclusion as:

$$R_\varphi (e^{\theta \circ \varphi^{-1}} - \Delta)^{-1}(\bar{\rho} \circ \varphi^{-1})$$

is the third component of  $\Theta^{-1}$ . □

Combining Lemmas 2.2-2.4 we have

**Proposition 2.5** *The map*

$$\mathcal{D}^s(\mathbb{R}^3) \times (1 + U_{\bar{\rho}}) \rightarrow H^s(\mathbb{R}^3; \mathbb{R}^3)$$

$$(\varphi, \rho_0) \mapsto R_\varphi \left( \nabla F^{-1} \left( \left( \frac{\rho_0}{\det(d\varphi)} \right) \circ \varphi^{-1} \right) \right)$$

*is analytic.*

**Proof** Applying  $\nabla$  to  $e^\theta - \Delta\theta = \rho$  gives

$$e^\theta \cdot \nabla\theta - \Delta\nabla\theta = \nabla\rho$$

or

$$\nabla\theta = (e^\theta - \Delta)^{-1}\nabla\rho$$

and

$$R_\varphi\nabla\theta = R_\varphi \left( (e^{(\theta \circ \varphi) \circ \varphi^{-1}} - \Delta)^{-1}\nabla\rho \right).$$

Substituting  $\rho$  gives

$$\nabla\rho = \left( [d\varphi^\top]^{-1}\nabla \left( \frac{\rho_0}{\det(d\varphi)} \right) \right) \circ \varphi^{-1}.$$

Since

$$\mathcal{D}^s(\mathbb{R}^3) \times (1 + U_{\bar{\rho}}) \rightarrow H^s(\mathbb{R}^3), \quad (\varphi, \rho_0) \mapsto \theta \circ \varphi$$

and

$$\mathcal{D}^s(\mathbb{R}^3) \times (1 + U_{\bar{\rho}}) \rightarrow H^{s-2}(\mathbb{R}^3; \mathbb{R}^3), \quad (\varphi, \rho_0) \mapsto [d\varphi^\top]^{-1}\nabla \left( \frac{\rho_0}{\det(d\varphi)} \right)$$

are analytic we get by Lemma 2.4 that

$$\mathcal{D}^s(\mathbb{R}^3) \times (1 + U_{\bar{\rho}}) \rightarrow H^s(\mathbb{R}^3; \mathbb{R}^3), \quad (\varphi, \rho_0) \mapsto \nabla\theta \circ \varphi$$

is analytic, which concludes the proof □

Consider the differential equation for the variables  $(\varphi, v) \in \mathcal{D}^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$

$$\frac{d}{dt} \begin{pmatrix} \varphi \\ v \end{pmatrix} = \begin{pmatrix} v \\ -R_\varphi \left( \nabla F^{-1} \left( \left( \frac{\rho_0}{\det(d\varphi)} \right) \circ \varphi^{-1} \right) \right) \end{pmatrix}. \tag{2.3}$$

By Proposition 2.5 this is an analytic ODE depending analytically on the parameter  $\rho_0$ . Consider the Cauchy problem with initial conditions  $\varphi(0) = \text{id}$  and  $v(0) = u_0$ . The solution  $\varphi$  provides via

$$u(t) := \varphi_t(t) \circ \varphi(t)^{-1} \quad \text{and} \quad \rho(t) = \left( \frac{\rho_0}{\det(d\varphi(t))} \right) \circ \varphi(t)^{-1}$$

a solution to (1.1) in  $(\rho, u) \in C([0, T]; (1 + U_{\bar{\rho}}) \times H^s(\mathbb{R}^3; \mathbb{R}^3))$ . Thus, by the existence and uniqueness result for ODEs and the composition properties of  $\mathcal{D}^s(\mathbb{R}^3)$  we get the local well-posedness of (1.1), see also [12]. Furthermore, as the trajectories of the ions are described by the analytic curves

$$t \mapsto \varphi(t, x)$$

we get Theorem 1.2.

### 3. Nonuniform dependence

We denote by  $\omega$  the vorticity of  $u$ , i.e

$$\omega = \text{curl } u.$$

Taking curl in the second equation in (1.1) gives

$$\omega_t + (u \cdot \nabla)\omega + \text{div } u \cdot \omega - (\omega \cdot \nabla)u = 0. \tag{3.1}$$

Now we take the  $t$  derivative in

$$\begin{aligned} \frac{d}{dt} (\det(d\varphi)[d\varphi]^{-1}\omega \circ \varphi) &= \det(d\varphi) \text{div } u \circ \varphi [d\varphi]^{-1}\omega \circ \varphi \\ &\quad - \det(d\varphi)[d\varphi]^{-1}[d\varphi_t][d\varphi]^{-1}\omega \circ \varphi \\ &\quad + \det(d\varphi)[d\varphi]^{-1}(\omega_t + (u \cdot \nabla)\omega) \circ \varphi. \end{aligned}$$

Using  $d\varphi_t = du \circ \varphi \cdot d\varphi$  and (3.1) gives

$$\frac{d}{dt} (\det(d\varphi)[d\varphi]^{-1}\omega \circ \varphi) = 0$$

or

$$\omega(t) = \left( \frac{1}{\det(d\varphi(t))} [d\varphi(t)]\omega_0 \right) \circ \varphi(t)^{-1}, \tag{3.2}$$

where  $\omega_0 = \text{curl } u_0$ . With the help of (3.2) we can express  $\omega(t)$  as some sort of “pullback” of  $\omega_0$ . This will be useful for our purpose. For the time  $T > 0$  we denote as above by  $U_T \subseteq (1 + U_{\bar{\rho}}) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$  the domain of definition of the time  $T$  solution map  $\Phi_T$  around the equilibrium point  $(1, 0) \in (1 + H^{s-1}(\mathbb{R}^3)) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$ . With  $\Psi_T$  we denote the solution map in Lagrangian coordinates, i.e.

$$\Psi_T : U_T \rightarrow \mathcal{D}^s(\mathbb{R}^3), \quad (\rho_0, u_0) \mapsto \varphi(T),$$

where  $\varphi(T)$  denotes the time  $T$  value of the  $\varphi$  component of the solution in (2.3) with initial conditions  $\varphi(0) = \text{id}$  and  $v(0) = u_0$ . Note that  $\Psi_T$  is analytic. Later we will use the following technical lemma.

**Lemma 3.1** *There is a dense subset  $S \subseteq U_T$  with the property that for each  $(\rho_\bullet, u_\bullet) \in S$  we have that  $u_\bullet$  is compactly supported and there are  $h = (h_\rho, h_u) \in H^{s-1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$  and  $x^* \in \mathbb{R}^3$  such that  $\text{dist}(x^*, \text{supp } u_\bullet) > 2$  (i.e. the distance of the point  $x^*$  to the support of  $u_\bullet$  is greater than 2) with*

$$(d_{(\rho_\bullet, u_\bullet)}\Psi_T(h))(x^*) \neq 0.$$

**Proof** Our strategy is to get an equation for  $d_{(1,0)}\Psi$ . Consider for small  $|\varepsilon|$ ,  $\bar{\rho} \in H^{s-1}(\mathbb{R}^3)$  and  $\bar{u} \in H^s(\mathbb{R}^3; \mathbb{R}^3)$  the ODE (2.3)

$$\frac{d}{dt} \begin{pmatrix} \varphi^{(\varepsilon)} \\ v^{(\varepsilon)} \end{pmatrix} = \begin{pmatrix} v^{(\varepsilon)} \\ G(\varphi^{(\varepsilon)}, 1 + \varepsilon\bar{\rho}) \end{pmatrix}, \quad \varphi^{(\varepsilon)}(0) = \text{id}, v^{(\varepsilon)}(0) = \varepsilon\bar{u},$$

where  $G$  is the corresponding expression from (2.3). In particular we have

$$\varphi^{(0)}(t) = \text{id}, \quad v^{(0)}(t) = 0 \quad \forall t \geq 0.$$

We consider the variation

$$\partial\varphi(t) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \varphi^{(\varepsilon)}(t), \quad \partial v(t) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} v^{(\varepsilon)}(t).$$

Calculating the variation in the ODE we get

$$\frac{d}{dt} \begin{pmatrix} \partial\varphi \\ \partial v \end{pmatrix} = \begin{pmatrix} \partial v \\ d_{(\text{id},1)}G(\partial\varphi, \bar{\rho}) \end{pmatrix}, \quad \partial\varphi(0) = 0, \partial v(0) = \bar{u}. \tag{3.3}$$

We calculate  $d_{(\text{id},1)}G$ . Note that with the maps from (2.1) and (2.2)

$$G(\varphi, 1 + \bar{\rho}) = (\Theta^{-1})_3 \left( \varphi, (\Pi^{-1})_2(\varphi, \frac{1 + \bar{\rho}}{\det(d\varphi)} - 1), [d\varphi^\top]^{-1} \nabla \frac{1 + \bar{\rho}}{\det(d\varphi)} \right).$$

Using the rules for differentiating determinants we have for  $k = 1, 2, 3$

$$\partial_k \frac{1}{\det(d\varphi)} = -\frac{1}{(\det(d\varphi))^2} \det(d\varphi) \operatorname{tr}([d\varphi]^{-1} \partial_k d\varphi) = -\frac{1}{\det(d\varphi)} \operatorname{tr}([d\varphi]^{-1} \partial_k d\varphi).$$

Thus, the derivative of

$$\varphi \mapsto [d\varphi^\top]^{-1} \nabla \frac{1}{\det(d\varphi)} = -\frac{1}{\det(d\varphi)} [d\varphi^\top]^{-1} \begin{pmatrix} \operatorname{tr}([d\varphi]^{-1} \partial_1 d\varphi) \\ \operatorname{tr}([d\varphi]^{-1} \partial_2 d\varphi) \\ \operatorname{tr}([d\varphi]^{-1} \partial_3 d\varphi) \end{pmatrix}$$

in direction of  $w \in H^s(\mathbb{R}^3; \mathbb{R}^3)$  at  $\varphi = \text{id}$  is given by

$$\operatorname{tr}(dw) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + dw^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \partial_1 w_1 \\ \partial_2 w_2 \\ \partial_3 w_3 \end{pmatrix} - \begin{pmatrix} \partial_1(\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3) \\ \partial_2(\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3) \\ \partial_3(\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3) \end{pmatrix},$$

which we denote by  $B(w)$ . We see that  $B : H^s(\mathbb{R}^3; \mathbb{R}^3) \rightarrow H^{s-2}(\mathbb{R}^3; \mathbb{R}^3)$  is a continuous linear map. The partial derivative of  $G$  with respect to  $\varphi$  at  $(\text{id}, 1)$  is then given by

$$\begin{aligned} \partial_\varphi|_{(\text{id},1)} G(w) &= [(d_{(\text{id},0)}\Theta)^{-1}]_{31}(w) + [(d_{(\text{id},0)}\Theta)^{-1}]_{32}(\ast) + [(d_{(\text{id},0)}\Theta)^{-1}]_{33}(Bw) \\ &= 0 + 0 + (1 - \Delta)^{-1}Bw = (1 - \Delta)^{-1}Bw \end{aligned}$$

for  $w \in H^s(\mathbb{R}^3; \mathbb{R}^3)$ . We introduce

$$A : H^s(\mathbb{R}^3; \mathbb{R}^3) \rightarrow H^s(\mathbb{R}^3; \mathbb{R}^3), \quad w \mapsto (1 - \Delta)^{-1}Bw.$$

It is a Fourier multiplier operator with a continuous  $L^\infty$  multiplier

$$m_A(\xi_1, \xi_2, \xi_3) = \frac{1}{(1 + |\xi|^2)} \begin{pmatrix} -3i\xi_1 + \xi_1^2 & -i\xi_1 - i\xi_2 + \xi_1\xi_2 & -i\xi_1 - i\xi_3 + \xi_1\xi_3 \\ -i\xi_1 - i\xi_2 + \xi_1\xi_2 & -3i\xi_2 + \xi_2^2 & -i\xi_2 - i\xi_3 + \xi_2\xi_3 \\ -i\xi_1 - i\xi_3 + \xi_1\xi_3 & -i\xi_2 - i\xi_3 + \xi_2\xi_3 & -3i\xi_3 + \xi_3^2 \end{pmatrix},$$

where  $\xi = (\xi_1, \xi_2, \xi_3)$ . The partial derivative of  $G$  with respect to  $\rho$  at  $(\text{id}, 1)$  in direction of  $\bar{\rho} \in H^{s-1}(\mathbb{R}^3)$  is

$$\partial_\rho|_{(\text{id},1)} G(\bar{\rho}) = (1 - \Delta)^{-1} \nabla \bar{\rho}.$$



Hence (3.3) reads as

$$\frac{d}{dt} \begin{pmatrix} \partial\varphi \\ \partial v \end{pmatrix} = \begin{pmatrix} 0 & \text{Id} \\ A & 0 \end{pmatrix} \begin{pmatrix} \partial\varphi \\ \partial v \end{pmatrix} + \begin{pmatrix} 0 \\ (1-\Delta)^{-1}\nabla\bar{\rho} \end{pmatrix}, \quad \partial\varphi(0) = 0, \partial v(0) = \bar{u}.$$

By Duhamel's principle the solution at time  $T > 0$  is given by

$$\begin{pmatrix} \partial\varphi(T) \\ \partial v(T) \end{pmatrix} = e^M \begin{pmatrix} 0 \\ \bar{u} \end{pmatrix} + \int_0^T e^{(T-s)M} \begin{pmatrix} 0 \\ (1-\Delta)^{-1}\nabla\bar{\rho} \end{pmatrix} ds,$$

where

$$M = \begin{pmatrix} 0 & \text{Id} \\ A & 0 \end{pmatrix}.$$

For  $\partial\varphi(T)$  the only contribution comes from the right up entry. We have

$$e^M = \begin{pmatrix} * & \sum_{k=0}^{\infty} \frac{T^{2k+1}}{(2k+1)!} A^k \\ * & * \end{pmatrix} \text{ resp. } e^{(T-s)M} = \begin{pmatrix} * & \sum_{k=0}^{\infty} \frac{(T-s)^{2k+1}}{(2k+1)!} A^k \\ * & * \end{pmatrix}.$$

Integrating gives

$$\partial\varphi(T) = \sum_{k=0}^{\infty} \frac{T^{2k+1}}{(2k+1)!} A^k \bar{u} + \frac{T^{2k+2}}{(2k+2)!} A^k (1-\Delta)^{-1}\nabla\bar{\rho}.$$

Consider the operator

$$K = \sum_{k=0}^{\infty} \frac{T^{2k+1}}{(2k+1)!} A^k.$$

It is a Fourier multiplier operator with multiplier

$$m(\xi) = \sum_{k=0}^{\infty} \frac{T^{2k+1}}{(2k+1)!} m_A(\xi)^k,$$

which is a continuous and bounded function with  $m(0) = T \cdot \text{Id}$ . Therefore,  $m$  is different from zero. Thus, there is  $\bar{u} \in H^s(\mathbb{R}^3; \mathbb{R}^3)$  with

$$K\bar{u} \neq 0.$$

Now fix  $(\rho_{\bullet}, u_{\bullet}) \in U_T$  with  $u_{\bullet}$  compactly supported. Take  $x^* \in \mathbb{R}^3$  with  $\text{dist}(x^*, \text{supp}(u_{\bullet})) > 2$ . Since  $K$  is a Fourier multiplier operator it is translation invariant. Thus, we can find  $h_u = \bar{u}(\cdot + \Delta x)$  with

$$(Kh_u)(x^*) \neq 0.$$

With this choice of  $h_u$  and  $h_{\rho} = 0$  we have

$$(d_{(1,0)}\Psi_T(h_{\rho}, h_u))(x^*) = (\partial\varphi(T))(x^*) = (Kh_u)(x^*) \neq 0.$$

Take an analytic curve  $\gamma = (\gamma_1, \gamma_2) : [0, 1] \rightarrow U_T$  connecting  $(1, 0)$  with  $(\rho_{\bullet}, u_{\bullet})$  with the property that the second component  $\gamma_2(t)$  is always compactly supported. This is clearly possible as we can take an arbitrary analytic curve and then multiply  $\gamma_2$  with a cut-off function. Now consider the analytic curve

$$\alpha : [0, 1] \rightarrow \mathbb{R}^3, \quad t \mapsto (d_{\gamma(t)}\Psi_T(h_{\rho}, h_u))(x^*).$$

As  $\alpha(0) \neq 0$  there is a sequence  $t_n \uparrow 1$  with  $\alpha(t_n) \neq 0$  for  $n \geq 1$ . We can put all these  $\gamma(t_n)$  to  $S$  after moving  $x^*$  outside of the support of the cut-off function if necessary. By this construction we see that  $S \subseteq U_T$  is dense.  $\square$

We can now prove Theorem 1.1.

**Proof** [Proof of Theorem 1.1] We fix  $(\rho_0, u_0) \in S$  where  $S \subseteq U_T$  is as in Lemma 3.1. In successive steps below we will choose  $R_* > 0$  and then show that for all  $0 < R \leq R_*$

$$\Phi_T|_{B_R((\rho_0, u_0))}$$

is not uniformly continuous, i.e. that the time  $T$  solution map  $\Phi_T$  restricted to  $B_R((\rho_0, u_0))$  is not uniformly continuous. As  $S$  is dense in  $U_T$  this suffices clearly to finish the proof.

Thus, denote by  $\varphi_0 = \Psi_T(\rho_0, u_0)$ . By continuity we can choose  $R_1 > 0$  and  $C_1 > 0$  such that

$$\frac{1}{C_1} \|\operatorname{curl} u\|_{s-1} \leq \left\| \frac{1}{\det(d\varphi)} [d\varphi](\operatorname{curl} u) \circ \varphi^{-1} \right\|_{s-1} \leq C_1 \|\operatorname{curl} u\|_{s-1} \tag{3.4}$$

for all  $u \in B_{R_1}(u_0) \subseteq H^s(\mathbb{R}^3; \mathbb{R}^3)$  and  $\varphi \in B_{R_1}(\varphi_0) \subseteq \mathcal{D}^s(\mathbb{R}^3)$  where  $B_r$  denotes the ball of radius  $r$  in the corresponding spaces, see e.g., [9, 12]. Consider the Taylor expansion

$$\Psi_T(\rho_\bullet + h_\rho, u_\bullet + h_u) = \Psi_T(\rho_\bullet, u_\bullet) + d_{(\rho_\bullet, u_\bullet)} \Psi_T(h) + \int_0^1 (1-t) d_{(\rho_\bullet + th_\rho, u_\bullet + th_u)}^2 \Psi_T(h, h),$$

where  $h = (h_\rho, h_u) \in H^{s-1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$ . In the following we use the norm

$$\| \|h\| \| = \| \|h_\rho\|_{s-1} + \| \|h_u\|_s \|.$$

By the smoothness of  $\Psi_T$  we can choose  $0 < R_2 \leq R_1$  and  $C_2 > 0$  with

$$\| \|d_{(\rho_\bullet, u_\bullet)}^2 \Psi_T(h_1, h_2)\| \|_s \leq C_2 \| \|h_1\| \| \cdot \| \|h_2\| \|$$

and

$$\| \|d_{(\tilde{\rho}_\bullet, \tilde{u}_\bullet)}^2 \Psi_T(h_1, h_2) - d_{(\rho_\bullet, u_\bullet)}^2 \Psi_T(h_1, h_2)\| \|_s \leq C_2 \| \|(\tilde{\rho}_\bullet - \rho_\bullet, \tilde{u}_\bullet - u_\bullet)\| \| \cdot \| \|h_1\| \| \cdot \| \|h_2\| \|$$

for all  $(\rho_\bullet, u_\bullet), (\tilde{\rho}_\bullet, \tilde{u}_\bullet) \in B_{R_2}((\rho_0, u_0))$  and  $h_1, h_2 \in H^{s-1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$ . Since  $(\rho_0, u_0) \in S$  there is by Lemma 3.1 a corresponding  $x^* \in \mathbb{R}^3$  with  $\operatorname{dist}(\operatorname{supp} u_0, x^*) > 2$  and  $h = (h_\rho, h_u) \in H^{s-1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3; \mathbb{R}^3)$  with

$$m := | (d_{(\rho_0, u_0)} \Psi_T(h))(x^*) | > 0,$$

which we fix. Here  $|\cdot|$  denotes the euclidean norm in  $\mathbb{R}^3$ . Consider the distance

$$d := \operatorname{dist}(\varphi_0(\operatorname{supp} u_0), \varphi_0(B_1(x^*))) > 0.$$

Because of  $s > 5/2$  we have by the Sobolev imbedding theorem

$$\| \|f\|_{C^1} \leq \tilde{C} \| \|f\|_s \|$$

for some  $\tilde{C} > 0$ . Thus, there is  $0 < R_3 \leq R_2$  with

$$| \Psi_T(\rho_\bullet, u_\bullet)(p) - \Psi_T(\rho_\bullet, u_\bullet)(q) | < L |p - q| \quad \forall p, q \in \mathbb{R}^3 \tag{3.5}$$

and

$$|\Psi_T(\rho_\bullet, u_\bullet)(p) - \varphi_0(p)| < d/4 \quad \forall p \in \mathbb{R}^3 \tag{3.6}$$

and for all  $(\rho_\bullet, u_\bullet) \in B_{R_*}((\rho_0, u_0)) \subseteq U_T$ . Finally we take  $0 < R_* \leq R_3$  small enough and  $N$  large enough to ensure

$$\tilde{C}C_2 \|h\| \cdot R_*^2/4 + \tilde{C}C_2 \frac{1}{n} \|h\| \cdot R_* + \tilde{C}C_2 \frac{1}{n^2} \|h\|^2 < \frac{m}{2n}$$

for all  $n \geq N$ .

Now fix  $0 < R \leq R_*$ . We will construct two sequences of initial data  $((\rho_0^{(n)}, u_0^{(n)}))_{n \geq 1}, ((\tilde{\rho}_0^{(n)}, \tilde{u}_0^{(n)}))_{n \geq 1} \subseteq B_R((u_0, \rho_0))$  with

$$\|(\rho_0^{(n)}, u_0^{(n)}) - (\tilde{\rho}_0^{(n)}, \tilde{u}_0^{(n)})\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

whereas

$$\limsup_{n \rightarrow \infty} \|\Phi_T((\rho_0^{(n)}, u_0^{(n)})) - \Phi_T((\tilde{\rho}_0^{(n)}, \tilde{u}_0^{(n)}))\| > 0.$$

Define the radii  $r_n = \frac{m}{8nL}$  with the Lipschitz constant from (3.5). With that take a sequence  $(w_n)_{n \geq 1} \subseteq H^s(\mathbb{R}^3; \mathbb{R}^3)$  with

$$\text{supp } w_n \subseteq B_{r_n}(x^*) \subseteq \mathbb{R}^3 \text{ and } \|w_n\|_s = R/2.$$

For some technical reason we assume additionally  $\text{div } w_n = 0$  which is not difficult to arrange. Finally define

$$\begin{pmatrix} \rho_0^{(n)} \\ u_0^{(n)} \end{pmatrix} = \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} + \begin{pmatrix} 0 \\ w_n \end{pmatrix}$$

resp.

$$\begin{pmatrix} \tilde{\rho}_0^{(n)} \\ \tilde{u}_0^{(n)} \end{pmatrix} = \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} + \begin{pmatrix} 0 \\ w_n \end{pmatrix} + \frac{1}{n} \begin{pmatrix} h_\rho \\ h_u \end{pmatrix}.$$

We clearly have that

$$(\tilde{\rho}_0^{(n)}, \tilde{u}_0^{(n)}), (\rho_0^{(n)}, u_0^{(n)}) \in B_R((\rho_0, u_0)) \quad \forall n \geq N,$$

where  $N$  is some large number. Taking  $N$  large enough we can assume  $r_n \leq 1$  for  $n \geq N$ . Furthermore, by construction

$$\|(\rho_0^{(n)}, u_0^{(n)}) - (\tilde{\rho}_0^{(n)}, \tilde{u}_0^{(n)})\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let

$$(\rho^{(n)}, u^{(n)}) = \Phi_T(\rho_0^{(n)}, u_0^{(n)}) \quad \text{resp.} \quad (\tilde{\rho}^{(n)}, \tilde{u}^{(n)}) = \Phi_T(\tilde{\rho}_0^{(n)}, \tilde{u}_0^{(n)}).$$

Similarly

$$\varphi^{(n)} = \Psi_T(\rho_0^{(n)}, u_0^{(n)}) \quad \text{resp.} \quad \tilde{\varphi}^{(n)} = \Psi_T(\tilde{\rho}_0^{(n)}, \tilde{u}_0^{(n)}).$$

We also introduce

$$\omega^{(n)} = \text{curl } u^{(n)} \quad \text{and} \quad \tilde{\omega}^{(n)} = \text{curl } \tilde{u}^{(n)}.$$

Since

$$\|\Phi_T((\rho_0^{(n)}, u_0^{(n)})) - \Phi_T((\tilde{\rho}_0^{(n)}, \tilde{u}_0^{(n)}))\| \geq \|u^{(n)} - \tilde{u}^{(n)}\|_s \geq C \|\omega^{(n)} - \tilde{\omega}^{(n)}\|_{s-1}$$

we get the claim by showing

$$\limsup_{n \rightarrow \infty} \|\omega^{(n)} - \tilde{\omega}^{(n)}\|_{s-1} > 0.$$

By (3.2) we have

$$\omega^{(n)} = \left( \frac{1}{\det(d\varphi^{(n)})} [d\varphi^{(n)}] \omega_0^{(n)} \right) \circ (\varphi^{(n)})^{-1}$$

and

$$\tilde{\omega}^{(n)} = \left( \frac{1}{\det(d\tilde{\varphi}^{(n)})} [d\tilde{\varphi}^{(n)}] \tilde{\omega}_0^{(n)} \right) \circ (\tilde{\varphi}^{(n)})^{-1},$$

where

$$\omega_0^{(n)} = \text{curl } u_0^{(n)} = \text{curl } u_0 + \text{curl } w_n$$

and

$$\tilde{\omega}_0^{(n)} = \text{curl } \tilde{u}_0^{(n)} = \text{curl } u_0 + \text{curl } w_n + \frac{1}{n} \text{curl } h_u.$$

By (3.4) we have

$$\left\| \left( \frac{1}{\det(d\tilde{\varphi}^{(n)})} [d\tilde{\varphi}^{(n)}] \frac{1}{n} \text{curl } h_u \right) \circ (\tilde{\varphi}^{(n)})^{-1} \right\|_{s-1} \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\omega^{(n)} - \tilde{\omega}^{(n)}\|_{s-1} &= \limsup_{n \rightarrow \infty} \left\| \left( \frac{1}{\det(d\varphi^{(n)})} [d\varphi^{(n)}] (\text{curl } u_0 + \text{curl } w_n) \right) \circ (\varphi^{(n)})^{-1} \right. \\ &\quad \left. - \left( \frac{1}{\det(d\tilde{\varphi}^{(n)})} [d\tilde{\varphi}^{(n)}] (\text{curl } u_0 + \text{curl } w_n) \right) \circ (\tilde{\varphi}^{(n)})^{-1} \right\|_{s-1}. \end{aligned}$$

Consider the supports of the above expressions. We have by (3.6)

$$\text{supp} \left( (\text{curl } u_0) \circ (\varphi^{(n)})^{-1} \right), \text{supp} \left( (\text{curl } u_0) \circ (\tilde{\varphi}^{(n)})^{-1} \right) \subseteq \varphi_0(\text{supp } u_0) + B_{d/4}(0),$$

where we use  $A + B = \{a + b \mid a \in A, b \in B\}$ . As  $\text{supp } w_n \subseteq B_1(x^*)$  for  $n \geq N$  we have again by (3.6)

$$\text{supp} \left( (\text{curl } w_n) \circ (\varphi^{(n)})^{-1} \right), \text{supp} \left( (\text{curl } w_n) \circ (\tilde{\varphi}^{(n)})^{-1} \right) \subseteq \varphi_0(B_1(x^*)) + B_{d/4}(0).$$

By the choice of  $d$  the supports are in fixed sets which are positive apart. Thus, we can “separate” the  $\|\cdot\|_{s-1}$  norms with a constant, see [13]. Thus, there exists a constant  $C > 0$  such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\omega^{(n)} - \tilde{\omega}^{(n)}\|_{s-1} &\geq C \limsup_{n \rightarrow \infty} \left\| \left( \frac{1}{\det(d\varphi^{(n)})} [d\varphi^{(n)}] (\text{curl } w_n) \right) \circ (\varphi^{(n)})^{-1} \right. \\ &\quad \left. - \left( \frac{1}{\det(d\tilde{\varphi}^{(n)})} [d\tilde{\varphi}^{(n)}] (\text{curl } w_n) \right) \circ (\tilde{\varphi}^{(n)})^{-1} \right\|_{s-1}. \end{aligned}$$

We claim that the supports of the above expressions are also apart. To show this we use the Taylor expansion

$$\begin{aligned} \tilde{\varphi}^{(n)} &= \Psi(\rho_0, u_0) + d_{(\rho_0, u_0)} \Psi \left( \begin{pmatrix} 0 \\ w_n \end{pmatrix} + \frac{1}{n} h \right) \\ &\quad + \int_0^1 (1-t) d_{(\rho_0 + t \frac{1}{n} h_\rho, u_0 + t w_n + t \frac{1}{n} h_u)} \Psi \left( \begin{pmatrix} 0 \\ w_n \end{pmatrix} + \frac{1}{n} h, \begin{pmatrix} 0 \\ w_n \end{pmatrix} + \frac{1}{n} h \right) dt \end{aligned}$$

and

$$\varphi^{(n)} = \Psi(\rho_0, u_0) + d_{(\rho_0, u_0)}\Psi\left(\begin{pmatrix} 0 \\ w_n \end{pmatrix}\right) + \int_0^1 (1-t)d_{(\rho_0, u_0+tw_n)}^2\Psi\left(\begin{pmatrix} 0 \\ w_n \end{pmatrix}, \begin{pmatrix} 0 \\ w_n \end{pmatrix}\right) dt.$$

The difference is

$$\tilde{\varphi}^{(n)} - \varphi^{(n)} = \frac{1}{n}d_{(\rho_0, u_0)}\Psi(h) + I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^1 (1-t)\left(d_{(\rho_0+t\frac{1}{n}h_\rho, u_0+tw_n+t\frac{1}{n}h_u)}^2\Psi - d_{(\rho_0, u_0+tw_n)}\Psi\right)\left(\begin{pmatrix} 0 \\ w_n \end{pmatrix}, \begin{pmatrix} 0 \\ w_n \end{pmatrix}\right) dt$$

and

$$I_2 = 2 \int_0^1 (1-t)d_{(\rho_0+t\frac{1}{n}h_\rho, u_0+tw_n+t\frac{1}{n}h_u)}^2\Psi\left(\begin{pmatrix} 0 \\ w_n \end{pmatrix}, \frac{1}{n}h\right) dt$$

and

$$I_3 = \int_0^1 (1-t)d_{(\rho_0+t\frac{1}{n}h_\rho, u_0+tw_n+t\frac{1}{n}h_u)}^2\Psi\left(\frac{1}{n}h, \frac{1}{n}h\right) dt.$$

Using the estimates for  $d^2\Psi$  from above we have

$$\|I_1\|_s \leq C_2 \frac{1}{n} \|h\| \cdot \|w_n\|_s^2 = C_2 \|h\| \cdot R^2/4$$

and

$$\|I_2\|_s \leq 2C_2 \frac{1}{n} \|h\| \cdot \|w_n\|_s = C_2 \frac{1}{n} \|h\| \cdot R$$

and

$$\|I_3\|_s \leq C_2 \frac{1}{n^2} \|h\|^2.$$

By the Sobolev imbedding we then have

$$|I_1(x^*)| + |I_2(x^*)| + |I_3(x^*)| \leq \tilde{C}C_2 \|h\| \cdot R^2/4 + \tilde{C}C_2 \frac{1}{n} \|h\| \cdot R + \tilde{C}C_2 \frac{1}{n^2} \|h\|^2 < \frac{m}{2n}$$

for  $n \geq N$  by the choice of  $R_*$ . Thus, we have

$$|\tilde{\varphi}^{(n)}(x^*) - \varphi^{(n)}(x^*)| \geq \frac{1}{n} |(d_{(\rho_0, u_0)}\Psi_T(h))(x^*)| - \frac{m}{2n} = \frac{m}{2n}.$$

By (3.5) we have

$$\text{supp}\left((\text{curl } w_n) \circ (\varphi^{(n)})^{-1}\right) \subseteq B_{Lr_n}(\varphi^{(n)}(x^*)) = B_{\frac{m}{8n}}(\varphi^{(n)}(x^*))$$

and

$$\text{supp}\left((\text{curl } w_n) \circ (\tilde{\varphi}^{(n)})^{-1}\right) \subseteq B_{Lr_n}(\tilde{\varphi}^{(n)}(x^*)) = B_{\frac{m}{8n}}(\tilde{\varphi}^{(n)}(x^*)).$$

So the supports are apart in such a way that we can separate the  $H^{s-1}$  norms (see [13]) with a constant  $\bar{C} > 0$  like

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left\| \left( \frac{[d\varphi^{(n)}]}{\det(d\varphi^{(n)})}(\text{curl } w_n) \right) \circ (\varphi^{(n)})^{-1} - \left( \frac{[d\tilde{\varphi}^{(n)}]}{\det(d\tilde{\varphi}^{(n)})}(\text{curl } w_n) \right) \circ (\tilde{\varphi}^{(n)})^{-1} \right\|_{s-1} \\ & \geq \bar{C} \limsup_{n \rightarrow \infty} \left\| \left( \frac{[d\varphi^{(n)}]}{\det(d\varphi^{(n)})}(\text{curl } w_n) \right) \circ (\varphi^{(n)})^{-1} \right\|_{s-1} + \left\| \left( \frac{[d\tilde{\varphi}^{(n)}]}{\det(d\tilde{\varphi}^{(n)})}(\text{curl } w_n) \right) \circ (\tilde{\varphi}^{(n)})^{-1} \right\|_{s-1}. \end{aligned}$$

Using (3.4) we can estimate this from below by

$$\frac{\bar{C}}{C_2} \limsup_{n \rightarrow \infty} \|\text{curl } w_n\|_{s-1} \geq \frac{\bar{C}\hat{C}}{C_2} \limsup_{n \rightarrow \infty} \|dw_n\|_{s-1},$$

where the last inequality with some  $\hat{C} > 0$  follows from the Biot-Savart law (see [3]) for divergence-free vector fields (Here is where we use  $\text{div } w_n = 0$ ). We have the following general equivalence

$$\|w_n\|_s \sim \|w_n\|_{L^2} + \|dw_n\|_{s-1}.$$

Moreover,

$$\|w_n\|_{L^2} \leq \|w_n\|_{L^\infty} \sqrt{\frac{4}{3}\pi r_n^3} \leq \tilde{C} \|w_n\|_s \sqrt{\frac{4}{3}\pi r_n^3} \rightarrow 0$$

since  $w_n$  is supported in  $B_{r_n}(x^*)$ . Therefore,

$$\limsup_{n \rightarrow \infty} \|dw_n\|_{s-1} \geq K \limsup_{n \rightarrow \infty} \|w_n\|_s \geq KR/2$$

for some  $K > 0$ . Altogether we have

$$\limsup_{n \rightarrow \infty} \|\tilde{\omega}^{(n)} - \omega^{(n)}\|_{s-1} \geq \frac{\bar{C}\hat{C}K}{2C_2}R.$$

The proof of Theorem 1.1 is now complete. □

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