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Polarization of neural codes

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Abstract: The neural rings and ideals as an algebraic tool for analyzing the intrinsic structure of neural codes were introduced by C. Curto, V. Itskov, A. Veliz-Cuba, and N. Youngs in 2013. Since then they were investigated in several papers, including the 2017 paper by S. Güntürkün, J. Jeffries, and J. Sun, in which the notion of polarization of neural ideals was introduced. In our paper we extend their ideas by introducing the notions of polarization of motifs and neural codes. We show that the notions that we introduce have very nice properties which allow the studying of the intrinsic structure of neural codes of length n via the square-free monomial ideals in $2n$ variables and interpreting the results back in the original neural code ambient space.

In the last section of the paper we introduce the notions of inactive neurons, partial neural codes, and partial motifs, as well as the notions of polarization of these codes and motifs. We use these notions to give a new proof of a theorem from the paper by Güntürkün, Jeffries, and Sun that we mentioned above.

Key words: Neural code, neural ideal, canonical form, minimal prime ideal, motifs, polarization, monomial ideal, pseudomonomial ideal, square-free monomial ideal, inactive neurons, partial word, partial motif, partial neural code

1. Introduction

One of the problems that neuroscience is faced with is to analyze the intrinsic structure of the so-called neural codes resulting from the activity of the certain type of neurons in the brain of an organism. An algebraic tool (in the form of neural rings and ideals) was introduced for that purpose by Curto et al. in their pioneering 2013 paper [5]. This area of mathematics is very active ever since; let us just mention the papers [2, 6, 8, 10–12], which are just a few of several papers that appeared in the last year or so.

The polarization of monomial ideals is a well-known operator in commutative algebra (see the 2018 paper [3] by Cimpoeaş for the most recent developments). As an analogue to that operator, Güntürkün et al. introduced in 2017 in [8] the notion of polarization of neural ideals. In our paper we extend their ideas by introducing the notions of polarization of motifs and neural codes. We show that these notions have very nice properties which allow the studying of the intrinsic structure of neural codes of length n via the square-free monomial ideals in $2n$ variables and interpreting the results back in the original neural code ambient space.

In the last section of the paper we introduce the notions of inactive neurons, partial neural codes, and partial motifs, as well as the notions of polarization of these codes and motifs. We use these notions to give a new proof of a theorem from the paper by Güntürkün et al. that we mentioned above.

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In order to make our paper self-contained, we will give in this section all the definitions and facts from [5] that we are going to use, which are related to neural codes. All other notions and facts (that we assume are well-known) can be found either in [5], or in the standard references [1] and [4].

Definition and basic facts 1.1 ([5]) *Elements of* $\mathbb{F}_2^n$ *will be written as vectors with concatenated coordinates, for example* $\mathbf{w} = w_1 \dots w_n$. *They are called words (of length* n*). A set* $\mathcal{C} \subseteq \mathbb{F}_2^n$ *is called a neural code, shortly code (of length* n*). We also call the subsets of* $\mathbb{F}_2^n$ *varietes in* $\mathbb{F}_2^n$. *The code* $\mathcal{D} = \mathbb{F}_2^n \setminus \mathcal{C}$ *is called the complement of the code* $\mathcal{C}$ *and is denoted by* ${}^c\mathcal{C}$. *We denote* $\mathbb{M} = \{0, 1, *\}$. *We say that this set is the set of motifs of length* 1. *We define a partial order on* $\mathbb{M}$ *by declaring that* $0 < *$ *and* $1 < *$. *A sequence* $\mathbf{a} = a_1 \dots a_n \in \mathbb{M}^n$ *is called a motif (of length* n*).We define a partial order on the set* $\mathbb{M}^n$ *by declaring that* $\mathbf{a} \le \mathbf{b}$ *if* $a_i \le b_i$ *for every* $i \in [n]$. *In other words,* $\mathbf{a} \le \mathbf{b}$ *if for each* $i \in [n]$, $b_i = 0$ *(resp.* 1*) implies* $a_i = 0$ *(resp.* 1*).*

For $\mathbf{a} \in \mathbb{M}^n$, *the subset* $V_\mathbf{a}$ *of* $\mathbb{F}_2^n$ *consisiting of all the words* $\mathbf{w}$ *obtained by replacing the stars of* $\mathbf{a}$ *by elements of* $\mathbb{F}_2$ *is called the variety of* $\mathbf{a}$. *We have*

$$\mathbf{a} \le \mathbf{b} \;\Leftrightarrow\; V_\mathbf{a} \subseteq V_\mathbf{b}.$$

For a code $\mathcal{C} \subseteq \mathbb{F}_2^n$, *a motif* $\mathbf{a}$ *of length* n *is called a motif of* $\mathcal{C}$ *if* $V_\mathbf{a} \subseteq \mathcal{C}$. *The set of all motifs of* $\mathcal{C}$ *is denoted by* $\mathrm{Mot}(\mathcal{C})$. *A motif* $\mathbf{a} \in \mathrm{Mot}(\mathcal{C})$ *is called a maximal motif of* $\mathcal{C}$ *if for any motif* $\mathbf{b} \in \mathrm{Mot}(\mathcal{C})$, $\mathbf{a} \le \mathbf{b}$ *implies* $\mathbf{a} = \mathbf{b}$. *The set of all maximal motifs of* $\mathcal{C}$ *is denoted by* $\mathrm{MaxMot}(\mathcal{C})$. *For any* $\mathbf{a} \in \mathrm{Mot}(\mathcal{C})$ *there is a* $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C})$ *such that* $\mathbf{a} \le \mathbf{b}$. *We have* $\mathcal{C} = \emptyset$ *if and only if* $\mathrm{MaxMot}(\mathcal{C}) = \emptyset$. *Moreover, for any two codes* $\mathcal{C}^1$ *and* $\mathcal{C}^2$,

$$\mathcal{C}^1 = \mathcal{C}^2 \Leftrightarrow \mathrm{MaxMot}(\mathcal{C}^1) = \mathrm{MaxMot}(\mathcal{C}^2).$$

Remark 1.2 ([5], pages 1593 and 1594) *We have*

$$\mathcal{C} = \cup \{V_\mathbf{a} \;:\; \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\};$$

however, it can happen that for a proper subset M *of* $\mathrm{MaxMot}(\mathcal{C})$ *we still have*

$$\mathcal{C} = \cup \{V_\mathbf{a} \;:\; \mathbf{a} \in M\}.$$

For example, consider the neural code $\mathcal{C} = \{000, 001, 011, 111\} \subseteq \mathbb{F}_2^3$. *Then* $\mathrm{MaxMot}(\mathcal{C}) = \{00*, 0*1, *11\}$*; however,* $\mathcal{C} = V_{00*} \cup V_{*11}$.

Definition 1.3 ([4], [5]) *For a variety* $V \subseteq \mathbb{F}_2^n$ *we define the ideal of* V, $\mathcal{I}(V) \subseteq \mathbb{F}_2[X_1, \dots, X_n]$, *in the following way:*

$$\mathcal{I}(V) = \{f \in \mathbb{F}_2[X_1, \dots, X_n] \;:\; f(\mathbf{w}) = 0 \textit{ for all } \mathbf{w} \in V\}.$$

Note that for any variety V *in* $\mathbb{F}_2^n$ *we have* $\mathcal{I}(V) \supseteq \mathcal{B}$, *where* $\mathcal{B} = (X_1^2 - X_1, \dots, X_n^2 - X_n)$ *is the Boolean ideal of* $\mathbb{F}_2[X_1, \dots, X_n]$. *Moreover, for* $V \subseteq \mathbb{F}_2^n$ *we have* $\mathcal{I}(V) = \mathcal{B}$ *if and only if* $V = \mathbb{F}_2^n$.

For an ideal $I \subseteq \mathbb{F}_2[X_1, \dots, X_n]$ *we define the variety of* I, $\mathcal{V}(I) \subseteq \mathbb{F}_2^n$, *in the following way:*

$$\mathcal{V}(I) = \{\mathbf{w} \in \mathbb{F}_2^n \;:\; f(\mathbf{w}) = 0 \textit{ for all } f \in I\}.$$

Theorem 1.4 ([5], [7]) *For every variety* $V \subseteq \mathbb{F}_2^n$ *we have*

$$\mathcal{V}(\mathcal{I}(V)) = V.$$

For every ideal $I \subseteq \mathbb{F}_2[X_1, \ldots, X_n]$ *we have*

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I} = I + \mathcal{B}.$$

The second formula in the previous theorem is called the *Hilbert's Nullstellensatz for* $\mathbb{F}_2$.

Definition 1.5 ([5, 7]) *For a motif* $\mathbf{a} \in \mathbb{M}^n$ *we define the Lagrange polynomial of* $\mathbf{a}$, $L_{\mathbf{a}} \in \mathbb{F}_2[X_1, \ldots, X_n]$, *in the following way:*

$$L_{\mathbf{a}} = \prod_{a_i=1} X_i \prod_{a_j=0} (1 - X_j).$$

Note that for any word $\mathbf{w} \in \mathbb{F}_2^n$, $L_{\mathbf{a}}(\mathbf{w}) = 1$ if and only if $\mathbf{w} \in V_{\mathbf{a}}$ (i.e. $L_{\mathbf{a}}(\mathbf{w}) = 0$ if and only if $\mathbf{w} \notin V_{\mathbf{a}}$).

Definition 1.6 ([5], page 1582) *For a neural code* $\mathcal{C} \subseteq \mathbb{F}_2^n$ *we define the neural ideal of* $\mathcal{C}$, $J_{\mathcal{C}} \subseteq \mathbb{F}_2[X_1, \ldots, X_n]$, *in the following way:*

$$J_{\mathcal{C}} = (\{L_{\mathbf{w}} \ : \ \mathbf{w} \in {}^{c}\mathcal{C}\}).$$

Proposition 1.7 ([5], Lemma 3.2) *For a neural code* $\mathcal{C} \subseteq \mathbb{F}_2^n$ *we have:*

$$\mathcal{V}(J_{\mathcal{C}}) = \mathcal{C},$$
$$\mathcal{I}(\mathcal{C}) = J_{\mathcal{C}} + \mathcal{B}.$$

Definition 1.8 ([5], page 1585) *A polynomial* $f \in \mathbb{F}_2[X_1, \ldots, X_n]$ *is called a pseudomonomial if it has the form*

$$f = \prod_{i \in \sigma} X_i \prod_{j \in \tau} (1 - X_j)$$

for some $\sigma, \tau \subseteq [n] = \{1, \ldots, n\}$ *with* $\sigma \cap \tau = \emptyset$.

An ideal $I \subseteq \mathbb{F}_2[X_1, \ldots, X_n]$ *is called a pseudomonomial ideal if* I *can be generated by (a finite set of) pseudomonomials.*

Definition 1.9 ([5], page 1585) *Let* I *be an ideal in* $\mathbb{F}_2[X_1, \ldots, X_n]$ *and* $f \in I$ *a pseudomonomial. We say that* f *is a minimal pseudomonomial of* I *if there does not exist another pseudomonomial* $g \in I$ *such that* $\deg(g) < \deg(f)$ *and* $g \mid f$ *in* $F_2[X_1, \ldots, X_n]$.

Definition 1.10 ([5], page 1585) *Let* I *be a pseudomonomial ideal in* $F_2[X_1, \ldots, X_n]$. *We call the (finite) set* $CF(I)$, *consisting of all minimal pseudomonomials of* I, *the canonical form of* I.

Remark 1.11 ([5], page 1585) *Clearly, for any pseudomonomial ideal* I *of* $F_2[X_1, \ldots, X_n]$, $CF(I)$ *is unique and* $I = (CF(I))$. *On the other hand,* $CF(I)$ *is not necessarily a minimal generating set of* I. *For example, consider the ideal* $I = (X_1(1 - X_2), X_2(1 - X_3))$. *This ideal contains a third minimal pseudomonomial:* $X_1(1-X_3) = (1-X_3) \cdot [X_1(1-X_2)] + X_1 \cdot [X_2(1-X_3)]$, *so that* $CF(I) = \{(X_1(1-X_2), X_2(1-X_3), X_1(1-X_3)\}$, *which is not a minimal generating set of* I.

Definition 1.12 *For a motif* $\mathbf{a} = a_1 \dots a_n \in \mathbb{M}^n$ *we define* $\overline{\mathbf{a}}$ *to be the motif* $\mathbf{b} = b_1 \dots b_n \in \mathbb{M}^n$ *which satisfies the following condition: for* $i = 1, 2 \dots, n$, *if* $a_i \neq *$ *then* $b_i = \overline{a_i} = 1 - a_i$, *and if* $a_i = *$ *then* $b_i = *$.

Example 1.13 $\overline{1 * 01} = 0 * 10$.

Proposition 1.14 ([5], Proposition 4.5) *Let* $\mathcal{C} \subseteq \mathbb{F}_2^n$ *be a code in* $\mathbb{F}_2^n$ *and* ${}^c\mathcal{C}$ *its complement. Let* $\mathrm{MaxMot}(\mathcal{C}) = \{\mathbf{a}^1, \dots, \mathbf{a}^l\}$. *Then*

$$\begin{aligned}\mathrm{MaxMot}({}^c\mathcal{C}) = \{\mathbf{b} = b_1 \dots b_n \in \mathbb{M}^n \ : \ & \\ & [\,(\forall b_i \neq *)(\exists \mathbf{a}^j)\ b_i = \overline{\mathbf{a}_i^j}\,] \ \text{and} \\ & [\,(\forall \mathbf{a}^j \neq * \cdots *)(\exists b_i \neq *)\ b_i = \overline{\mathbf{a}_i^j}\,] \ \text{and} \\ & [\,\mathbf{b} \textit{ is maximal with respect to these two properties}]\}.\end{aligned}$$

In particular, if $\mathrm{MaxMot}(\mathcal{C}) = \{* \cdots *\}$, *then* $\mathrm{MaxMot}({}^c\mathcal{C}) = \emptyset$.

Proof The proposition follows from Proposition 4.5 and Corollary 5.5 from [5]. □

Proposition 1.15 ([5], Lemma 5.7) *Let* $\mathcal{C} \subseteq \mathbb{F}_2^n$ *be a neural code and* $J_{\mathcal{C}}$ *the neural ideal of* $\mathcal{C}$. *Then*

$$CF(J_{\mathcal{C}}) = \{L_{\mathbf{a}} \ : \ \mathbf{a} \in \mathrm{MaxMot}({}^c\mathcal{C})\}.$$

Remark 1.16 ([5], page 1594) *Note that it can happen that* $J_{\mathcal{C}} = (\{L_{\mathbf{a}} \ : \ \mathbf{a} \in M\})$, *where* M *is a proper subset of* $\mathrm{MaxMot}({}^c\mathcal{C})$. *For example, for the neural code* $\mathcal{C} = \{000, 001, 011, 111\} \subseteq \mathbb{F}_2^3$, *we have* ${}^c\mathcal{C} = \{100, 010, 110, 101\}$, *so that* $\mathrm{MaxMot}({}^c\mathcal{C}) = \{10*, *10, 1*0\}$. *Hence,* $CF(J_{\mathcal{C}}) = \{L_{10*}, L_{*10}, L_{1*0}\}$. *However,* $J_{\mathcal{C}} = (L_{10*}, L_{*10})$.

Definition 1.17 ([5], page 1594) *For a motif* $\mathbf{a} \in \mathbb{M}^n$ *we define a prime ideal of* $\mathbf{a}$, $\mathfrak{p}_{\mathbf{a}} \subseteq \mathbb{F}_2[X_1, \dots, X_n]$, *in the following way:*

$$\mathfrak{p}_{\mathbf{a}} = (\{X_i \ : \ a_i = 0\} \cup \{1 - X_j \ : \ a_j = 1\}).$$

If a prime ideal $\mathfrak{p}$ *in* $\mathbb{F}_2[X_1, \dots, X_n]$ *is equal to* $\mathfrak{p}_{\mathbf{a}}$ *for some motif* $\mathbf{a}$, *we say that* $\mathfrak{p}$ *is a motivic prime ideal.*

Proposition 1.18 ([5], page 1594) *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$ *be two motifs of length* n. *We have:*

$$\begin{aligned} V_{\mathbf{a}} \subseteq V_{\mathbf{b}} &\Leftrightarrow \mathfrak{p}_{\mathbf{b}} \subseteq \mathfrak{p}_{\mathbf{a}}, \\ \mathcal{I}(V_{\mathbf{a}}) &= \mathfrak{p}_{\mathbf{a}} + \mathcal{B}, \\ \mathcal{V}(\mathfrak{p}_{\mathbf{a}}) &= V_{\mathbf{a}}. \end{aligned}$$

For an ideal I in $\mathbb{F}_2[X_1, \dots, X_n]$ we denote by $\mathrm{Min}(I)$ the set of all minimal prime ideals of I.

Proposition 1.19 ([5], Lemma 5.1, Lemma 5.3 and Corollary 5.5) *Let* $\mathcal{C} \subseteq \mathbb{F}_2^n$ *be a neural code and* $\mathbf{a} \in \mathbb{M}^n$ *a motif. We have:*

$$\begin{aligned} \mathbf{a} \in \mathrm{Mot}(\mathcal{C}) &\Leftrightarrow \mathfrak{p}_{\mathbf{a}} \supseteq J_{\mathcal{C}}, \\ \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C}) &\Leftrightarrow \mathfrak{p}_{\mathbf{a}} \in \mathrm{Min}(J_{\mathcal{C}}). \end{aligned}$$

Moreover,

$$\mathrm{Min}(J_{\mathcal{C}}) = \{\mathfrak{p}_{\mathbf{a}} \,:\, \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\}. \tag{1.1}$$

Proposition 1.20 ([5], Corollary 5.5) *Let $\mathcal{C} \subseteq \mathbb{F}_2^n$ be a nonempty neural code. Then*

$$J_{\mathcal{C}} = \cap\{\mathfrak{p}_{\mathbf{a}} \,:\, \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\}$$

is the unique irredundant primary decomposition of $J_{\mathcal{C}}$.

The notions of polarization of pseudomonomials and pseudomonomial ideals were introduced in 2017 in the paper [8] by Güntürkün et al..

Definition 1.21 ([8], page 6) *For a pseudomonomial*

$$f = \prod_{i\in\sigma} X_i \prod_{j\in\tau}(1 - X_j) \in \mathbb{F}_2[X_1, \ldots, X_n],$$

where σ, τ are two disjoint subsets of $[n]$, we define its polarization f^p to be the square-free monomial

$$f^p = \prod_{i\in\sigma} X_i \prod_{j\in\tau} Y_j \in \mathbb{F}_2[X_1, \ldots, X_n, Y_1, \ldots, Y_n].$$

Proposition 1.22 ([8], Lemma 3.1) *Let $f, g \in \mathbb{F}_2[X_1, \ldots, X_n]$ be two pseudomonomials. Then*

$$f \mid g \;\Leftrightarrow\; f^p \mid g^p.$$

Definition 1.23 ([8], Definition 3.3) *Let J be a pseudomonomial ideal in $\mathbb{F}_2[X_1, \ldots, X_n]$ and let $CF(J) = \{f_1, \ldots, f_l\}$ be its canonical form. We define the polarization of J to be the ideal*

$$J^p = (f_1^p, \ldots, f_l^p) \subseteq \mathbb{F}_2[X_1, \ldots, X_n, Y_1, \ldots, Y_n].$$

Remark 1.24 (about definitions and notation related to $\mathbb{F}_2^{2n}$ vs those related to $\mathbb{F}_2^n$) *Square-free monomial ideals are the ideals generated by square-free monomials and they are easier to deal with than pseudomonomial ideals. The previous two definitions show that, in order to get some conclusions about the pseudomonomial ideals in n variables $X_1, \ldots, X_n$, we can consider some related square-free monomial ideals in $2n$ variables, which, however, are not denoted by $X_1, \ldots, X_{2n}$, but by $X_1, \ldots, X_n, Y_1, \ldots, Y_n$. Because of this difference in the notation for variables, we should be aware that, for example, a pseudomonomial in $\mathbb{F}_2[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ is a polynomial of the form*

$$f = \prod_{i\in\sigma} X_i \prod_{j\in\tau}(1 - X_j) \prod_{k\in\mu} Y_k \prod_{l\in\nu}(1 - Y_l),$$

where $\sigma, \tau, \mu, \nu \subseteq [n]$, $\sigma \cap \tau = \emptyset$, $\mu \cap \nu = \emptyset$. Similarly, we have, for example, that for a motif $\mathbf{a} = b_1 \ldots b_n\, c_1 \ldots c_n \in \mathbb{M}^{2n}$, the Lagrange polynomial $L_{\mathbf{a}}$ of $\mathbf{a}$ and the prime ideal $\mathfrak{p}_{\mathbf{a}}$ of $\mathbf{a}$ are respectively given in the following way:

$$L_{\mathbf{a}} = \prod_{a_i=1} X_i \prod_{a_j=0}(1 - X_j) \prod_{b_i=1} Y_i \prod_{b_j=0}(1 - Y_j), \tag{1.2}$$

$$\mathfrak{p}_{\mathbf{a}} = (\{X_i \,:\, b_i = 0\} \cup \{1 - X_j \,:\, b_j = 1\} \cup \{Y_i \,:\, c_i = 0\} \cup \{1 - Y_j \,:\, c_j = 1\}). \tag{1.3}$$

Thus, the definitions of these notions with respect to $\mathbb{F}_2^{2n}$ are the same as the ones with respect to $\mathbb{F}_2^n$, we just need to take into account the notation for the variables. This works for other notions as well (for example, minimal pseudomonomials in an ideal, the neural ideal of a code, the canonical form of a pseudomonomial ideal), while some notions (for example, minimal primes of an ideal) can be given in the form that does not depend on the notation for the variables.

From now on we have the following convention: if the length of motifs and codes is denoted by n, then the associated rings and ideals will always be in n variables $X_1, \dots, X_n$, while for the length denoted by $2n$ the associated rings and ideals will always be in $2n$ variables $X_1, \dots, X_n, Y_1, \dots, Y_n$. For lengths given by concrete numbers it will always be clear from the context if the number is n or $2n$.

2. Definitions of the polarizations of motifs and codes

We would like to define the motif $\mathbf{a}^p$ which is the polarization of a motif $\mathbf{a} = a_1 \dots a_n \in \mathbb{M}^n$. Since for the motifs from $\mathbb{M}^n$ we have that the Lagrange polynomials and the prime ideals of motifs are in n variables, and the polarizations of those Lagrange polynomials and prime ideals of motifs are in $2n$ variables, it is natural to try to define $\mathbf{a}^p$ to be an element of M^{2n}. Moreover, for any $M \subseteq \mathbb{M}^n$ we will use the notation

$$M^p = \{\mathbf{a}^p \,:\, \mathbf{a} \in M\}.$$

After defining the polarization of the motifs, we would define the polarization of a neural code $\mathcal{C} \subseteq \mathbb{F}_2^n$ in the following way:

$$\mathcal{C}^p = \cup \{V_{\mathbf{a}^p} \,:\, \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\} \subseteq \mathbb{F}_2^{2n}.$$

This would imply that

$$\mathrm{MaxMot}^p(\mathcal{C}) = \mathrm{MaxMot}(\mathcal{C}^p). \tag{2.1}$$

We would also like that the formula

$$\mathrm{Min}(J_{\mathcal{C}^p}) = \mathrm{Min}^p(J_{\mathcal{C}}) \tag{2.2}$$

holds. Here $\mathrm{Min}^p(J_{\mathcal{C}})$ denotes the set of ideals consisting of polarized elements of $\mathrm{Min}(J_{\mathcal{C}})$. Our construction of $\mathcal{C}^p$ will be a step toward finding the code corresponding to the polarization of the neural ideal, that goal will eventually be realized in Theorem 3.17 below.

In the next definition we introduce a naturally looking candidate for the polarization of a motif, and it will turn out that this definition works well.

Definition 2.1 *Let $\mathbf{a} = a_1 \dots a_n \in \mathbb{M}^n$. We define its polarization*

$$\mathbf{a}^p = b_1 \dots b_n | c_1 \dots c_n \in \mathbf{M}^{2n}$$

in the following way:

$$\textit{if } a_i = 0 \textit{ in } \mathbf{a}, \textit{ then } b_i = 0,\ c_i = * \textit{ in } \mathbf{a}^p;$$

$$\textit{if } a_i = 1 \textit{ in } \mathbf{a}, \textit{ then } b_i = *,\ c_i = 0 \textit{ in } \mathbf{a}^p;$$

$$\textit{if } a_i = * \textit{ in } \mathbf{a}, \textit{ then } b_i = *,\ c_i = * \textit{ in } \mathbf{a}^p.$$

Schematically:

$$\begin{aligned}\dots 0 \dots &\quad\mapsto\quad \dots 0 \dots | \cdots * \dots \\ \dots 1 \dots &\quad\mapsto\quad \cdots * \dots | \dots 0 \dots \\ \cdots * \dots &\quad\mapsto\quad \cdots * \dots | \cdots * \dots\end{aligned} \tag{2.3}$$

We now define the polarization of a code.

Definition 2.2 *For any code $\mathcal{C} \subseteq \mathbb{F}_2^n$ we define its polarization $\mathcal{C}^p \subseteq \mathbb{F}_2^{2n}$ in the following way:*

$$\mathcal{C}^p = \cup\{V_{\mathbf{a}^p} \mid \mathbf{a} \in MaxMot(\mathcal{C})\}.$$

Example 2.3 *Let us determine $\mathcal{C}^p$ and $\mathcal{D}^p$ for $\mathcal{C} = \{10\} \subseteq \mathbb{F}_2^2$ and $\mathcal{D} = {}^c\mathcal{C}$:*

$$\begin{aligned}\mathcal{C}^p &= \{10\}^p \\ &= V_{(10)^p} \\ &= V_{*00*} \\ &= \{0000, 1000, 0001, 1001\},\end{aligned}$$

$$\begin{aligned}\mathcal{D}^p =&\{00, 01, 11\}^p \\ =&V_{(0*)^p} \cup V_{(*1)^p} \\ =&V_{0***} \cup V_{***0} \\ =&\{0000, 0100, 0010, 0110, 0001, 0101, \\ &\ 0011, 0111, 1000, 1100, 1010, 1110\}.\end{aligned}$$

Note that here $\mathcal{C}^p \cap \mathcal{D}^p = \{0000\}$ and $\mathcal{C}^p \cup \mathcal{D}^p = \mathbb{F}_2^4 \setminus \{1111\}$. In general, $\mathcal{C}^p \cap \mathcal{D}^p$ can contain several words and the complement of $\mathcal{C}^p \cup \mathcal{D}^p$ in $\mathbb{F}_2^{2n}$ can, as well, contain several words.

Definition 2.4 *We say that a motif $\mathbf{b} \in \mathbb{M}^{2n}$ is a polar motif if there is a motif $\mathbf{a} \in \mathbb{M}^n$ such that $\mathbf{b} = \mathbf{a}^p$. The motif $\mathbf{a}$ is unique and we then denote $\mathbf{a} = \mathbf{b}^d$.*

Note that we have

$$\begin{aligned}\mathbf{a}^{pd} = \mathbf{a} &\quad \text{for every } \mathbf{a} \in \mathbb{M}^n, \\ \mathbf{b}^{dp} = \mathbf{b} &\quad \text{for every polar motif } \mathbf{b} \in \mathbb{M}^{2n}.\end{aligned}$$

3. Properties of the polarization of motifs and codes

Proposition 3.1 *Let $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$. Then*

$$\mathbf{a} \leq \mathbf{b} \;\Leftrightarrow\; \mathbf{a}^p \leq \mathbf{b}^p.$$

Proof Suppose $\mathbf{a} \leq \mathbf{b}$. Let $i \in [n]$. If $(\mathbf{b}^p)_i = 0$, then $(\mathbf{b}^p)_{n+i} = *$ and $b_i = 0$; hence, $a_i = 0$; hence, $(\mathbf{a}^p)_i = 0$. If $(\mathbf{b}^p)_{n+i} = 0$, then $(\mathbf{b}^p)_i = *$ and $b_i = 1$; hence, $a_i = 1$; hence, $(\mathbf{a}^p)_{n+i} = 0$. Thus, $\mathbf{a}^p \leq \mathbf{b}^p$.

Suppose $\mathbf{a}^p \leq \mathbf{b}^p$. Let $i \in [n]$. If $b_i = 0$, then $(\mathbf{b}^p)_i = 0$; hence, $(\mathbf{a}^p)_i = 0$; hence, $a_i = 0$. If $b_i = 1$, then $(\mathbf{b}^p)_{n+i} = 0$; hence, $(\mathbf{a}^p)_{n+i} = 0$; hence, $a_i = 1$. Thus, $\mathbf{a} \leq \mathbf{b}$. □

Corollary 3.2 *For any code $\mathcal{C} \subseteq \mathbb{F}_2^n$ we have*

$$\mathrm{Mot}^p(\mathcal{C}) \;\subseteq\; \mathrm{Mot}(\mathcal{C}^p).$$

Proof Let $\mathbf{a} \in \mathrm{Mot}(\mathcal{C})$ and let $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C})$ such that $\mathbf{a} \leq \mathbf{b}$. By Proposition 3.1, $\mathbf{a}^p \leq \mathbf{b}^p$. Since (by the definition of $\mathcal{C}^p$) $\mathbf{b}^p \in \mathrm{Mot}(\mathcal{C}^p)$, we have $\mathbf{a}^p \in \mathrm{Mot}(\mathcal{C}^p)$. □

Theorem 3.3 *For any code $\mathcal{C} \subseteq \mathbb{F}_2^n$ we have*

$$\mathrm{MaxMot}(\mathcal{C}^p) = \mathrm{MaxMot}^p(\mathcal{C}).$$

Proof Claim 1. $\mathrm{MaxMot}(\mathcal{C}^p) \subseteq \{0, *\}^{2n}$.

Proof of Claim 1. Suppose to the contrary, i.e. that $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C}^p)$ has a component $b_\alpha = 1$ for some $\alpha \in [2n]$. Let $\mathbf{w} \in V_{\mathbf{b}}$. Then $w_\alpha = 1$. We have $\mathbf{w} \in V_{\mathbf{a}^p}$ for some $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$. Then $(\mathbf{a}^p)_\alpha = *$; hence, the word $\mathbf{w}'$ obtained by replacing w_α in $\mathbf{w}$ by 0 is also in $V_{\mathbf{a}^p}$; hence, in $\mathcal{C}$. Hence, the motif $\mathbf{b}'$ obtained by replacing b_α by $*$ is also a motif of $\mathcal{C}^p$, contradicting to the maximality of $\mathbf{b}$. Claim 1 is proved.

Claim 2. Let $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C}^p)$. Then there is no $i \in [n]$ such that $b_i = b_{n+i} = 0$.

Proof of Claim 2. Suppose to the contrary. Let

$$A = \{j \in [n] \;:\; b_j = b_{n+j} = *\},$$

$$B = \{j \in [n] \;:\; b_j = 0,\, b_{n+j} = *\},$$

$$C = \{j \in [n] \;:\; b_j = *,\, b_{n+j} = 0\},$$

$$D = \{j \in [n] \;:\; b_j = b_{n+j} = 0\}.$$

Then the sets A, B, C, D form a partition of $[n]$ and $i \in D$. Let $\mathbf{w} \in V_{\mathbf{b}}$ be defined in the following way:

$$(\forall\, j \in A)\; w_j = w_{n+j} = 1;$$

$$(\forall\, j \in B)\; w_j = 0,\, w_{n+j} = 1;$$

$$(\forall\, j \in C)\; w_j = 1,\, w_{n+j} = 0;$$

$$(\forall\, j \in D)\; w_j = w_{n+j} = 0.$$

Since $\mathbf{w} \in \mathcal{C}^p$, there is an $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$ such that $\mathbf{w} \in V_{\mathbf{a}^p}$. Since $\mathbf{a}^p$ is a polar motif, we have:

$$(\forall\, j \in A)\; (\mathbf{a}^p)_j = (\mathbf{a}^p)_{n+j} = *;$$

$$(\forall\, j \in B)\; (\mathbf{a}^p)_j = 0 \text{ or } *,\; (\mathbf{a}^p)_{n+j} = *;$$

$$(\forall\, j \in C)\; (\mathbf{a}^p)_j = *,\; (\mathbf{a}^p)_{n+j} = 0 \text{ or } *;$$

$$(\forall\, j \in D) \text{ at least one of } (\mathbf{a}^p)_j,\, (\mathbf{a}^p)_{n+j} \text{ is } *, \text{ the other one is } 0 \text{ or } *.$$

Since D contains at least one element, these relations imply $\mathbf{a}^p > \mathbf{b}$, contradicting to the maximality of $\mathbf{b}$. Claim 2 is proved.

Claim 3. $\mathrm{MaxMot}(\mathcal{C}^p) \subseteq \mathrm{MaxMot}^p(\mathcal{C})$.

Proof of Claim 3. Let $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C}^p)$. By the Claims 1 and 2, for each $i \in [n]$ we have one the following three cases: $b_i = b_{n+i} = *$, or, $b_i = 0$, $b_{n+i} = *$, or, $b_i = *$, $b_{n+i} = 0$. Let $\mathbf{w} \in V_{\mathbf{b}}$ be a word defined in the following way: if $b_i = b_{n+i} = *$, then $w_i = w_{n+i} = 1$; if $b_i = 0$ and $b_{n+i} = *$, then $w_i = 0$, $w_{n+i} = 1$; if $b_i = *$ and $b_{n+i} = 0$, then $w_i = 1$, $w_{n+i} = 0$. This word belongs to some $V_{\mathbf{a}^p}$, where $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$. Since $\mathbf{a}^p$ is a polar motif, we have the following cases: when $w_i = w_{n+i} = 1$, then $(\mathbf{a}^p)_i = (\mathbf{a}^p)_{n+i} = *$; when $w_i = 0$ and $w_{n+i} = 1$, then $(\mathbf{a}^p)_i = 0$, $(\mathbf{a}^p)_{n+i} = *$; when $w_i = 1$ and $w_{n+i} = 0$, then $(\mathbf{a}^p)_i = *$, $(\mathbf{a}^p)_{n+i} = 0$. Hence, $\mathbf{a}^p \geq \mathbf{b}$. Since $\mathbf{a}^p \in \mathrm{Mot}(\mathcal{C})$ and $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C})$, we have $\mathbf{b} = \mathbf{a}^p$. Claim 3 is proved.

Claim 4. $\mathrm{MaxMot}^p(\mathcal{C}) \subseteq \mathrm{MaxMot}(\mathcal{C}^p)$.

Proof of Claim 4. Suppose to the contrary. Let $\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$ such that $\mathbf{a}^p \notin \mathrm{MaxMot}(\mathcal{C}^p)$. By the definition of $\mathcal{C}^p$, $\mathbf{a}^p \in \mathrm{Mot}(\mathcal{C}^p)$; hence, there is a $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{C}^p)$ such that $\mathbf{a}^p < \mathbf{b}$. By Claim 3, $\mathbf{b} = \mathbf{c}^p$ for some $\mathbf{c} \in \mathrm{MaxMot}(\mathcal{C})$. Now by Proposition 3.1, from $\mathbf{a}^p < \mathbf{c}^p$ we get $\mathbf{a} < \mathbf{c}$, which is a contradiction since both $\mathbf{a}$ and $\mathbf{c}$ are maximal motifs of $\mathcal{C}$. Claim 4 is proved.

Now the statement of the theorem follows from Claim 3 and Claim 4. □

Note that for two motifs $\mathbf{a}, \mathbf{b} \in \mathbb{M}^n$ we have

$$\mathbf{b} = \overline{\overline{\mathbf{a}}^p} \;\Leftrightarrow\; \mathbf{a} = \overline{\overline{\mathbf{b}}^d}. \tag{3.1}$$

Moreover, if for any code $\mathcal{C} \subseteq \mathbb{F}_2^n$ and any $M \subseteq \mathrm{Mot}(\mathcal{C})$ we denote $\overline{M} = \{\overline{\mathbf{a}} \,:\, \mathbf{a} \in M\}$, then

$$\mathrm{Mot}(\overline{\mathcal{C}}) = \overline{\mathrm{Mot}(\mathcal{C})}, \tag{3.2}$$

$$\mathrm{MaxMot}(\overline{\mathcal{C}}) = \overline{\mathrm{MaxMot}(\mathcal{C})}. \tag{3.3}$$

Proposition 3.4 *For any motif* $\mathbf{a} \in \mathbb{M}^n$ *we have*

$$L^p_{\mathbf{a}} = L_{\overline{\overline{\mathbf{a}}^p}}.$$

Proof This follows from the definitions (1.21), (1.2), (2.1), and (1.12). □

Example 3.5 *Let n=4 and let* $\mathbf{a} = 11*0 \in \mathbf{M}^4$. *Then*

$$L_{\mathbf{a}} = X_1X_2(1 - X_4).$$

Hence,

$$L^p_{\mathbf{a}} = X_1X_2Y_4.$$

On the other side, we have:

$$\begin{aligned}
\overline{\overline{\mathbf{a}}^p} &= \overline{\overline{11*0}^p} \\
&= \overline{00*1 1^p} \\
&= \overline{00*0|**{*}0} \\
&= 11**|**{*}1.
\end{aligned}$$

Hence,

$$L_{\overline{\mathbf{a}^p}} = X_1X_2Y_4 = L^p_{\mathbf{a}}.$$

Definition 3.6 *We say that two motifs* $\mathbf{a},\mathbf{b} \in \mathbb{M}^n$ *are disjoint if there is an* $i \in [n]$ *such that* $a_i \neq *$ *and* $a_i = \overline{b_i}$.

Definition 3.7 *On the set* $\mathbb{M}$ *of motifs of length* 1 *we introduce a commutative operation of addition in the following way:*

$$0+0=0,$$

$$0+1=1,$$

$$1+1=0,$$

$$0+*=*,$$

$$1+*=*,$$

$$*+*=*.$$

The first three lines represent the arithmetic in the field $\mathbb{F}_2$, *while the last three lines represent the max-arithmetic. We then define the addition in* $\mathbb{M}^n$ *by adding two motifs componentwise.*

It is easy to verify that with this operation and the partial order that we introduced before, $\mathbb{M}^n$ is a *partially ordered monoid*.

The importance of above definition lies in the fact that the sum $\mathbf{a}+\mathbf{b}$ of two motifs $\mathbf{a},\mathbf{b} \in \mathbb{M}^n$ has a 1-component if and only if the motifs $\mathbf{a}$ and $\mathbf{b}$ are disjoint. Thus, we can recognize the disjointness of two motifs algebraically by considering their sum.

Proposition 3.8 *Let* $\mathbf{a} \in \mathrm{Mot}(\mathcal{C})$ *for some code* $\mathcal{C} \subseteq \mathbb{F}_2^n$ *and* $\mathbf{b} \in \mathbb{M}^n$. *Then* $\mathbf{b} \in \mathrm{Mot}({}^c\mathcal{C})$ *if and only if* $\mathbf{b}$ *is disjoint with* $\mathbf{a}$. *Moreover, the maximal motifs of* ${}^c\mathcal{C}$ *are the motifs* $\mathbf{b}$ *that are maximal among the motifs from* $\mathbb{M}^n$ *that are disjoint with all the maximal motifs of* $\mathcal{C}$.

Proof Easy to see. □

Proposition 3.9 *Let* $\mathbf{a},\mathbf{b},\mathbf{b}' \in \mathbb{M}^n$. *If* $\mathbf{a}+\mathbf{b}$ *has an* 1*-component and* $\mathbf{b}' \leq \mathbf{b}$, *then* $\mathbf{a}+\mathbf{b}'$ *has an* 1*-component.*

In particular, if $\mathcal{C}$ *is a code in* $\mathbb{M}^n$, *the maximal motifs of* ${}^c\mathcal{C}$ *are the maximal elements* $\mathbf{b} \in \mathbb{M}^n$ *such that each* $\mathbf{a}+\mathbf{b}$ *(*$\mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})$*) has an* 1*-component.*

Proof Easy to see. □

Corollary 3.10 *Let* $\mathcal{C} \subseteq \mathbb{F}_2^n$ *be a code in* $\mathbb{F}_2^n$. *If* $\boldsymbol{b} \in \mathrm{MaxMot}({}^c(\mathcal{C}^p))$, *then every* b_i *which is different from* $*$ *is equal to* 1.

Proof The statement follows from the previous proposition as each 0 could be replaced by a $*$, which would result in a strictly bigger motif disjoint with all maximal motifs of $\mathcal{C}^p$. □

Proposition 3.11 *The motifs* $\mathbf{a}$ *and* $\mathbf{b}$ *from* $\mathbb{M}^n$ *are disjoint if and only if the motifs* $\mathbf{a}^p$ *and* $\overline{\overline{\mathbf{b}}^p}$ *from* $\mathbb{M}^{2n}$ *are disjoint.*

Proof $\Rightarrow$) Suppose that $\mathbf{a}$ and $\mathbf{b}$ are disjoint. We first consider the case $a_i = 1,\ b_i = 0$ for some $i \in [n]$. Then $(\mathbf{a}^p)_i = *$ and $(\mathbf{a}^p)_{n+i} = 0$, while $(\overline{\overline{\mathbf{b}}^p})_i = *$ and $(\overline{\overline{\mathbf{b}}^p})_{n+i} = 1$. Hence, $\mathbf{a}^p$ and $\overline{\overline{\mathbf{b}}^p}$ are disjoint. The case $a_i = 0,\ b_i = 1$ for some $i \in [n]$ is similar.

$\Leftarrow$) Suppose that $\mathbf{a}^p$ and $\overline{\overline{\mathbf{b}}^p}$ are disjoint. We first consider the case $(\mathbf{a}^p)_i = 0,\ (\overline{\overline{\mathbf{b}}^p})_i = 1$ for some $i \in [n]$. Then $a_i = 0$ and $(\overline{\mathbf{b}}^p)_i = 0$; hence, $(\overline{\mathbf{b}})_i = 0$. Hence, $b_i = 1$, so that $\mathbf{a}$ and $\mathbf{b}$ are disjoint. The case $(\mathbf{a}^p)_i = 1,\ (\overline{\overline{\mathbf{b}}^p})_i = 0$ for some $i \in [n]$ is similar. □

Proposition 3.12 *Let* $\mathcal{C}, \mathcal{D}$ *be two codes in* $\mathbb{F}_2^n$*. Then:*

$$\mathcal{D} \subseteq {}^c\mathcal{C} \ \Leftrightarrow\ \overline{\overline{\mathcal{D}}^p} \subseteq {}^c(\mathcal{C}^p).$$

Proof The next equivalences follow from Proposition 3.8, Proposition 3.11, Theorem 3.3, and Proposition 3.8, respectively.

$$\begin{aligned}
\mathcal{D} \subseteq {}^c\mathcal{C} &\Leftrightarrow (\forall\, \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C}))(\forall\, \mathbf{b} \in \mathrm{MaxMot}(\mathcal{D}))\ \mathbf{a} \text{ and } \mathbf{b} \text{ are disjoint} \\
&\Leftrightarrow (\forall\, \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C}))(\forall\, \mathbf{b} \in \mathrm{MaxMot}(\mathcal{D}))\ \mathbf{a}^p \text{ and } \overline{\overline{\mathbf{b}}^p} \text{ are disjoint} \\
&\Leftrightarrow (\forall\, \mathbf{c} \in \mathrm{MaxMot}(\mathcal{C}^p))(\forall\, \mathbf{d} \in \mathrm{MaxMot}(\overline{\overline{\mathcal{D}}^p}))\ \mathbf{c} \text{ and } \mathbf{d} \text{ are disjoint} \\
&\Leftrightarrow \overline{\overline{\mathcal{D}}^p} \subseteq {}^c(\mathcal{C}^p).
\end{aligned}$$

□

Remark 3.13 *Note that in the previous proposition the equality on the left hand side is not equivalent with the equality on the right hand side, as we are going to see in Example 3.18.*

Corollary 3.14 *Let* $\mathcal{D} = {}^c\mathcal{C}$*. Then*

$$\mathcal{C}^p \subseteq {}^c\overline{\overline{\mathcal{D}}^p}.$$

Proof It follows immediately from the previous proposition. □

The reason for giving the next definition and using the terminology introduced in it will become clear later, after Theorem 3.17 and Example 3.18.

Definition 3.15 *Let* $\mathcal{C}$ *be a code in* $\mathbb{F}_2^n$ *and let* $\mathcal{D}$ *be its complement. We call the code* $\mathcal{C}^{[p]}$*, defined by*

$$\mathcal{C}^{[p]} = {}^c\overline{\overline{\mathcal{D}}^p}, \tag{3.4}$$

the formal polarization of the code $\mathcal{C}$*.*

Proposition 3.16 *Let* $\mathcal{C}$ *be a code in* $\mathbb{F}_2^n$ *and* $\mathcal{D}$ *its complement. We have:*

$$\mathrm{MaxMot}(\mathcal{C}^p) \subseteq \mathrm{MaxMot}(\mathcal{C}^{[p]}), \tag{3.5}$$

$$\mathrm{MaxMot}({}^c(\mathcal{C}^{[p]})) = \overline{\overline{\mathrm{MaxMot}(\mathcal{D})}^p} \subseteq \mathrm{MaxMot}({}^c(\mathcal{C}^p)). \tag{3.6}$$

Proof By Theorem 3.3 and the formula (3.4), (3.5) is equivalent with showing that

$$(\forall\, \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C}))(\forall\, \mathbf{b} \in \mathrm{MaxMot}(\mathcal{D}))\ \ \mathbf{a}^p \ \text{ and } \ \overline{\overline{\mathbf{b}}^p} \ \text{ are disjoint,}$$

which is true by Proposition 3.11.

We now show (3.6). Let $\mathbf{d} \in \mathrm{MaxMot}({}^c(\mathcal{C}^{[p]})) = \mathrm{MaxMot}(\overline{\overline{\mathcal{D}}^p})$. Then $\mathbf{d} = \overline{\overline{\mathbf{b}}^p}$ for some $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{D})$. Let $\mathbf{e} \in \mathrm{MaxMot}({}^c(\mathcal{C}^p))$ such that $\mathbf{d} \leq \mathbf{e}$, i.e. $\overline{\overline{\mathbf{b}}^p} \leq \mathbf{e}$. Hence, $\overline{\mathbf{b}}^p \leq \overline{\mathbf{e}}$. Since $\mathbf{e}$ is bigger than or equal to a polar motif, $\mathbf{e}$ is a polar motif too, so $\mathbf{e} = \overline{\mathbf{f}}^p$ for some motif $\mathbf{f}$. Hence, $\overline{\mathbf{b}}^p \leq \overline{\mathbf{f}}^p$, so that $\overline{\overline{\mathbf{b}}^p} \leq \overline{\overline{\mathbf{f}}^p} = \mathbf{e}$. Hence, since $\mathbf{e}$ is disjoint with all the maximal motifs of $\mathcal{C}^p$, then by Proposition 3.11, $\mathbf{f}$ is disjoint with all the maximal motifs of $\mathcal{C}$ and $\mathbf{f} \geq \mathbf{b}$, where $\mathbf{b}$ is one of the maximal motifs among the motifs that are disjoint with all maximal motifs of $\mathcal{C}$. Hence, $\mathbf{f} = \mathbf{b}$, so that $\mathbf{d} = \mathbf{e}$. $\square$

Theorem 3.17 *Let $\mathcal{C}$ be a code in $\mathbb{F}_2^n$. We have:*

$$CF(J_{\mathcal{C}}^p) = CF^p(J_{\mathcal{C}}) = CF(J_{\mathcal{C}^{[p]}}) \subseteq CF(J_{\mathcal{C}^p}).$$

Proof Let $CF(J_{\mathcal{C}}) = \{f_1, \ldots, f_k\}$. By definition, $J_{\mathcal{C}}^p = (f_1^p, \ldots, f_k^p)$. Here f_1^p, ..., f_k^p are square-free monomials. By [9, Corollary 1.10], the set $\{f_1^p, \ldots, f_k^p\}$ contains a minimal subset S (with respect to inclusion) which generates $J_{\mathcal{C}}^p$. By [9, Corollary 1.8], if $f_i^p \notin S$, then $f_i^p \mid f_j^p$ for some $f_j^p \in S$. Then by Proposition 1.22, $f_i \mid f_j$, a contradiction. Thus, $S = \{f_1^p, \ldots, f_k^p\}$. Hence, by [9, Proposition 1.11], $CF(J_{\mathcal{C}}^p) = \{f_1^p, \ldots, f_k^p\} = CF^p(J_{\mathcal{C}})$.

Let $\mathcal{D}$ be the complement of $\mathcal{C}$. We have:

$$\begin{aligned}
CF^p(J_{\mathcal{C}}) &= \{L_{\mathbf{a}}^p \ : \ \mathbf{a} \in \mathrm{MaxMot}(\mathcal{D})\} && \text{(by Proposition 1.15)}\\
&= \{L_{\overline{\overline{\mathbf{a}}^p}} \ : \ \mathbf{a} \in \mathrm{MaxMot}(\mathcal{D})\} && \text{(by Proposition 3.4)}\\
&= \{L_{\mathbf{b}} \ : \ \overline{\overline{\mathbf{b}}}^d \in \mathrm{MaxMot}(\mathcal{D})\} && \text{(by (3.1))}\\
&= \{L_{\mathbf{b}} \ : \ \overline{\mathbf{b}}^d \in \mathrm{MaxMot}(\overline{\mathcal{D}})\} &&\\
&= \{L_{\mathbf{b}} \ : \ \overline{\mathbf{b}} \in \mathrm{MaxMot}(\overline{\mathcal{D}}^p)\} &&\\
&= \{L_{\mathbf{b}} \ : \ \mathbf{b} \in \mathrm{MaxMot}(\overline{\overline{\mathcal{D}}^p})\} &&\\
&= CF(J_{\mathcal{C}^{[p]}}). && \text{(by Proposition 1.15)}
\end{aligned}$$

Finally, the inclusion in the statement of the theorem follows from Proposition 3.16 and Proposition 1.15. $\square$

Example 3.18 *Consider the neural codes*

$$\mathcal{C} = \{000, 100, 110, 011\} \ \textit{and} \ \mathcal{D} = {}^c\mathcal{C} = \{001, 010, 101, 111\}$$

in $\mathbb{F}_2^3$. We have

$$\mathrm{MaxMot}(\mathcal{C}) = \{{*}00, 1{*}0, 011\} \ \textit{and} \ \mathrm{MaxMot}(\mathcal{D}) = \{{*}01, 1{*}1, 010\}.$$

Then by Theorem 3.3,

$$\mathrm{MaxMot}(\mathcal{C}^p) = \{*00***,\ **00**,\ 0***00\}, \tag{3.7}$$

$$\mathrm{MaxMot}(\overline{\overline{\mathcal{D}}^p}) = \{**1*1*,\ 1*1***,\ *1*1*1\}. \tag{3.8}$$

By Proposition 3.12 we have $\overline{\overline{\mathcal{D}}^p} \subseteq {}^c(\mathcal{C}^p)$. *From (3.8) we have by Proposition 1.15,*

$$CF(J_{\mathcal{C}^{[p]}}) = \{X_3Y_2,\ X_1X_3,\ X_2Y_1Y_3\} = CF^p(J_{\mathcal{C}}).$$

By Proposition 3.8, a motif $\mathbf{b} \in \mathbb{M}^6$ *is a maximal motif of* ${}^c(\mathcal{C}^p)$ *if and only if it is a maximal motif from* $\mathbb{M}^6$ *disjoint with all the maximal motifs of* $\mathcal{C}^p$. *The sets*

$$A_1 = \{2,3\},\quad A_2 = \{3,4\},\quad A_3 = \{1,5,6\},$$

are the sets of coordinates which the maximal motifs $\mathbf{a}^1 = *00***$, $\mathbf{a}^2 = **00**$, *and* $\mathbf{a}^3 = 0***00$ *of* $\mathcal{C}^p$ *have zeros at, respectively. To get a set* B *of coordinates where a motif* $\mathbf{b} \in \mathrm{MaxMot}({}^c(\mathcal{C}^p))$ *has ones, we need to take one element from each of the sets* A_1, A_2, A_3, *and then, out of all sets* B *obtained in that way* ($2 \times 2 \times 3 = 12$ *of them) select the minimal ones with respect to inclusion. In that way we get the sets* $B_1 = \{2,4,1\}$, $B_2 = \{2,4,5\}$, $B_3 = \{2,4,6\}$, $B_4 = \{3,1\}$, $B_5 = \{3,5\}$, *and* $B_6 = \{3,6\}$. *For each of these sets* B_i *we get an element* $\mathbf{b}^i \in \mathrm{MaxMot}({}^c(\mathcal{C}^p))$ *by putting ones at all the coordinates of* B_i *and stars at all other coordinates. Thus,*

$$\mathrm{MaxMot}({}^c(\mathcal{C}^p)) = \{**1*1*,\ \ 1*1***,\ \ *1*1*1,\ \ **1**1,$$
$$*1*11*,\ \ 11*1**\}.$$

Hence, by Proposition 1.15,

$$CF(J_{\mathcal{C}^p}) = \{X_3Y_2,\ X_1X_3,\ X_2Y_1Y_3,\ X_3Y_3,\ X_2Y_1Y_2,\ X_1X_2Y_1\}.$$

In particular,

$$\mathcal{C}^p \subset \mathcal{C}^{[p]}.$$

(One can check that $\mathcal{C}^p$ *has* 29 *words, while* $\mathcal{C}^{[p]}$ *has* 35 *words.)*

Example 3.19 *Consider again the code* $\mathcal{C} = \{10\} \subseteq \mathbb{F}_2^2$ *and its complement* $\mathcal{D} = \{00, 01, 11\}$ *from Example 2.3. We have:*

$$\mathrm{MaxMot}(\mathcal{C}) = \{10\},$$
$$\mathrm{MaxMot}(\mathcal{D}) = \{0*,\ *1\},$$
$$CF(J_{\mathcal{C}}) = \{1 - X_1,\ X_2\},$$
$$CF(J_{\mathcal{C}^{[p]}}) = CF^p(J_{\mathcal{C}}) = \{X_2,\ Y_1\},$$
$$\mathcal{C}^p = \{0000, 1000, 0001, 1001\},$$
$$\mathrm{MaxMot}({}^c(\mathcal{C}^p)) = \{*1**,\ **1*\},$$
$$CF(J_{\mathcal{C}^p}) = \{X_2,\ Y_1\}.$$

Thus in this example $\mathcal{C}^p = \mathcal{C}^{[p]}$.

Definition 3.20 *The prime ideals* $\mathfrak{p} \subseteq \mathbb{F}_2[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ *such that* $\mathfrak{p} = \mathfrak{p}_{\mathbf{a}^p}$ *for some* $\mathbf{a} \in \mathbb{M}^n$ *are called polar motivic primes.*

For polar motivic primes we have the following formula:

$$\mathfrak{p}_{\mathbf{a}^p} = \mathfrak{p}_{\mathbf{a}}^p. \tag{3.9}$$

Indeed, if $\mathbf{a} = a_1 \ldots a_n \in \mathrm{MaxMot}(\mathcal{C})$ and $\mathbf{a}^p = b_1 \ldots b_n c_1 \ldots c_n$, then

$$\begin{aligned} \mathfrak{p}_{\mathbf{a}^p} &= (\{X_i \ : \ b_i = 0\} \cup \{Y_j \ : \ c_j = 0\}) \\ &= (\{X_i \mid a_i = 0\} \cup \{Y_j \ : \ a_j = 1\}) \\ &= \mathfrak{p}_{\mathbf{a}}^p. \end{aligned}$$

Theorem 3.21 *For any code* $\mathcal{C} \subseteq \mathbb{F}_2^n$ *we have:*

$$\mathrm{Min}(J_{\mathcal{C}^p}) = \mathrm{Min}^p(J_{\mathcal{C}}) \subseteq \mathrm{Min}(J_{\mathcal{C}^{[p]}}).$$

Proof We have

$$\begin{aligned} \mathrm{Min}(J_{\mathcal{C}^p}) &= \{\mathfrak{p}_{\mathbf{d}} \ : \ \mathbf{d} \in \mathrm{MaxMot}(\mathcal{C}^p)\} && \text{(by Proposition 1.19)} \\ &= \{\mathfrak{p}_{\mathbf{a}^p} \ : \ \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\} && \text{(by Theorem 3.3)} \\ &= \{\mathfrak{p}_{\mathbf{a}^p} \ : \ \mathfrak{p}_{\mathbf{a}} \in \mathrm{Min}(J_{\mathcal{C}})\} && \text{(by Proposition 1.19)} \\ &= \{\mathfrak{p}_{\mathbf{a}}^p \ : \ \mathfrak{p}_{\mathbf{a}} \in \mathrm{Min}(J_{\mathcal{C}})\}. && \text{(by the formula (3.9))} \end{aligned}$$

Hence,

$$\mathrm{Min}(J_{\mathcal{C}^p}) = \mathrm{Min}^p(J_{\mathcal{C}}).$$

The inclusion part of the statement follows from (3.5) and the relation (1.1) from Proposition 1.19. □

Example 3.22 *We continue Example 3.18. By (3.8) we have*

$$\mathrm{MaxMot}(^c(\mathcal{C}^{[p]})) = \{{*}{*}1{*}1{*},\, 1{*}1{*}{*}{*},\, {*}1{*}1{*}1\}.$$

Now using the same technique as in Example 3.18 (for finding $\mathrm{MaxMot}(^c(\mathcal{C}^p))$ *given* $\mathrm{MaxMot}(\mathcal{C}^p)$*) we find here that*

$$\begin{aligned} \mathrm{MaxMot}(\mathcal{C}^{[p]}) = \{&{*}00{*}{*}{*},\ \ {*}{*}00{*}{*},\ \ {*}{*}0{*}{*}0,\ \ 00{*}{*}0{*}, \\ &0{*}{*}00{*},\ \ 0{*}{*}{*}00\}. \end{aligned}$$

Hence, by Proposition 1.19 we have:

$$\begin{aligned} \mathrm{Min}(J_{\mathcal{C}^p}) &= \{(X_2, X_3),\ (X_3, Y_1),\ (X_1, Y_2, Y_3)\} = \mathrm{Min}^p(J_{\mathcal{C}}), \\ \mathrm{Min}(J_{\mathcal{C}^{[p]}}) &= \{(X_2, X_3),\ (X_3, Y_1),\ (X_1, Y_2, Y_3), \\ &\qquad (X_3, Y_3),\ (X_1, X_2, Y_2),\ (X_1, Y_1, Y_2)\}. \end{aligned}$$

The minimal prime ideals of $J_{\mathcal{C}}^p$ *(i.e.* $J_{\mathcal{C}^{[p]}}$*) were also calculated in [8, Example 5.4] in a different way.*

Note that among the minimal primes of $J_{\mathcal{C}[p]}$ *we have, in addition to all the minimal primes of* $J_{\mathcal{C}^p}$*, three nonpolar minimal primes, namely* $\mathfrak{p}_{**0**0} = (X_3, Y_3)$*,* $\mathfrak{p}_{00**0*} = (X_1, X_2, Y_2)$*, and* $p_{0**00*} = (X_1, Y_1, Y_2)$*. A natural question to ask is the following one: if for an* $\mathbf{a} \in \mathbb{M}^{2n}$ *we have* $\mathfrak{p}_{\mathbf{a}} \in \mathrm{Min}(J_{\mathcal{C}[p]})$*, how is then the motif* $\mathbf{a}$ *related to* $\mathcal{C}$*? A statement related to this question is given in the next section in Theorem 4.5, which is [8, Theorem 5.1]. We will give a different proof of this theorem.*

Theorem 3.23 *For any code* $\mathcal{C} \subseteq \mathbb{F}_2^n$*, the ideal* $J_{\mathcal{C}^p}$ *has the unique irredundant primary decomposition and it is obtained by polarizing the prime ideals from the unique irredundant primary decomposition of* $J_{\mathcal{C}}$*.*

Proof By Proposition 1.20, the ideals $J_{\mathcal{C}}$ and $J_{\mathcal{C}^p}$ have the unique irredundant primary decompositions

$$J_{\mathcal{C}} = \cap\{\mathfrak{p}_{\mathbf{a}} \ : \ \mathbf{a} \in \mathrm{MaxMot}(\mathcal{C})\},$$

$$J_{\mathcal{C}} = \cap\{\mathfrak{p}_{\mathbf{b}} \ : \ \mathbf{b} \in \mathrm{MaxMot}(\mathcal{C}^p)\}.$$

Hence, the statement follows from Theorem 3.3 and the formula (3.9). □

4. Partial motifs

Definition 4.1 *We denote* $\mathbb{PM} = \{0, 1, *, \_\}$*. We say that this set is the set of partial motifs of length* 1*. We define a partial order on* $\mathbb{PM}$ *by declaring that* $0 < *$ *and* $1 < *$*. Note that* $\_$ *is comparable only with itself (the same holds for* 0 *and* 1*). We define a partial order on the set* $\mathbb{PM}^n$ *by declaring that* $\mathbf{a} \le \mathbf{b}$ *if* $a_i \le b_i$ *for every* $i \in [n]$*. A partial motif (of length* n*) is an element of* $\mathbb{PM}^n$*. A partial word (of length* n*) is an element of* $\mathbb{PW}^n = \{0, 1, \_\}^n$*. The neurons* $i \in [n]$ *for which* $w_i = \_$ *are said to be inactive. A partial code (of length* n*) is a subset of* $\mathbb{PW}^n$*. The variety of a partial motif* $\mathbf{a}$ *is the set of all partial words obtained by replacing all the stars in* $\mathbf{a}$ *by zeros and ones. It is denoted by* $V_{\mathbf{a}}$*. If* $\mathcal{C} \subseteq \mathbb{PW}^n$ *is a a partial code, then* $\mathbf{a} \in \mathbb{PM}^n$ *is a partial motif of* $\mathcal{C}$ *if* $V_{\mathbf{a}} \subseteq \mathcal{C}$*. The set of all partial motifs of a partial code* $\mathcal{C}$ *is denoted by* $\mathrm{ParMot}(\mathcal{C})$*. The set of all maximal partial motifs of a partial code* $\mathcal{C}$ *is denoted by* $\mathrm{MaxParMot}(\mathcal{C})$*.*

Example 4.2 *Intuitively, we can think of the partial word* $\mathbf{w} = \_01\_00\_1$ *as of a statement that the neurons* 3 *and* 8 *are firing, the neurons* $2, 5$ *and* 6 *are not firing, and the neurons* $1, 4,$ *and* 7 *are also "participating in the neural activity"; however, their status is not defined. That could happen, for example, when we delete certain neurons, but we want to keep their spots for a possibility of their reactivation.*

The set of all partial motifs $\mathbf{a} \in \mathbb{PM}^n$ (resp. partial words $\mathbf{w} \in \mathbb{PW}^n$) such that $a_{i_1} = \cdots = a_{i_k} = \_$ (resp. $w_{i_1} = \cdots = w_{i_k} = \_$) and all the remaining neurons are active, is denoted by $\mathbb{PM}^n_{i_1,\ldots,i_k}$ (resp. $\mathbb{PW}^n_{i_1,\ldots,i_k}$). It is naturally in a bijective correspondence with the set $\mathbb{M}^{n-k}$ (resp. $\mathbb{F}_2^{n-k}$). If $\mathbf{a}$ (resp. $\mathbf{w}$) is a motif (resp. word), then the partial motif (resp. partial word) obtained by replacing each a_i (resp. w_i), $i = i_1, \ldots, i_k$, by $\_$ is called the *partial motif* (resp. *partial word*) *obtained by deactivating the neurons* $i_1, \ldots, i_k$ and is denoted by $a^-_{i_1,\ldots,i_k}$ (resp. $w^-_{i_1,\ldots,i_k}$).

If $\mathcal{C} \subseteq \mathbb{F}_2^n$ is a code and $\{i_1, \ldots, i_k\} \subseteq [n]$, then the code obtained by replacing each w_{i_r} $(r = 1, \ldots, k)$ by $\_$ in each word $\mathbf{w} \in \mathcal{C}$ is called the *partial code obtained by deactivating the neurons* $i_1, \ldots, i_k$ and is denoted by $\mathcal{C}^-_{i_1,\ldots,i_k}$. The partial code $\mathcal{C}^-_{i_1,\ldots,i_k}$ is naturally in a bijective correspondence with the code $\mathcal{C}_{i_1,\ldots,i_k}$ obtained by deleting the neurons $i_1, \ldots, i_k$.

Proposition 4.3 *Let $\mathcal{C}$ be a code in $\mathbb{F}_2^n$ and let $\mathbf{w} \in \mathbb{PW}^n_{i_1,\dots,i_k}$. If $\mathbf{w}$ is not an element of $\mathcal{C}^{-}_{i_1,\dots,i_k}$, then the motif $\mathbf{a}$ obtained from $\mathbf{w}$ by replacing each _ by $*$ belongs to $\mathrm{Mot}({}^c\mathcal{C})$.*

Proof Easy to see. □

Definition 4.4 *If $\mathbf{a} \in \mathbb{PM}^n$, then we define its polarization $\mathbf{a}^p = b_1 \dots b_n b_{n+1} \dots b_{2n} \in \mathbb{PM}^{2n}$ by:*

$$b_i = 0,\ b_{n+i} = *, \quad \text{when } a_i = 0;$$

$$b_i = *,\ b_{n+i} = 0, \quad \text{when } a_i = 1;$$

$$b_i = *,\ b_{n+i} = *, \quad \text{when } a_i = *;$$

$$b_i = \_,\ \ b_{n+i} = \_, \quad \text{when } a_i = \_.$$

A partial motif $\mathbf{b} \in \mathbb{PM}^{2n}$ is called a polar partial motif if $\mathbf{b} = \mathbf{a}^p$ for some partial motif $\mathbf{a} \in \mathbb{PM}^n$. Then $\mathbf{a} = \mathbf{b}^d$ is called the depolarization of the polar partial motif $\mathbf{b}$.

Note that we have

$$\mathbf{a}^{pd} = \mathbf{a} \quad \text{for every } \mathbf{a} \in \mathbb{PM}^n,$$

$$\mathbf{b}^{dp} = \mathbf{b} \quad \text{for every polar partial motif } \mathbf{b} \in \mathbb{PM}^{2n}.$$

The next theorem is a slight reformulation of Theorem 5.1 from [8]. We give a different proof.

Theorem 4.5 ([8], Theorem 5.1) *Let $\mathbf{c} \in \mathbb{M}^{2n}$ and let $\mathbf{a} \in \mathbb{M}^{2n}$ be the motif obtained by replacing all ones in $\mathbf{c}$ by stars. Let $\{i_1, \dots, i_k\}$ be the set of all elements i of $[n]$ such that $a_i = a_{i+n}$. Then $\mathfrak{p}_{\mathbf{c}} \supseteq J_{\mathcal{C}^{[p]}}$ if and only if $\mathbf{a}^{-\,d}_{i_1,\dots,i_k,i_1+n,\dots,i_k+n} \in \mathrm{ParMot}(\mathcal{C}^{-}_{i_1,\dots,i_k})$.*

Proof Let $\mathcal{D} = {}^c\mathcal{C}$. Since

$$\mathrm{MaxMot}({}^c(\mathcal{C}^{[p]})) = \{\, \overline{\overline{\mathbf{b}}^p} \;:\; \mathbf{b} \in \mathrm{MaxMot}(\mathcal{D})\},$$

the motifs $\mathbf{c}$ of $\mathcal{C}^{[p]}$ are the motifs from $\mathbb{M}^{2n}$ that are disjoint with all $\overline{\overline{\mathbf{b}}^p}$, $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{D})$. They give the primes $\mathfrak{p}_{\mathbf{c}}$, and these are all the motivic primes that contain $J_{\mathcal{C}^{[p]}}$. Note, however, that $\mathfrak{p}_{\mathbf{c}} \supset J_{\mathcal{C}^{[p]}}$ if and only if $\mathfrak{p}_{\mathbf{a}} \supset J_{\mathcal{C}^{[p]}}$. It follows that a motivic prime $\mathfrak{p}_{\mathbf{c}}$ contains $J_{\mathcal{C}^{[p]}}$ if and only if $\mathbf{a} + \overline{\overline{\mathbf{b}}^p}$ has at least one component equal to 1, or, equivalently, such that

$$\mathbf{a} + \overline{\overline{\mathbf{b}}^p} < * \cdots * \tag{4.1}$$

(as $\mathbf{a} \in \{0, *\}^{2n}$ and $\overline{\overline{\mathbf{b}}^p} \in \{1, *\}^{2n}$) for every $\mathbf{b} \in \mathrm{MaxMot}(\mathcal{D})$. Let $\{i_1, \dots, i_k\}$ be the set of elements i of $[n]$ such that $a_i = a_{i+n} = 0$. (This includes the possibility $k = 0$.) Let $\mathbf{a}^{-\,d}$ denote $\mathbf{a}^{-\,d}_{i_1,\dots,i_k,i_1+n,\dots,i_k+n}$. The statement of the theorem follows if we now justify the claim that $\mathbf{a}$ satisfies (4.1) if and only if each partial word $\mathbf{w} \le \mathbf{a}^{-\,d}$ belongs to $\mathcal{C}^{-}_{i_1,\dots,i_k}$, i.e. if and only if $\mathbf{a}^{-\,d} \in \mathrm{ParMot}(\mathcal{C}^{-}_{i_1,\dots,i_k})$. The necessity is clear. For the sufficiency, there would otherwise be a partial word $\mathbf{w} \le \mathbf{a}^{-\,d}$ which is not coming from any partial word in $\mathcal{C}^{-}_{i_1,\dots,i_k}$. Let $\mathbf{w}^*$ be the motif obtained by replacing each _ in $\mathbf{w}$ by $*$. Then $\mathbf{w}^* \in \mathrm{Mot}(\mathcal{D})$; hence, $\overline{\overline{\mathbf{w}^*}^p} \in \mathrm{Mot}({}^c(\mathcal{C}^{[p]}))$. This motif would have stars at all the components at which $\mathbf{a}$ has zeros and ones or stars at all other components. Hence, $\mathbf{a} + \overline{\overline{\mathbf{w}^*}^p} = * \cdots *$, contradicting to the asumption that $\mathbf{a}$ satisfies (4.1). □

Example 4.6 *In the context of the examples (3.18) and (3.22), let* $\mathbf{c} = 00{*}{*}0{*} \in \mathbb{M}^6$. *Then* $\mathbf{a} = \mathbf{c}$ *and*

$$\mathbf{a}_{\overline{2,2}}^{d} \;=\; 0\_{*}{*}\_{*}^{d} \;=\; 0\_{*}\,.$$

Moreover,

$$\mathcal{C}_{\overline{2}} = \{000,\ 100,\ 110,\ 011\}_{\overline{2}} = \{0\_0,\ 1\_0,\ 0\_1\}.$$

Hence,

$$\mathrm{MaxParMot}(\mathcal{C}_{\overline{2}}) \;=\; \{{*}\_0,\ 0\_{*}\}.$$

Thus,

$$\mathbf{a}_{\overline{2,2}}^{d} \in \mathrm{MaxParMot}(\mathcal{C}_{\overline{2}}),$$

so that

$$\mathfrak{p}_{\mathbf{c}} = (X_1, X_2, Y_2) \supseteq J_{\mathcal{C}^{[p]}}.$$

In fact we have

$$\mathfrak{p}_{\mathbf{c}} = (X_1, X_2, Y_2) \in \mathrm{Min}(J_{\mathcal{C}^{[p]}}).$$

Let now $\mathbf{c} = 0{*}0{*}{*}0 \in \mathbb{M}^6$. *Then* $\mathbf{a} = \mathbf{c}$ *and*

$$\mathbf{a}_{\overline{3,3}}^{d} \;=\; 0{*}\_{*}{*}\_^{d} \;=\; 0{*}\_.$$

Moreover,

$$\mathcal{C}_{\overline{3}} = \{000,\ 100,\ 110,\ 011\}_{\overline{3}} = \{00\_,\ 10\_,\ 11\_,\ 01\_\}.$$

Hence,

$$\mathrm{MaxParMot}(\mathcal{C}_{\overline{3}}) \;=\; \{{*}{*}\_\}.$$

Thus,

$$\mathbf{a}_{\overline{3,3}}^{d} \in \mathrm{ParMot}(\mathcal{C}_{\overline{3}}),$$

so that

$$\mathfrak{p}_{\mathbf{c}} = (X_1, X_3, Y_3) \supseteq J_{\mathcal{C}^{[p]}}.$$

Note that

$$\mathfrak{p}_{\mathbf{c}} = (X_1, X_3, Y_3) \notin \mathrm{Min}(J_{\mathcal{C}^{[p]}})$$

since it was shown in Example 3.22 that the prime (X_3, Y_3) *is a minimal prime of* $J_{\mathcal{C}^{[p]}}$.

Finally, let $\mathbf{c} = 100{*}0{*} \in \mathbb{M}^6$. *Then* $\mathbf{a} = {*}00{*}0{*}$ *and*

$$\mathbf{a}_{\overline{2,2}}^{d} \;=\; {*}\_0{*}\_{*}^{d} \;=\; {*}\_0.$$

Moreover,

$$\mathcal{C}_{\overline{2}} = \{000,\ 100,\ 110,\ 011\}_{\overline{2}} = \{0\_0,\ 1\_0,\ 0\_1\}.$$

Hence,

$$\mathrm{MaxParMot}(\mathcal{C}_{\overline{2}}) \;=\; \{{*}\_0,\ 0\_1\}.$$

Thus,

$$\mathbf{a}_{2,2}^{-d} \in \mathrm{MaxParMot}(\mathcal{C}_2^-),$$

so that

$$\mathfrak{p}_{\mathbf{a}} = (X_2, X_3, Y_2) \supseteq J_{\mathcal{C}[p]}$$

and

$$\mathfrak{p}_{\mathbf{c}} = (1 - X_1, X_2, X_3, Y_2) \supseteq J_{\mathcal{C}[p]}.$$

Note that

$$\mathfrak{p}_{\mathbf{a}} \notin \mathrm{Min}(J_{\mathcal{C}[p]})$$

even though $\mathbf{a}_{2,2}^{-d} \in \mathrm{MaxParMot}(\mathcal{C}_2^-)$ *since it was shown in Example 3.22 that the prime* (X_2, X_3) *is a minimal prime of* $J_{\mathcal{C}[p]}$ *.*

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