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On the product of dilation of truncated Toeplitz operators

Zohra BENDAOUD, Nafissa SAOULI*
Department of Mathematics, Faculty of Science, University Amar Theliji, Laghouat, Algeria

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Abstract: In this paper we study when the product of two dilations of truncated Toeplitz operators gives a dilation of a truncated Toeplitz operator. We will use an approach established in a recent paper written by Ko and Lee. This approach allows us to represent the dilation of the truncated Toeplitz operator via a 2×2 block operator.

Key words: Model space, truncated Toeplitz operator, dilation of truncated Toeplitz operator

1. Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. We start by recalling that the Hilbert space $L^2 = L^2(\mathbb{T})$ is the space of all square-integrable functions on the unit circle $\mathbb{T}$ equipped with the normalized Lebesgue measure $dm(e^{i\theta}) = \frac{d\theta}{2\pi}$. This space is endowed with the scalar product $\langle f, g\rangle = \int_{\mathbb{T}} f\bar{g}dm$.

An orthonormal basis of L^2 is given by the set $\{e_n(\theta) : n \in \mathbb{Z}\}$, where $e_n(\theta) = e^{in\theta}$ for $\theta \in \mathbb{R}$. The following orthonormal expansions are the classical Fourier series:

$$f = \sum_{n=-\infty}^{+\infty} f_n e_n = \sum_{n=-\infty}^{+\infty} f_n e^{in\theta},$$

$$f_n = \langle f, e_n\rangle = \int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\frac{d\theta}{2\pi}, n \in \mathbb{Z}.$$

For all $f, g \in L^2$, the tensor product $f \otimes g$ is the rank one operator in L^2 and is defined by

$$(f \otimes g)\, h = \langle h, g\rangle f$$

for $h \in L^2$. Let L^∞ be the Banach space of essentially bounded functions on $\mathbb{T}$. For any $\varphi \in L^\infty$, the bounded multiplication operator M_φ is defined by the formula

$$M_\varphi f = \varphi f, f \in L^2.$$

An operator A is a multiplication operator if and only if $AM_z = M_zA$. It is well known that, for all $\varphi \in L^\infty$, the multiplication operator M_φ is invertible if and only if φ is invertible in L^∞. Moreover, $(M_\varphi)^{-1} = M_{\varphi^{-1}}$. The Hardy space of the circle H^2 is the set of functions $f \in L^2$ such that $f_n = 0$ for all $n < 0$, and let

*Correspondence: nafissa.saouli@gmail.com

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H^∞ be the set of functions $f \in L^\infty$ such that $f_n = 0$ for all $n < 0$. We introduce now an important class of operators on spaces of analytic functions, which is the class of Toeplitz operators. Let P and $Q = I - P$ indicate the orthogonal projections that map L^2 onto H^2 and $(H^2)^\perp = \overline{zH^2}$, respectively. Given that $\varphi \in L^\infty$, the Toeplitz operator $T_\varphi : H^2 \to H^2$ is defined by

$$T_\varphi f = P(\varphi f), f \in H^2$$

and the Hankel operator $H_\varphi : H^2 \to (H^2)^\perp$ is defined by

$$H_\varphi f = Q(\varphi f), f \in H^2.$$

Hankel operators play an important role in the study of Toeplitz operators, and vice versa. Note that the Toeplitz operator becomes bounded if and only if $\varphi \in L^\infty$. In this case, we have $\|T_\varphi\| = \|\varphi\|_\infty$ (see [1]). For any $\varphi, \psi \in L^\infty$, the singular integral operator $S_{\varphi,\psi} : L^2 \to L^2$ is defined by

$$S_{\varphi,\psi}(f) = \varphi P(f) + \psi Q(f), f \in L^2.$$

With respect to the decomposition $L^2 = H^2 \oplus (H^2)^\perp$, the operator $S_{\varphi,\psi}$ can be represented as follows:

$$S_{\varphi,\psi} = \begin{pmatrix} T_\varphi & \widetilde{H_\psi} \\ H_\varphi & \widetilde{T_\psi} \end{pmatrix},$$

where T_φ and H_φ are the Toeplitz operator and Hankel operator, respectively. For more information about the operators $\widetilde{T_\psi}$ and $\widetilde{H_\psi}$, see [6]. Ko and Lee concluded that the operator $S_{\varphi,\psi}$ is the dilation of a Toeplitz operator on L^2 [5].

An inner function is an H^∞ function that has unit modulus almost everywhere on $\mathbb{T}$. For a nonconstant inner function u, the model space K_u^2 is defined by

$$K_u^2 \quad = \quad H^2 \ominus uH^2 = \{f \in H^2 : \langle f, ug\rangle = 0, \forall g \in H^2\}.$$

The space K_u^∞ is defined by $K_u^\infty = K_u^2 \cap L^\infty$, which is dense in K_u^2. For any $\varphi \in L^\infty$ and an inner function u, the truncated Toeplitz operator A_φ^u on K_u^2 is defined by

$$A_\varphi^u f = P_u(\varphi f), f \in K_u^2, \tag{1.1}$$

where $P_u = P - M_u P M_{\overline{u}}$ denotes the orthogonal projection that maps L^2 onto K_u^2.

For any $\varphi \in L^\infty$ and an inner function u, the dual of truncated Toeplitz operator $\widetilde{A_\varphi^u}$ is the operator on $(K_u^2)^\perp$ defined as follows:

$$\widetilde{A_\varphi^u} = Q_u(\varphi f), f \in (K_u^2)^\perp, \tag{1.2}$$

where $Q_u = I - P_u$ refers to the orthogonal projection that maps L^2 onto $(K_u^2)^\perp = L^2 \ominus K_u^2 = \overline{zH^2} \oplus uH^2$. The truncated Hankel operator $\Gamma_\varphi^u : K_u^2 \to (K_u^2)^\perp$ is defined by

$$\Gamma_\varphi^u f = Q_u(\varphi f), f \in K_u^2. \tag{1.3}$$

Let $\widetilde{\Gamma^u_\varphi}$ be the operator of $(K_u^2)^\perp$ to K_u^2 such that

$$\widetilde{\Gamma^u_\varphi} f = P_u(\varphi f), f \in (K_u^2)^\perp. \tag{1.4}$$

From [5], we will use what can be helpful to us in our following work, notably the following identity:

$$\widetilde{\Gamma^u_\varphi} = (\Gamma^u_{\overline{\varphi}})^*. \tag{1.5}$$

In 1963, in a famous paper on algebraic properties of Toeplitz operators [1], Brown and Halmos studied when the product of two Toeplitz operators itself becomes a Toeplitz operator. The same issue about truncated Toeplitz operators was solved by Sedlock in 2010 [8]. In 2015, Gu in [3] proved that the product $S_{\varphi_1,\psi_1}S_{\varphi_2,\psi_2}$ on L^2 is a singular integral operator if and only if $\varphi_2 \in H^\infty, \psi_2 \in \overline{H^\infty}$.

Definition 1.1 *[5] For $\varphi, \psi \in L^\infty$ and an inner function u, the dilation of truncated Toeplitz operator $S^u_{\varphi,\psi} : L^2 \to L^2$ is defined by the formula*

$$S^u_{\varphi,\psi}(f) = \varphi P_u(f) + \psi Q_u(f), f \in L^2.$$

Obviously, the operator $S^u_{\varphi,\psi}$ is a bounded operator if and only if $\varphi, \psi \in L^\infty$, such that

$$\|S^u_{\varphi,\psi}(f)\| \quad \leq \quad \|\varphi P_u(f)\| + \|\psi Q_u(f)\| \leq (\|\varphi\|_\infty + \|\psi\|_\infty)\|f\|.$$

Note that for $f \in L^2$, we have

$$S^u_{\varphi,\psi} f \quad = \quad \varphi P_u f + \psi Q_u f = \varphi P_u f + \psi[f - P_u f] = (\varphi - \psi)P_u f + \psi f.$$

Hence, it is easy to see that $S^u_{\varphi,\psi} = M_\psi + S^u_{\varphi-\psi,0}$ and $S^u_{\varphi,\varphi} = M_\varphi$.

The class of dilation of truncated Toeplitz operators was introduced in 2015 by Ko and Lee. For further details of the introduction of this class of operators, see [5]. Moreover, relying on the decomposition $L^2 = K_u^2 \oplus (K_u^2)^\perp$, they proved that the operator $S^u_{\varphi,\psi}$ has the following matrix representation:

$$S^u_{\varphi,\psi} = \begin{pmatrix} A^u_\varphi & \widetilde{\Gamma^u_\psi} \\ \Gamma^u_\varphi & \widetilde{A^u_\psi} \end{pmatrix}, \tag{1.6}$$

where $A^u_\varphi, \widetilde{A^u_\psi}, \Gamma^u_\varphi$, and $\widetilde{\Gamma^u_\psi}$ are defined by equations (1.1), (1.2), (1.3), and (1.4), respectively. We refer to [5, Lemma 3.2] for more details about this representation.

Recently, Gu and Kang gave in [4] a complete characterization when $S^u_{\varphi,\psi}$ is a self-adjoint, isometric, coisometric, and normal operator using their important key observation where $S^u_{\varphi,\psi}$ and M_z are almost commuting. As shown in [4, lemma 3.1], Gu and Kang proved that the operator $S^u_{\varphi,\psi}$ satisfies the following equation:

$$S^u_{\varphi,\psi} - M_z S^u_{\varphi,\psi} M_z^* = (\varphi - \psi) \otimes e_0 - (\varphi - \psi)u \otimes ue_0. \tag{1.7}$$

In this work, we study the product of two dilations of truncated Toeplitz operators $S^u_{\varphi_1,\psi_1}$ and $S^u_{\varphi_2,\psi_2}$.

2. Characterization

Let $B(L^2)$ be the algebra of all bounded linear operators on L^2. For an operator $A \in B(L^2)$, the operator A^* is called the adjoint of A. For an inner function $u \in H^2$, D_u denotes the set of all dilations of truncated Toeplitz operators on L^2:

$$D_u = \{S^u_{\varphi,\psi} \in B(L^2), \varphi, \psi, \in L^\infty\}.$$

In [4] Gu and Kang gave a full characterization of the class of operators D_u as described in the following lemma.

Lemma 2.1 *[4] Let $A \in B(L^2)$. Then $A \in D_u$ if and only if there exists a $\chi \in L^\infty$ such that*

$$A - M_z A M_z^* = \chi \otimes e_0 - \chi u \otimes u e_0. \tag{2.1}$$

In this case, $A = S^u_{\chi+\theta,\theta}$ for some $\theta \in L^\infty$.

Remark 2.2 *[4] Let φ, ψ be in L^∞. Then for all $f, g \in L^2$ we have*

$$\langle S^u_{\varphi,\psi} f, g\rangle \quad = \quad \langle \varphi P_u(f) + \psi Q_u(f), g\rangle = \langle f, P_u(\overline{\varphi} g)\rangle + \langle f, Q_u(\overline{\psi} g)\rangle.$$

Therefore,

$$(S^u_{\varphi,\psi})^* f = P_u(\overline{\varphi} f) + Q_u(\overline{\psi} f), f \in L^2.$$

Proposition 2.3 *Let $\varphi \in L^\infty$ and let $S^u_{1,0}, S^u_{\varphi,0} \in D_u$. Then*

$$(S^u_{1,0} S^u_{\varphi,0})^* = S^u_{1,0} S^u_{\overline{\varphi},0}.$$

Proof Since $S^u_{\varphi,0} = M_\varphi S^u_{1,0}$ and $(S^u_{1,0})^* = S^u_{1,0}$, we obtain

$$(S^u_{1,0} S^u_{\varphi,0})^* \quad = \quad (S^u_{1,0} M_\varphi S^u_{1,0})^* = (S^u_{1,0})^* M_\varphi^* (S^u_{1,0})^* = S^u_{1,0} M_{\overline{\varphi}} S^u_{1,0} = S^u_{1,0} S^u_{\overline{\varphi},0}.$$

□

3. Product of dilation of truncated Toeplitz operators

To arrive at the main result of this work, we need the following lemma and proposition.

Lemma 3.1 *Letting $\varphi \in L^\infty$, the following statements hold:*

1. $A^u_\varphi = 0$ *if and only if* $\varphi \in uH^\infty + \overline{uH^\infty}$.
2. $\widetilde{A^u_\varphi} = 0$ *if and only if* $\varphi = 0$.
3. $\Gamma^u_\varphi = 0$ *if and only if* $\varphi \in K^\infty_u$.
4. $\widetilde{\Gamma^u_\varphi} = 0$ *if and only if* $\varphi \in \overline{K^\infty_u}$.

Proof

1. This statement is an important result in Sarason's paper; see [7, Theorem 3.1].

2. Since $\varphi \in L^\infty$, it follows from Property 2.1 in [2] that $\widetilde{A^u_\varphi}$ is a bounded operator and $\|\widetilde{A^u_\varphi}\| = \|\varphi\|_\infty$. Then $\widetilde{A^u_\varphi} = 0$ if and only if $\varphi = 0$.

According to the proof of Theorem 3.14 in [5, p. 15] and equation (1.5), we deduce statements 3) and 4).

□

Proposition 3.2 *Let u be an inner function, $\varphi_1, \psi_1, \varphi_2, \psi_2 \in L^\infty$. Let $S^u_{\varphi_1,\psi_1}$, $S^u_{\varphi_2,\psi_2} \in D_u$, and then the following statements hold:*

1. *$S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} \in D_u$ if and only if $M_{\varphi_1-\psi_1} S^u_{1,0} S^u_{\varphi_2,\psi_2} \in D_u$.*

2. *If $\varphi_1 - \psi_1$ is invertible in L^∞ then $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} \in D_u$ if and only if $S^u_{1,0} S^u_{\varphi_2,\psi_2} \in D_u$.*

Proof

1. It is clear that

$$S^u_{\varphi_1,\psi_1} = M_{\psi_1} + S^u_{\varphi_1-\psi_1,0} = M_{\psi_1} + M_{\varphi_1-\psi_1} S^u_{1,0}.$$

Therefore,

$$S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = \left(M_{\psi_1} + M_{\varphi_1-\psi_1} S^u_{1,0}\right) S^u_{\varphi_2,\psi_2} = S^u_{\varphi_2\psi_1,\psi_2\psi_1} + M_{\varphi_1-\psi_1} S^u_{1,0} S^u_{\varphi_2,\psi_2}.$$

We deduce that $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} \in D_u$ if and only if $M_{\varphi_1-\psi_1} S^u_{1,0} S^u_{\varphi_2,\psi_2} \in D_u$.

2. From the above, we obtain that

$$M_{\varphi_1-\psi_1} S^u_{1,0} S^u_{\varphi_2,\psi_2} = S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} - S^u_{\varphi_2\psi_1,\psi_2\psi_1}.$$

If $\varphi_1 - \psi_1$ is invertible, then

$$S^u_{1,0} S^u_{\varphi_2,\psi_2} = M_{(\varphi_1-\psi_1)^{-1}} (S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} - S^u_{\varphi_2\psi_1,\psi_2\psi_1}).$$

Thus, we conclude that $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} \in D_u$ if and only if $S^u_{1,0} S^u_{\varphi_2,\psi_2} \in D_u$.

□

The main result of this paper is the following theorem.

Theorem 3.3 *Let $\varphi, \psi \in L^\infty$ and let u be a nonconstant inner function. Then $S^u_{1,0} S^u_{\varphi,\psi} \in D_u$ if and only if $\varphi \in K^\infty_u + uH^\infty + \overline{uH^\infty}$, $\psi \in \overline{K^\infty_u}$. In this case,*

$$S^u_{1,0} S^u_{\varphi,\psi} = S^u_{P_u\varphi,0}.$$

Proof By the representation (1.6), we have

$$S^u_{\varphi,\psi} = \begin{pmatrix} A^u_\varphi & \widetilde{\Gamma^u_\psi} \\ \Gamma^u_\varphi & \widetilde{A^u_\psi} \end{pmatrix}$$

and

$$S^u_{1,0} = \begin{pmatrix} A^u_1 & \widetilde{\Gamma^u_0} \\ \Gamma^u_1 & \widetilde{A^u_0} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that

$$S^u_{1,0} S^u_{\varphi,\psi} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^u_\varphi & \widetilde{\Gamma^u_\psi} \\ \Gamma^u_\varphi & \widetilde{A^u_\psi} \end{pmatrix} = \begin{pmatrix} A^u_\varphi & \widetilde{\Gamma^u_\psi} \\ 0 & 0 \end{pmatrix}.$$

For each $\Phi, \Psi \in L^\infty$, we put

$$S^u_{1,0} S^u_{\varphi,\psi} = S^u_{\Phi,\Psi} = \begin{pmatrix} A^u_\Phi & \widetilde{\Gamma^u_\Psi} \\ \Gamma^u_\Phi & \widetilde{A^u_\Psi} \end{pmatrix}.$$

Then

$$\begin{pmatrix} A^u_{\Phi-\varphi} & \widetilde{\Gamma^u_{\Psi-\psi}} \\ \Gamma^u_\Phi & \widetilde{A^u_\Psi} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$A^u_{\Phi-\varphi} = 0, \widetilde{A^u_\Psi} = 0, \Gamma^u_\Phi = 0, \widetilde{\Gamma^u_{\Psi-\psi}} = 0.$$

Since $A^u_{\Phi-\varphi} = 0$ and $\tilde{A}^u_\Psi = 0$, it follows from Lemma 3.1 that $\Phi - \varphi \in uH^\infty + \overline{uH^\infty}$ and $\Psi = 0$. In the same way, since $\Gamma^u_\Phi = 0$ and $\widetilde{\Gamma^u_{\Psi-\psi}} = 0$ and seeing that

$$0 = \widetilde{\Gamma^u_{\Psi-\psi}} = (\Gamma^u_{\overline{\Psi-\psi}})^*$$

is equivalent to $\Gamma^u_{\overline{\Psi-\psi}} = 0$, it results from Lemma 3.1 that $\Phi \in K^\infty_u$ and $\overline{\Psi - \psi} \in K^\infty_u$. From the above, we conclude that

$$\varphi = \Phi + \varphi_1$$

for $\Phi \in K^\infty_u$ and $\varphi_1 \in uH^\infty + \overline{uH^\infty}$, and

$$\psi \in \overline{K^\infty_u}.$$

At last, we have

$$\varphi \in K^\infty_u + uH^\infty + \overline{uH^\infty}$$

and

$$\psi \in \overline{K_u^\infty}.$$

Observe that $\Phi = P_u\varphi$ and $\Psi = Q_u(\overline{\psi})$. In light of this,

$$S^u_{1,0}S^u_{\varphi,\psi} \quad = \quad S^u_{\Phi,\Psi} = S^u_{P_u\varphi, Q_u(\overline{\psi})} = S^u_{P_u\varphi,0}.$$

This finishes the proof of the theorem. □

Corollary 3.4 *Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in L^\infty$ such that $\varphi_1 - \psi_1$ is invertible in L^∞. Let $S^u_{\varphi_1,\psi_1}, S^u_{\varphi_2,\psi_2} \in D_u$, and then $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} \in D_u$ if and only if $\varphi_2 \in K_u^\infty + uH^\infty + \overline{uH^\infty}$ and $\psi_2 \in \overline{K_u^\infty}$. In this case,*

$$S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = S^u_{S^u_{\varphi_1,\psi_1}\varphi_2,\psi_1\psi_2}$$

.

Proof The result easily follows from Proposition 3.2 and Theorem 3.3, and we also have

$$\begin{aligned} S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} & \\ &= S^u_{\varphi_2\psi_1,\psi_2\psi_1} + M_{\varphi_1-\psi_1}S^u_{1,0}S^u_{\varphi_2,\psi_2} \\ &= S^u_{\varphi_2\psi_1+(\varphi_1-\psi_1)P_u(\varphi_2),\psi_2\psi_1+(\varphi_1-\psi_1)Q_u(\overline{\psi_2})} \\ &= S^u_{\varphi_1 P_u\varphi_2+\psi_1 Q_u\varphi_2,\psi_2\psi_1} \\ &= S^u_{\varphi_1 P_u\varphi_2+\psi_1 Q_u\varphi_2,\psi_2\psi_1}. \end{aligned}$$

□

Remark 3.5 *1) If $S^u_{\varphi_1,\psi_1}$ is a multiplication operator $S^u_{\varphi_1,\psi_1} = M_{\varphi_1}$, then $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} \in D_u$ for all $S^u_{\varphi_2,\psi_2}$ and $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = S^u_{\varphi_1\varphi_2,\varphi_1\psi_2}$.*

2) Let $\varphi_1, \psi_1 \in L^\infty$ such that $\varphi_1 - \psi_1$ is invertible in L^∞. If $S^u_{\varphi_1,\psi_1}$ is not a multiplication operator and $S^u_{\varphi_2,\psi_2} = M_{\varphi_2}$, and if $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} \in D_u$, then, by Theorem 3.3, we have the following two cases:

(a) If $u(0) = 0$, then $\lambda \in K_u^\infty \cap \overline{K_u^\infty}$ for some complex number λ. Therefore, $\varphi_2 = \lambda$ and $S^u_{\varphi_1,\psi_1}M_{\varphi_2} = S^u_{\lambda\varphi_1,\lambda\psi_1}$.

(b) If $u(0) \neq 0$, then $\lambda \notin K_u^\infty$ and $\lambda \notin \overline{K_u^\infty}$ for some complex number λ. Therefore, $\varphi_2 = 0$.

To study particular cases of the product of dilation of truncated Toeplitz operators, we need to construct the subsets K_1 and K_2 described below:

$$K_1 = \{S^u_{\varphi,\psi} \in D_u, \varphi \in K_u^\infty, \psi \in \overline{K_u^\infty}\}$$

$$K_2 = \{S^u_{\varphi,\psi} \in D_u, \varphi \in uH^\infty + \overline{uH^\infty}, \psi \in \overline{K_u^\infty}\}.$$

Proposition 3.6 *Let $\varphi_1, \psi_1 \in L^\infty$ such that $\varphi_1 - \psi_1$ is invertible in L^∞. For $S^u_{\varphi_1,\psi_1} \in D_u$, we have the following cases:*

(a) If $S^u_{\varphi_2,\psi_2} \in K_1$ then

$$S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = S^u_{\varphi_1\varphi_2,\psi_1\psi_2}.$$

(b) If $S^u_{\varphi_2,\psi_2} \in K_2$ then

$$S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = S^u_{\psi_1\varphi_2,\psi_1\psi_2}.$$

Proof

(a) If $\varphi_2 \in K^\infty_u$ and $\psi_2 \in \overline{K^\infty_u}$, then by theorem 3.3 we have

$$S^u_{1,0} S^u_{\varphi_2,\psi_2} \quad = \quad S^u_{P_u\varphi_2,0} = S^u_{\varphi_2,0}.$$

Therefore,

$$S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} \quad = \quad S_{\psi_1\varphi_2+(\varphi_1-\psi_1)\varphi_2,\psi_1\psi_2} = S^u_{\varphi_1\varphi_2,\psi_1\psi_2}.$$

□

We are now able to give a sufficient condition under which the operator $S^u_{\varphi,\psi} \in D_u$ becomes invertible and whose inverse is also in D_u.

In all the following results we will assume that $\varphi_1 - \psi_1$ is invertible in L^∞.

Corollary 3.7 *Assume that $S^u_{\varphi,\psi}$ is not a multiplication operator. If $S^u_{\varphi,\psi} \in K_1$ and $\varphi, \overline{\psi}$ are invertible in K^∞_u, then $S^u_{\varphi,\psi}$ is invertible operator. In this case,*

$$(S^u_{\varphi,\psi})^{-1} = S^u_{\varphi^{-1},\psi^{-1}}.$$

Proof Let $S^u_{\varphi_1,\psi_1} \in D_u$ be the inverse of $S^u_{\varphi,\psi}$. Then $S^u_{\varphi_1,\psi_1} S^u_{\varphi,\psi} = S^u_{1,1}$. Supposing that $\varphi, \overline{\psi} \in K^\infty_u$ are invertible functions, then by Proposition 3.6 we have

$$S^u_{\varphi_1,\psi_1} S^u_{\varphi,\psi} = S^u_{\varphi_1\varphi,\psi_1\psi} = S^u_{1,1}.$$

Therefore, $\varphi_1 = \varphi^{-1}$ and $\psi_1 = \psi^{-1}$.

□

According to Proposition 3.6, we get the following results.

Corollary 3.8 *Assuming that $S^u_{\varphi_1,\psi_1} \in D_u$ is not a multiplication operator, we have the following two cases:*

1) If $S^u_{\varphi_2,\psi_2} \in K_1$ then the operator $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2}$ is a multiplication operator if and only if $\varphi_1\varphi_2 = \psi_1\psi_2$. In this case,

$$S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = M_{\varphi_1\varphi_2} = M_{\psi_1\psi_2}.$$

2) If $S^u_{\varphi_2,\psi_2} \in K_2$ *then the operator* $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2}$ *is a multiplication operator if and only if* $\psi_1\varphi_2 = \psi_1\psi_2$. *In this case,*

$$S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = M_{\psi_1\varphi_2} = M_{\psi_1\psi_2}.$$

The next corollary tells us when $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = 0$.

Corollary 3.9 *Assuming that* $S^u_{\varphi_1,\psi_1} \in D_u$ *is not a multiplication operator, we have the following:*

1) If $S^u_{\varphi_2,\psi_2} \in K_1$ *and* $S^u_{\varphi_2,\psi_2} \neq 0$ *then*

$$S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = 0$$

if and only if one of the following two assertions holds:

(a) $\varphi_1 \neq 0, \psi_1 = 0, \varphi_2 = 0, \psi_2 \in \overline{K^\infty_u}$,

(b) $\psi_1 \neq 0, \varphi_1 = 0, \psi_2 = 0, \varphi_2 \in K^\infty_u$.

2) If $S^u_{\varphi_2,\psi_2} \in K_2$ *and* $S^u_{\varphi_2,\psi_2} \neq 0$ *then*

$$S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = 0$$

if and only if one of the following two assertions holds

(a) $\psi_1 = 0, \varphi_2 \neq 0, \psi_2 \neq 0$,

(b) $\psi_1 \neq 0, \varphi_2 = 0, \psi_2 = 0$.

Proof

1) Since $S^u_{\varphi_2,\psi_2} \in K_1$, it follows from Proposition 3.6 that $\varphi_2 \in K^\infty_u$ and $\psi_2 \in \overline{K^\infty_u}$ and the equation $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = 0$ is equivalent to $\varphi_1\varphi_2 = \psi_1\psi_2 = 0$.

2) Again using Proposition 3.6, we obtain that the equation $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = 0$ is equivalent to $\psi_1\varphi_2 = \psi_1\psi_2 = 0$.

□

The following corollary shows when $S^u_{\varphi_1,\psi_1}$ commutes with $S^u_{\varphi_2,\psi_2}$.

Corollary 3.10 *The following statements hold:*

1) Let $S^u_{\varphi_1,\psi_1}, S^u_{\varphi_2,\psi_2} \in K_1$. *Then* $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = S^u_{\varphi_2,\psi_2}S^u_{\varphi_1,\psi_1}$.

2) Let $S^u_{\varphi_1,\psi_1}, S^u_{\varphi_2,\psi_2} \in K_2$. *Then* $S^u_{\varphi_1,\psi_1}S^u_{\varphi_2,\psi_2} = S^u_{\varphi_2,\psi_2}S^u_{\varphi_1,\psi_1}$ *if and only if* $\psi_1\varphi_2 = \varphi_1\psi_2$.

References

[1] Brown A, Halmos P. Algebraic properties of Toeplitz operators. Journal für die Reine und Angewandte Mathematik 1963; 213: 89-102.

[2] Ding X, Sang Y. Dual truncated Toeplitz operators. Journal of Mathematical Analysis and Applications 2018; 461 (1): 929-946. doi: 10.1016/j.jmaa.2017.12.032

[3] Gu C. Algebraic properties of Cauchy singular integral operators on the unit circle. Taiwanese Journal of Mathematics 2016; 20 (1): 161–189. doi: 10.11650/tjm.19.2015.6188

[4] Gu C, Kang DO. A commutator approach to truncated singular integral operators. Integral Equations and Operator Theory 2018; 90: 16. doi: 10.1007/s00020-018-2429-7

[5] Ko E, Lee JE. On the dilation of truncated Toeplitz operators. Complex Analysis and Operator Theory 2016; 10 (4): 815–833.

[6] Nakazi T, Yamamoto T. Normal singular integral operators with Cauchy kernel on L2. Integral Equations and Operator Theory 2014; 78 (2): 233–248.

[7] Sarason D. Algebraic properties of truncated Toeplitz operators. Operators and Matrices 2007; 1 (4): 491–526.

[8] Sedlock NA. Algebras of truncated Toeplitz operators. Operators and Matrices 2011; 5 (2): 309–326.