Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 152 – 168
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doi:10.3906/mat-1902-57

Sensitivity analysis in parametric vector optimization in Banach spaces via τ^w-contingent derivatives

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Received: 13.02.2019 • **Accepted/Published Online:** 14.11.2019 • **Final Version:** 20.01.2020

Abstract: This paper is concerned with sensitivity analysis in parametric vector optimization problems via τ^w-contingent derivatives. Firstly, relationships between the τ^w-contingent derivative of the Borwein proper perturbation map and the τ^w-contingent derivative of feasible map in objective space are considered. Then, the formulas for estimating the τ^w-contingent derivative of the Borwein proper perturbation map via the τ^w-contingent of the constraint map and the Hadamard derivative of the objective map are obtained.

Key words: Parametric vector optimization problem, τ^w-contingent derivative, Borwein perturbation map, Borwein efficient solution map, sensitivity analysis

1. Introduction

Sensitivity analysis is a quantitative analysis, i.e. the study of derivatives of perturbation maps. Due to its importance not only for theoretical aspect, but also for practical application, sensitivity analysis has been considered by numerous researchers. To deal with the nonsmooth perturbation maps, the generalized derivatives in the primal space and coderivatives in the dual space were utilized in sensitivity analysis. In dual space approach, many interesting results in sensitivity analysis via Mordukhovich coderivatives were obtained; see the books [14, 15] for comprehensive expositions. In primal space approach, one of the first results for sensitivity analysis via contingent derivatives was given by Tanino in [24]. The paper [22] presented TP-derivative and this derivative was put to use to weaken some assumptions in [10]. In [6, 9], the Clarke derivatives were employed for analyzing sensitivity. Properties of the contingent derivatives of some types of proper perturbation maps of a parameterized optimization problem were discussed in [1, 7, 16, 23, 25]. Some results in the proto-differentiability and semidifferentiability of the perturbation maps were obtained in [11, 13, 17, 26].

When the sensitivity analysis was considered in Banach space, the weak/the weak star coderivatives were utilized in [14, 15]. In primal space approach, the τ^w-contingent epiderivative has been introduced and applied to consider the optimality conditions for a set-valued optimization problem in [18]. In [8], the weak subdifferentials were presented and applied to obtain the optimality conditions for nonconvex optimization problems in reflexive Banach spaces. However, to the best of our knowledge, the sensitivity analysis terms of the τ^w-contingent derivatives was not considered. Motivated by the above notices, we aim to have a consideration

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2010 *AMS Mathematics Subject Classification:* 49Q12, 90C26, 90C29, 90C30 of the τ^w-contingent derivatives of the Borwein perturbation map in this paper. The paper is organized as follows. In Section 2, we recollect some important notions and present some auxiliary results, which will be useful hereafter. Then, the relations between the τ^w-contingent derivative of the Borwein proper perturbation map and the τ^w-contingent derivative of feasible map in objective space are derived in Section 3. In Section 4, we investigate the formulas for computing the τ^w-contingent derivative of the Borwein proper perturbation map via the τ^w-contingent of the constraint map and the Hadamard derivative of the objective map.

2. Preliminaries

Throughout this paper, let P, X, and Y be Banach spaces, where the space Y is partially ordered by a pointed, closed, and convex cone with apex at the origin K. The closed ball centered at origin of radius $\lambda > 0$ is denoted by $B(0,\lambda)$. The Cartesian product of Banach spaces of P and Y, denoted by $P\times Y$, is a Banach space with the norm $\|(p,y)\| = \|p\|_P + \|y\|_Y$. For $A \subseteq X$; $\text{int}A$, $\text{cl}A$, ∂A, and $\text{cone}\,A$ denote its interior, closure, boundary, and the cone $\{\lambda a \mid \lambda \geq 0,\ a \in A\}$, respectively. A set $B \subset Y$ is called a base for K if $0 \notin \text{cl}B$ and $K = \{\lambda b : \lambda > 0, b \in B\}$. If B is compact we say that K has a compact base B. The cone K has a compact base if and only if $K \cap \partial B$ is compact (see in [22]). The set of all neighborhoods of $y \in Y$ is represented by $\mathcal{N}(y)$. For the set-valued map $G : P \rightrightarrows Y$, the domain, graph, and epigraph of G are respectively defined by:

$$\begin{array}{lll} \text{dom}G & := & \{p \in P \mid G(p) \neq \emptyset\}, \\ \text{gr}G & := & \{(p,y) \in P \times Y \mid y \in G(p)\}, \\ \text{epi}G & := & \{(p,y) \in P \times Y \mid p \in \text{dom}F, y \in G(p) + K\}. \end{array}$$

The profile map of G is $G+K$, defined by $(G+K)(p) := G(p)+K$. We recall notions of efficiency in set-valued vector optimization, for $\bar{y} \in \Omega \subseteq Y$.

(i) $\bar{y}$ is said to be a local (Pareto) efficient/minimal point of Ω with respect to (shortly wrt) K, and denoted by $\bar{y} \in \text{locMin}_K\Omega$, iff there exists $U \in \mathcal{N}(\bar{y})$ such that

$$(\Omega \cap U - \bar{a}) \cap -K = \{0\}.$$

(ii) $\bar{y}$ is said to be a local Borwein efficient/minimal [5] of Ω wrt K, and denoted by $\bar{y} \in \text{locBoMin}_K\Omega$, iff there exists $U \in \mathcal{N}(\bar{y})$ such that

$$\text{cl cone}(\Omega \cap U - \bar{a}) \cap (-K) = \{0\}.$$

If $U = Y$, the word "local" is dropped. In this case, the minimal point sets and the Borwein minimal point sets of Ω are denoted by $\text{Min}_K\Omega$ and $\text{BoMin}_K\Omega$, respectively. It is easy to check that $\text{BoMin}_K\Omega \subset \text{Min}_K\Omega$ and the inclusion may be strict as in the following example.

Example 2.1 *Let* $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ *and* $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2^2 \leq x_1 \leq 1\}$. *Then, we can check that*

$$\text{Min}_K\Omega = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = x_2^2, 0 \leq x_1 \leq 1, -1 \leq x_2 \leq 0\},$$

$$\text{BoMin}_K\Omega = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = x_2^2, 0 < x_1 \leq 1, -1 \leq x_2 < 0\}.$$

Hence,

$$\text{BoMin}_K\Omega \subsetneqq \text{Min}_K\Omega.$$

In the sequel by $\to / \xrightarrow[w]{} / \xrightarrow[w*]{}$ we denote the convergence with respect to the norm topology/the weak topology/the weak star topology. Given $(p_n, y_n) \in P \times Y$ and $(\bar{p}, \bar{y}) \in P \times Y$, by $(p_n, y_n) \xrightarrow[s,w]{} (\bar{p}, \bar{y})$ ($(p_n, y_n) \xrightarrow[s,w*]{} (\bar{p}, \bar{y})$) we mean $p_n \to \bar{p}, y_n \xrightarrow[w]{} \bar{y}$ ($p_n \to \bar{p}, y_n \xrightarrow[w*]{} \bar{y}$, resp).

Definition 2.2 *Let $G : P \rightrightarrows Y$ and $(\bar{p}, \bar{y}) \in grG$.*

(i) *[2] The* contingent derivative *of G at $(\bar{p}, \bar{y})$ is the set-valued map $DG(\bar{p}, \bar{y}) : P \rightrightarrows Y$ defined by*

$$DG(\bar{p}, \bar{y})(p) := \{y \in Y \mid \exists t_n > 0, \exists (p_n, y_n) \in grG : (p_n, y_n) \to (\bar{p}, \bar{y}), t_n(p_n - \bar{p}, y_n - \bar{y}) \to (p, y)\}.$$

(ii) *[18] The τ^w-*contingent derivative *of G at $(\bar{p}, \bar{y})$ is the set-valued map $D^w G(\bar{p}, \bar{y}) : P \rightrightarrows Y$ defined by*

$$\begin{aligned} D^w G(\bar{p}, \bar{y})(p) :=& \{y \in Y \mid \exists t_n > 0, \exists (p_n, y_n) \in grG \\ & \text{such that } (p_n, y_n) \xrightarrow[s,w]{} (\bar{p}, \bar{y}), t_n(p_n - \bar{p}, y_n - \bar{y}) \xrightarrow[s,w]{} (p, y)\}. \end{aligned}$$

(iii) *The $\tau^{w*}-$*contingent derivative *of G at $(\bar{p}, \bar{y})$ is the set-valued map $D^w G(\bar{p}, \bar{y}) : P \rightrightarrows Y$ defined by*

$$\begin{aligned} D^{w*} G(\bar{p}, \bar{y})(p) :=& \{y \in Y \mid \exists t_n > 0, \exists (p_n, y_n) \in grG \\ & \text{such that } (p_n, y_n) \xrightarrow[s,w*]{} (\bar{p}, \bar{y}), t_n(p_n - \bar{p}, y_n - \bar{y}) \xrightarrow[s,w*]{} (p, y)\}. \end{aligned}$$

Remark 2.3 *It is easy to see that*

(i) $DG(\bar{p}, \bar{y}))(p) \subset D^w G(\bar{p}, \bar{y}))(p) \subset D^{w*} G(\bar{p}, \bar{y}))(p), \forall p \in P.$

(ii)

$$\begin{aligned} D^w G(\bar{p}, \bar{y})(p) =& \{(y \in Y \mid \exists t_n \downarrow 0, \exists (p_n, y_n) \in grG \\ & \text{such that } (p_n, y_n) \xrightarrow[s,w]{} (p, y) \text{ with } \bar{y} + t_n y_n \in G(\bar{p} + t_n p_n), \forall n \in \mathbb{N}\}. \end{aligned}$$

Definition 2.4 *The* lower τ^w-contingent derivative *of a set-valued map $G : P \rightrightarrows Y$ at $(\bar{p}, \bar{y})$ is the set-valued map $\underline{D}^w G(\bar{p}, \bar{y}) : P \rightrightarrows Y$ such that*

$$\begin{aligned} \underline{D}^w G(\bar{p}, \bar{y})(p) :=& \{y \in Y \mid \forall t_n > 0, \exists \{(p_n, y_n)\}_n \subset grG \\ & \text{such that } (p_n, y_n) \xrightarrow[s,w]{} (\bar{p}, \bar{y}), t_n(p_n - \bar{p}, y_n - \bar{y}) \xrightarrow[s,w]{} (p, y), \forall n \in \mathbb{N}\}. \end{aligned}$$

If $D^w G(\bar{p}, \bar{y})(p)) = \underline{D}^w G(\bar{p}, \bar{y})(p))$ for any $p \in \mathrm{dom}\underline{D}^w G(\bar{p}, \bar{y})$, then G is said to have a weak contingent proto-derivative at $(\bar{p}, \bar{y})$.

Definition 2.5 *Let $(\bar{p}, \bar{y}) \in grG$.*

(i) *The* weak radial-contingent cone *of G at $(\bar{p}, \bar{y})$, denoted by $T^w_S(grG; (\bar{p}, \bar{y}))$, is defined by*

$$\begin{aligned} T^w_S(grG; (\bar{p}, \bar{y})) :=& \{(p, y) \in P \times Y \mid \exists t_n > 0, \exists (p_n, y_n) \in grG \\ & \text{such that } (p_n, y_n) \xrightarrow[s,w]{} (\bar{p}, \bar{y}), \text{ with } \bar{y} + t_n y_n \in G(\bar{p} + t_n p_n), \forall n \in \mathbb{N}, t_n p_n \to 0\}. \end{aligned}$$

(ii) *The* τ^w-TP-derivative *of a set-valued map* $G : P \rightrightarrows Y$ *at* $(\bar{p},\bar{y})$ *is the set-valued map* $D^w_S G(\bar{p},\bar{y}) : P \rightrightarrows Y$ *such that*

$$gph\,(D^w_S G(\bar{p},\bar{y})) = T^w_S(grG;(\bar{p},\bar{y})).$$

Definition 2.6 *[21]*

(i) *The set* $\Omega \subset Y$ *is said to have the* domination *property if*

$$\Omega \subset Min_K \Omega + K.$$

(ii) *We say the* domination property *satisfies for* $G : P \rightrightarrows Y$ *around* $\bar{p} \in P$ *if there exists a neighborhood* $U \in \mathcal{N}(\bar{p})$ *such that*

$$G(p) \subset Min_K G(p) + K, \forall p \in U.$$

Based on the notion of *directional compact* [3] of a set-valued map at a point of its graph, we propose the notion of *weak directional compact* as follows.

Definition 2.7 G *is called* weak/weak* directional compact *at* $(\bar{p},\bar{y}) \in grG$ *in the direction* $p \in P$ *if for every sequence* $\{t_n\}_n \subset (0,+\infty), t_n \to 0$ *and for any sequence* $\{p_n\}_n \subset P, p_n \to p \in P$*, any sequence* $\{y_n\}_n \subset Y$ *with* $\bar{y}+t_n y_n \in G(\bar{p}+t_n p_n)$ *for each* n *includes a* weak/weak* convergent subsequence*. If* G *is* weak/weak* directional compact *at* $(\bar{p},\bar{y})$ *for every* $p \in P$*, then* G *is said to be weak/weak* directional compact at* $(\bar{p},\bar{y})$*.*

Example 2.8 *Let* $X = \mathbb{R}_+$ *and* $Y = l^2$ *be the space of all scalar sequences* $x = \{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ *with* $\sum\limits_{i=1}^{\infty} |x_i|^2 < +\infty$*. By* $\{e_i\}_{i\in\mathbb{N}} \subset l^2$ *we indicate its standard unit basis. We note the ordering cone on* l^2 *as follows*

$$K = \left\{y = \{y_i\}_{i\in\mathbb{N}} \in l^2 : y_i \geq 0 \text{ for every } i \in \mathbb{N}\right\}.$$

K *is a closed, convex, and pointed cone with* $intK = \emptyset$*. Let the set-valued map* $G : X \rightrightarrows 2^Y$ *be defined by*

$$G(x) = \begin{cases} \{-2xe_n\}, & \text{if } x = \dfrac{1}{n}, \\ \{x^2(e_1+e_2)\}, & \text{elsewhere in } \mathbb{R}_+, \end{cases}$$

and $(\bar{p},\bar{x}) = (0,0) \in \mathrm{gr}G$*. Then, we can check that* G *is weak directional compact at* $(\bar{p},\bar{y})$*. Let* $u_n = u = 1$*,* $t_n = \frac{1}{n}$*. Then, for sequence* v_n *with* $\bar{y}+t_n v_n \in G(\bar{x}+t_n u_n)$*, one has*

$$0 + \frac{1}{n}v_n \in G(0+\frac{1}{n}.1) = -2\frac{1}{n}e_n,$$

i.e. $v_n = -2e_n$ *and* v_n *has no convergent subsequence. Hence,* G *is not directional compact at* $(\bar{p},\bar{y})$*.*

Example 2.9 *Let* $X = \mathbb{R}_+$ *and* $Y = l^1$ *be the space of all scalar sequences* $x = \{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ *with* $\sum\limits_{i=1}^{\infty} |x_i| < +\infty$*. We designate by* $\{e_i\}_{i\in\mathbb{N}} \subset l^1$ *its standard unit basis. The ordering cone on* l^1 *is considered as follows*

$$K = \left\{y = \{y_i\}_{i\in\mathbb{N}} \in l^1 : y_i \geq 0 \text{ for every } i \in \mathbb{N}\right\}.$$

K is a closed, convex, and pointed cone with $intK = \emptyset$. The set-valued map $G : X \rightrightarrows 2^Y$ is given by

$$G(x) = \begin{cases} \{3xe_n\}, & \text{if } x = \dfrac{1}{n}, \\ \{|x|e_1\}, & \text{elsewhere in } \mathbb{R}_+, \end{cases}$$

and $(\bar{p}, \bar{x}) = (0, 0) \in \mathrm{gr}G$. Then, we can check that G is weak* directional compact at $(\bar{p}, \bar{y})$. Let $u_n = u = 1$, $t_n = \frac{1}{n}$. Then, for sequence v_n with $\bar{y} + t_n v_n \in G(\bar{x} + t_n u_n)$, one has

$$0 + \frac{1}{n} v_n \in G(0 + \frac{1}{n}.1) = 3\frac{1}{n} e_n,$$

i.e. $v_n = 3e_n$ and v_n has no weak convergent subsequence. Hence, G is not weak directional compact at $(\bar{p}, \bar{y})$.

In the line of [12], we propose the following notion.

Definition 2.10 *A set-valued map $G : X \rightrightarrows Y$ is said to be weak lower semidifferentiable at $(\bar{p}, \bar{y}) \in \mathrm{gr}G$ in the direction $p \in P$ iff for any sequence $h_n > 0$ and any sequence $x_n \to \bar{p}$ with $h_n(x_n - \bar{p}) \to p$, there exists a sequence $v_n \in F(x_n)$ in order that $h_n(v_n - \bar{y})$ has a weak convergence subsequence. If G is weak lower semidifferentiable at $(\bar{p}, \bar{y})$ for every $p \in P$, then G is said to be weak lower semidifferentiable at $(\bar{p}, \bar{y})$.*

Definition 2.11 *A set-valued map $G : X \rightrightarrows Y$ is said to be stable [19] (or local Lipschitz calm) at $(\bar{p}, \bar{y}) \in \mathrm{gr}G$ if there exist a real constant $M > 0$ and a neighborhood U of $\bar{p}$ such that*

$$G(p) \subset \{\bar{y}\} + M\|p - \bar{p}\|B(0, 1), \forall p \in U \setminus \{\bar{p}\}.$$

Lemma 2.12 *[19] Let $G(\bar{p}) = \{\bar{y}\}$ and let G be stable at $(\bar{p}, \bar{y})$. Then,*

$$D^w G(\bar{p}, \bar{y})(p) + K = D^w (G + K)(\bar{p}, \bar{y})(p), \forall p \in P.$$

Lemma 2.13 *Let $G : P \rightrightarrows Y, (\bar{p}, \bar{y}) \in \mathrm{gr}G$ and $T^w(epi(G), (\bar{p}, \bar{y})) = T(epi(G), (\bar{p}, \bar{y}))$. Then,*

(i) *[20] If G is directional compact at $(\bar{p}, \bar{y})$, then $DG(\bar{p}, \bar{y}) = D^w G(\bar{p}, \bar{y})$.*

(ii) *If G is weak directional compact at $(\bar{p}, \bar{y})$, then $D^w G(\bar{p}, \bar{y}) = D^{w*} G(\bar{p}, \bar{y})$.*

Lemma 2.14 *Let $G : X \rightrightarrows Y$, $(\bar{p}, \bar{y}) \in \mathrm{gr}G$.*

(i) *If Y is a reflexive Banach space and G is stable at $(\bar{p}, \bar{y})$ then $D^w_S G(\bar{p}, \bar{y})(0) = \{0\}$.*

(ii) *If G is weak lower semidifferentiable at $(\bar{p}, \bar{y})$ then G is weak directional compact at $(\bar{p}, \bar{y})$.*

Proof (i) Consider an arbitrary $y \in D^w_S G(\bar{p}, \bar{y})(0)$. Then, there exist $y_n \underset{w}{\to} y$, $x_n \to 0$ and $t_n > 0$ in order that $\bar{y} + t_n y_n \in G(\bar{p} + t_n x_n)$ and $t_n x_n \to 0$. Since G is stable at $(\bar{p}, \bar{y})$, we imply that for n large enough, there exists $M > 0$ satisfying

$$\bar{y} + t_n y_n \in \bar{y} + M\|t_n x_n\|B(0, 1).$$

Consequently,

$$y_n \in M\|x_n\|B(0, 1).$$

Taking the above equation into account, $x_n \to 0$ and $y_n \underset{w}{\longrightarrow} y$, one infers that $y = 0$.

(ii) Let $p \in P$, $t_n \downarrow 0$, $p_n \to p \in P$, and $\{y_n\}_n$ be arbitrary sequence in Y satisfying $\bar{y} + t_n y_n \in G(\bar{p} + t_n p_n)$ for all n. Setting $h_n := \frac{1}{t_n}$, $x_n := \bar{p} + t_n p_n$, $v_n := \bar{y} + t_n y_n$, then $h_n > 0$, $y_n = h_n(v_n - \bar{y})$, $x_n \to \bar{p}$, and $h_n(x_n - \bar{p}) = p_n \to p$. As G is weak lower semidifferentiable at $(\bar{p}, \bar{y})$, $h_n > 0$, $x_n \to \bar{p}$ and $h_n(x_n - \bar{p}) \to p$, one can find a sequence $v_n \in G(x_n)$ such that $y_n = h_n(v_n - \bar{y})$ has a weak convergence subsequence. □

Proposition 2.15 *For all $p \in P$, one has*

$$D^w G(\bar{p}, \bar{y})(p) + K \subseteq D^w(G + K)(\bar{p}, \bar{y})(p). \tag{2.1}$$

Proof Let $z = y + k$ for some $y \in D^w G(\bar{p}, \bar{y})(p)$ and $k \in K$. Then, there exist sequence $t_n \downarrow 0$ and $\{(p_n, y_n)\}_n \subset \mathrm{gr}G$ with $(p_n, y_n) \underset{s,w}{\longrightarrow} (p, y)$ such that $\bar{y} + t_n y_n \in G(\bar{p} + t_n p_n)$ for al n. Setting $y'_n := y_n + k$, one has $y'_n \underset{w}{\longrightarrow} y + k$ and $\bar{y} + t_n y'_n \in (G + K)(\bar{p} + t_n p_n)$. Therefore, $z = y + k \in D^w(G + K)(\bar{p}, \bar{y})(p)$. □

The following example shows that the inverse inclusion of (2.1) does not hold.

Example 2.16 *Let $X = \mathbb{R}_+$ and $Y = l^1$ be the space of all scalar sequences $x = \{x_i\}_{i \in \mathbb{N}} \subset \mathbb{R}$ with $\sum_{i=1}^{\infty} |x_i| < +\infty$. The standard unit basis of l_1 is denoted by $\{e_i\}_{i \in \mathbb{N}}$. The ordering cone of l_1 is*

$$K = \left\{ y = \{y_i\}_{i \in \mathbb{N}} \in l^1 : y_1 \geq 0 \text{ for every } i \in \mathbb{N} \right\}.$$

K is a closed, convex, and pointed cone with $\mathrm{int}K = \emptyset$. We consider the following set-valued map $G : X \rightrightarrows 2^Y$ as

$$G(x) = \begin{cases} \{2xe_n\}, & \text{if } x = \dfrac{1}{n}, \\ \{xe_2 - x^2 e_1\}, & \text{otherwise in } \mathbb{R}_+ \end{cases}.$$

Then, for $(\bar{p}, \bar{y}) = (0, 0)$ and $p = 1$,

$$D^w(G + K)(0, 0)(1) = K \neq D^w G(0, 0)(1) + K = e_2 + K.$$

Proposition 2.17 *Assume that either of the following conditions holds:*

(i) *G has the weak directional compact property at $(\bar{p}, \bar{y})$;*

(ii) *K has a compact base and $D^w_S G(\bar{p}, \bar{y})(0) \cap (-K) = \{0\}$;*

(iii) *K has a compact base and $D^w(G + K)(\bar{p}, \bar{y})(p)$ has domination property.*

Then, for all $p \in P$,

$$D^w G(\bar{p}, \bar{y})(p) + K = D^w(G + K)(\bar{p}, \bar{y})(p), \forall p \in P.$$

Proof By Proposition 2.15, it is sufficient to show the converse inclusion of (2.1).

(i) Now we prove $D^w(G + K)(\bar{p}, \bar{y})(p) \subset D^w G(\bar{p}, \bar{y})(p) + K, \forall p \in P$. Let $y \in D^w(G + K)(\bar{p}, \bar{y})(p)$ be chosen arbitrarily. By definition there exist sequences $t_n \downarrow 0$ and $(p_n, y_n) \in \mathrm{gr}G$ with $(p_n, y_n) \underset{s,w}{\longrightarrow} (p, y)$ such that

$\overline{y}+t_ny_n \in G(\overline{p}+t_np_n)$. This deduces the existence of $\{k_n\}_n \subset K$ in order that $\overline{y}+t_n(y_n-\frac{k_n}{t_n}) \in G(\overline{p}+t_np_n)$. Because G is weak directionally compact at $(\overline{p},\overline{y})$, we ensure that $y_n-\frac{k_n}{t_n}\underset{w}{\longrightarrow}\overline{y}\in Y$. Then, $\frac{k_n}{t_n}\underset{w}{\longrightarrow}\overline{k}=y-\overline{y}\in K$ and $y\in D^wG(\overline{p},\overline{y})(p)+K$.

(ii) Let $p\in P$ and $y\in D^w(G+K)(\overline{p},\overline{y})(p)$ be chosen arbitrarily. According to definition, there are sequences $t_n\downarrow 0$ and $\{(p_n,y_n)\}_n\subset \text{gr}G$ with $(p_n,y_n)\underset{s,w}{\longrightarrow}(p,y)$ and the sequence $k_n\in K$ such that $\overline{y}+t_ny_n\in G(\overline{p}+t_np_n)+k_n$. If there exists n_0 such that $k_n=0$ for all $n>n_0$, then $y\in D^w(G)(\overline{p},\overline{y})(p)\subset D^w(G)(\overline{p},\overline{y})(p)+K$. Now, assume that $k_n\neq 0$. Since K has a compact base, we can denote by $k_n=\alpha_nb_n$ with $\alpha_n>0$ and $b_n\to b\neq 0$. One gets, $\frac{k_n}{||k_n||}=\frac{b_n}{||b_n||}\to b$ with $b\neq 0$. Thus, $\frac{k_n}{||k_n||}\underset{w}{\longrightarrow}b$.

Case 1: $\frac{||k_n||}{t_n}\to+\infty$. We obtain $||k_n||\left(\frac{t_n}{||k_n||}\right)p_n=t_np_n\to 0$. Since

$$\overline{y}+||k_n||\left(\frac{t_n}{||k_n||}y_n-\frac{k_n}{||k_n||}\right)\in G\left(\overline{p}+||k_n||\frac{t_n}{||k_n||}p_n\right),$$

$\frac{t_n}{||k_n||}y_n-\frac{k_n}{||k_n||}\underset{w}{\longrightarrow}-b$ and $\frac{t_n}{||k_n||}p_n\to 0$, one has $-b\in D_S^wG(\overline{p},\overline{y})(0)$, which contradicts with $D_S^wG(\bar{p},\bar{y})(0)\cap(-K)=\{0\}$.

Case 2: $\frac{||k_n||}{t_n}$ is bounded. Since K has a compact base, we can write that $k_n=\alpha_nb_n$ with $\alpha_n>0$ and $b_n\to b\neq 0$. One has, $\frac{||k_n||}{t_n}=\frac{\alpha_n}{t_n}||b_n||\to\frac{\alpha_n}{t_n}||b||$ with $b\neq 0$. Setting $\frac{\alpha_n}{t_n}||b||=\lambda$, we have $\frac{||k_n||}{t_n}\underset{w}{\longrightarrow}\lambda\geq 0$. Then, since

$$\overline{y}+t_n\left(y_n-\frac{||k_n||}{t_n}\frac{k_n}{||k_n||}\right)\in G\left(\overline{p}+t_np_n\right),$$

$y_n-\frac{||k_n||}{t_n}\frac{k_n}{||k_n||}\underset{w}{\longrightarrow}y-\lambda k$ and $p_n\to p$, one gets, $y-\lambda k\in D^wG(\overline{p},\overline{y})(p)$; hence, $y\in D^wG(\overline{p},\overline{y})(p)+K$.

(iii) Since $D^w(G+K)(\bar{p},\bar{y})(p)$ has domination property, for any $p\in P$,

$$D^w(G+K)(\bar{p},\bar{y})(p)\subset \text{Min}_KD^w(G+K)(\bar{p},\bar{y})(p)+K.$$

We will prove that $\text{Min}_KD^w(G+K)(\bar{p},\bar{y})(p)\subset D^wG(\bar{p},\bar{y})(p)$, for all $p\in P$. Indeed, let $y\in\text{Min}_KD^w(G+K)(\bar{p},\bar{y})(p)$. The definition gives us the existence of the sequences $t_n\downarrow 0$ and $\{(p_n,y_n)\}_n\subset\text{gr}G$ with $(p_n,y_n)\underset{s,w}{\longrightarrow}(p,y)$ and $k_n\in K$ such that $\overline{y}+t_n(y_n-k_n)\in G(\overline{p}+t_np_n)$. Since K has a compact base, we conclude that $k_n=\alpha_nb_n$ with $\alpha_n>0$ and $b_n\to b\neq 0$. Then, $b_n\underset{w}{\longrightarrow}b\neq 0$. Now we prove that $\alpha_n\to 0$. Reasoning by contraposition, assume that $\alpha_n\nrightarrow 0$. This provides a positive scalar $\varepsilon>0$ such that $\alpha_n\geq\varepsilon$ for all n. Setting $k_n'=\frac{\varepsilon}{\alpha_n}k_n$. Then, for any n, $k_n-k_n'=\left(1-\frac{\varepsilon}{\alpha_n}\right)k_n\in K$ and

$$\overline{y}+t_n(y_n-k_n')=\overline{y}+t_n(y_n-k_n)+t_n(k_n-k_n')\in G(\overline{p}+t_np_n)+K=(G+K)(\overline{p}+t_np_n).$$

Since $y_n - k'_n = y_n - \dfrac{\varepsilon}{\alpha_n} k_n = y_n - \varepsilon b_n \underset{w}{\rightarrow} y - \varepsilon b$, we have $v - \varepsilon b \in D^w(G+K)(\bar{p},\bar{y})(p)$ and $y - (y - \varepsilon b) = \varepsilon b \in K \backslash \{0\}$, which contradicts $y \in \mathrm{Min}_K D^w(G+K)(\bar{p},\bar{y})(p)$. Therefore, $\alpha_n \to 0$ and $y_n - k_n = v_n - \alpha_n b_n \underset{w}{\rightarrow} v$. Hence, $y \in D^w G(\bar{p},\bar{y})(p)$. Thus, $\mathrm{Min}_K D^w(G+K)(\bar{p},\bar{y})(p) \subset D^w G(\bar{p},\bar{y})(p)$ and $D^w(G+K)(\bar{p},\bar{y})(p) \subset \mathrm{Min}_K D^w(G+K)(\bar{p},\bar{y})(p) + K$. It follows that $D^w(G+K)(\bar{p},\bar{y})(p) \subset D^w G(\bar{p},\bar{y})(p) + K$.

This completes the proof. $\square$

Corollary 2.18 *Let $(\bar{p},\bar{y}) \in grG$ and suppose that G is weak directionally compact at $(\bar{p},\bar{y})$. Then, for any $y \in D^w(G+K)(\bar{p},\bar{y})(p)$, there exists $y' \in D^w G(\bar{p},\bar{y})(p)$, such that $y - y' \in K$.*

Definition 2.19 *Let $\phi : X \to Y$ be a vector-valued map.*

(i) *ϕ is said to be Fréchet differentiable [2] at $\overline{x} \in X$, iff there exists a linear continuous operator $\phi'_F(\overline{x}) : X \to Y$, such that*

$$\phi(x) = \phi(\overline{x}) + \phi'_F(\overline{x})(x - \overline{x}) + o(||x - \overline{x}||),$$

where $o(||x-\overline{x}||)$ satisfies $\dfrac{o(||x-\overline{x}||)}{||x-\overline{x}||} \to 0$ when $x \to \overline{x}$.

(ii) *ϕ is said to be Hadamard differentiable [4] at $\overline{x} \in X$ in a direction $u \in X$ iff there exist a linear continuous operator $\phi'_H(\overline{x}) : X \to Y$, for any sequence $u_n \in X$ with $u_n \to u$ and any sequence $t_n \downarrow 0$:*

$$\phi'_H(\bar{x})(u) = \lim_{u_n \to u, t_n \downarrow 0} \frac{\phi(\bar{x} + t_n u_n) - \phi(\bar{x})}{t_n}.$$

If ϕ is Hadamard differentiable at $\overline{x} \in X$ in any direction $u \in X$, then ϕ is said to be Hadamard differentiable at $\overline{x}$.

Note that if ϕ is be Fréchet differentiable at $\overline{x}$, then ϕ is be Hadamard differentiable at $\overline{x}$ and $\phi'_H(\overline{x})(u) = \phi'_F(\overline{x})(u)$. The following example establishes the statement that the inversion is not true in general.

Example 2.20 *Let $Y = \mathbb{R}$ and $X = l^2 = \{x = \{x_i\}_{i\in\mathbb{N}} \mid \sum\limits_{i=1}^{\infty} |x_i|^2 < +\infty\}$ and with standard unit basis $\{e_i\}_{i\in\mathbb{N}} \subset l^2$. Let $\phi : X \to Y$ be a vector-valued map given by*

$$\phi(x) = \begin{cases} \left(\dfrac{1}{m}\right)^{\frac{1}{m}+1}, & \textit{if } x = \dfrac{e_m}{m}, m = 1, 2, ..., \\ 0, & \textit{otherwise}. \end{cases}$$

Then, ϕ is Hadamard differentiable at $\overline{x} = 0$ and $\phi'_H(\bar{x}) = 0_{l^2}$. However, with sequence $u_n = \dfrac{e_n}{n}, ||u_n||_{l^2} = \dfrac{1}{n} \to 0$, one has

$$\lim_{||u_n||_{l^2} \to 0} \frac{|\phi(u_n) - \phi(0) - 0_{l^2}(u_n)|}{||u_n||_{l^2}} = \lim_{n\to\infty} \frac{\left(\dfrac{1}{n}\right)^{\frac{1}{n}+1}}{\dfrac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1 \neq 0.$$

Hence, ϕ is not Fréchet differentiable at $\overline{x} = 0$.

Now, let $f : P \times X \to Y$ be the objective function, $C : P \rightrightarrows X$ be the feasible decision set-valued map and the feasible set-valued map $F : P \rightrightarrows Y$ be defined by

$$F(p) := f\left(p, C(p)\right) = \left\{f(p,x) : x \in C(p)\right\}. \tag{2.2}$$

In this paper, the following parameterized vector optimization problem is discussed:

$$(\mathrm{PVO}_p) \qquad \mathrm{Min}_K\{f(p,x) : x \in C(p)\} = \mathrm{Min}_K F(p),$$

where x is a decision variable and p is a parameter.

The Borwein perturbation/frontier map $\mathcal{B} : P \rightrightarrows Y$ of a family of parametric vector optimization problem is given by

$$\mathcal{B}(p) := \mathrm{BoMin}_K\{f(p,x) \mid x \in C(p)\} = \mathrm{BoMin}_K F(p), \tag{2.3}$$

and the Borwein efficient solution map $\mathcal{S} : P \rightrightarrows X$ is defined by

$$\mathcal{S}(p) := \{x \in C(p) \mid f(p,x) \in \mathcal{B}(p)\}. \tag{2.4}$$

3. The τ^w-contingent derivative of the Borwein frontier map without constraints

In this part, we derive only the formula for computing the τ^w-contingent derivative of the Borwein perturbation solution map $\mathcal{B}$ via the Borwein efficient point of the τ^w-contingent derivative of F. However, by some suitable changes, most of the results of this part and the next one are still true for τ^{w*}-contingent derivative.

Lemma 3.1 *Suppose that $(\bar{p},\bar{y}) \in \mathrm{gr}F$ and F is weak directionally compact at $(\bar{p},\bar{y})$. Then,*

$$\mathrm{BoMin}_K D^w(F+K)(\bar{p},\bar{y})(p) \subset \mathrm{Min}_K D^w(F+K)(\bar{p},\bar{y})(p) \subset D^wF(\bar{p},\bar{y})(p), \forall p \in P.$$

Proof The first inclusion is from the definition. Suppose that $y \in \mathrm{Min}_K D^w(F+K)(\bar{p},\bar{y})(p)$. Then, $y \in D^w(F+K)(\bar{p},\bar{y})(p)$. According to Corollary 2.18, there exist $y' \in D^wF(\bar{p},\bar{y})(p) \subset D^w(F+K)(\bar{p},\bar{y})(p)$, which satisfies $y - y' = k' \in K$. We will prove that $k' = 0$. Suppose to the contrary that $k' \neq 0$. Then, we derive that $y \notin \mathrm{Min}_K D^w(F+K)(\bar{p},\bar{y})(p)$, a contradiction. Thus, $y = y' \in D^wF(\bar{p},\bar{y})(p)$. □

Lemma 3.2 *Let $(\bar{p},\bar{y}) \in \mathrm{gr}F$. If F has the weak directionally compact property at $(\bar{p},\bar{y})$ then*

$$\mathrm{BoMin}_K D^wF(\bar{p},\bar{y})(p) \subset \mathrm{BoMin}_K D^w(F+K)(\bar{p},\bar{y})(p), \forall p \in P.$$

Proof Let $y \in \mathrm{BoMin}_K D^wF(\bar{p},\bar{y})(p)$. One has, $y \in D^wF(\bar{p},\bar{y})(p) \subset D^w(F+K)(\bar{p},\bar{y})(p)$. Reasoning ad absurdum, assume that $y \notin \mathrm{BoMin}_K D^w(F+K)(\bar{p},\bar{y})(p)$. This arrives at the existence of $\hat{y}_m \in D^w(F+K)(\bar{p},\bar{y})(p)$, $h_m > 0$ such that

$$\lim_{m\to\infty} h_m(\hat{y}_m - y) \in K \setminus \{0\}. \tag{3.1}$$

From Corollary 2.18, there exist $\hat{y}'_m \in D^wF(\bar{p},\bar{y})(p)$, such that $\hat{y}_m - \hat{y}'_m \in K$, for all m. Thus,

$$h_m(y' - \hat{y}'_m) = h_m(y' - \hat{y}_m) + h_m(\hat{y}_m - \hat{y}'_m) \in K + K \setminus \{0\} \subset K \setminus \{0\}.$$

Consequently,

$$\lim_{m\to\infty} h_m(\hat{y}_m - y) \in K,$$

which contradicts (3.1). Thus, $y \in \mathrm{BoMin}_K D^w(F+K)(\bar{p},\bar{y})(p)$. □

Definition 3.3 *We say that* F *is* K*-minicomplete by* $\mathcal{B}$ *around* $\bar{p}$*, iff there exists a neighborhood* U *of* $\bar{p}$ *in order that,* $F(p) \subset \mathcal{B}(p) + K, \forall p \in U$*.*

Proposition 3.4 *Let* $(\bar{p}, \bar{y}) \in \mathrm{gr}\mathcal{B}$*. If* F *is* K*-minicomplete by* $\mathcal{B}$ *around* $\bar{p}$ *and* F *is weak directionally compact at* $(\bar{p}, \bar{y})$*, then,*

$$\mathrm{BoMin}_K D^w F(\bar{p}, \bar{y})(p) \subset D^w \mathcal{B}(\bar{p}, \bar{y})(p), \forall p \in P.$$

Proof Since $\mathcal{B}(p) \subset F(p)$ for any $p \in P$ and the domination property fulfills for F around $\bar{p}$, there is a set $U \in \mathcal{N}(\bar{p})$ such that

$$\mathcal{B}(p) + K = F(p) + K, \forall p \in U.$$

Therefore,

$$D^w(\mathcal{B} + K)(\bar{p}, \bar{y})(p) = D^w(F + K)(\bar{p}, \bar{y})(p), \forall p \in P.$$

It follows from the weak directionally compactness of F at $(\bar{p}, \bar{y})$ that $\mathcal{B}$ is weak directionally compact at $(\bar{p}, \bar{y})$. Hence,

$$\begin{aligned} \mathrm{BoMin}_K D^w F(\bar{p}, \bar{y})(p) &\subset \mathrm{BoMin}_K D^w (F + K)(\bar{p}, \bar{y})(p) \\ &= \mathrm{BoMin}_K D^w (\mathcal{B} + K)(\bar{p}, \bar{y})(p) \\ &\subset \mathrm{Min}_K D^w (\mathcal{B} + K)(\bar{p}, \bar{y})(p) \\ &\subset D^w \mathcal{B}(\bar{p}, \bar{y})(p), \forall p \in P. \end{aligned}$$

Here the first inclusion follows from Lemma 3.2, and the second one is attained from Lemma 3.1. □

Proposition 3.5 *Let* $(\bar{p}, \bar{y}) \in \mathrm{gr}\mathcal{B}$*. Suppose that the following provisos are fulfilled:*

(i) F *has the local Lipschitzness at* $\bar{p}$*;*

(ii) F *has a weak contingent proto-derivative at* $(\bar{p}, \bar{y})$*;*

(iii) F *is* K*-minicomplete by* $\mathcal{B}$ *around* $\bar{p}$*;*

(iv) *there is a set* $U \in \mathcal{U}(\bar{p})$ *in order that for every* $p \in U, \mathcal{B}(p)$ *includes only one element.*

Then,

$$D^w \mathcal{B}(\bar{p}, \bar{y})(p) \subset \mathrm{BoMin}_K D^w F(\bar{p}, \bar{y})(p), \forall p \in P.$$

Proof Let $y \in D^w \mathcal{B}(\bar{p}, \bar{y})(p)$. Then, it amounts to the existence of the sequence $t_n \downarrow 0$ and the sequence $(p_n, y_n) \xrightarrow[s,w]{} (p, y)$ satisfying

$$\bar{y} + t_n y_n \in \mathcal{B}(\bar{p} + t_n p_n) \subset F(\bar{p} + t_n p_n), \forall n.$$

Consequently, $y \in D^w F(\bar{p}, \bar{y})(p)$. Arguing by contradiction, suppose that $y \notin \mathrm{BoMin}_K D^w F(\bar{p}, \bar{y})(p)$. Then, there exist $h_m > 0$, $\hat{y}_m \in D^w F(\bar{p}, \bar{y})(p)$ such that

$$\lim_{m \to \infty} h_m(\hat{y}_m - y) \in -K \setminus \{0\}. \tag{3.2}$$

It follows from (ii) and $\hat{y}_m \in D^w F(\bar{p},\bar{y})(p)$ that, for the preceding sequence t_n, there exists sequence $(\hat{p}_{m_n}, \hat{y}_{m_n}) \xrightarrow[s,w]{} (\bar{p}, \hat{y}_m)$ in order that

$$\bar{y} + t_n \hat{y}_{m_n} \in F(\bar{p} + t_n \hat{p}_n), \forall n. \tag{3.3}$$

Since F is K-dominated by $\mathcal{B}$ near $\bar{p}$, there exists $U_1 \in \mathcal{U}(\bar{p})$ such that, for all $p \in U_1$,

$$F(p) \subseteq \mathcal{B}(u) + K. \tag{3.4}$$

By using the locally Lipschitz of F, one concludes that there exist $U_2 \in \mathcal{U}(\bar{p})$ and $L > 0$ such that, for all $u_1, u_2 \in U_2$ and

$$F(p_1) \subseteq F(p_2) + L\|p_1 - p_2\| B_Y. \tag{3.5}$$

Naturally, since $t_n \downarrow 0$, there exists $N > 0$ such that

$$\bar{p} + t_n \hat{p}_{m_n}, \bar{p} + t_n p_n \in U \cap U_1 \cap U_2, \forall n > N, \forall m. \tag{3.6}$$

Therefore, from (3.3), (3.6), (3.5), and (3.4), there exists $b_n \in B_Y$ in order that, for every n large enough,

$$\bar{y} + t_n(\hat{y}_{m_n} - L\|\hat{p}_{m_n} - p_n\| b_n) \in F(\bar{p} + t_n p_n) \subseteq \mathcal{B}(\bar{p} + t_n p_n) + K, \forall m. \tag{3.7}$$

Thus, it follows from (3.7), and assumption (iv), one gets

$$\bar{y} + t_n(\hat{y}_{m_n} - L\|\hat{p}_{m_n} - p_n\| b_n) - (\bar{y} + t_n y_n) = t_n(\hat{y}_{m_n} - L\|\hat{p}_{m_n} - p_n\| b_n - y_n) \in K, \forall m.$$

Thus, $\hat{y}_{m_n} - L\|\hat{p}_{m_n} - p_n\| b_n - y_n \xrightarrow[w]{} \hat{y}_m - y$ for all m. Since K is a pointed closed convex cone in Banach space Y (locally convex space), K is also weak closed; hence, $\hat{y}_m - y \in K$ for all m. Therefore, we derive from $h_m > 0$ and K is a pointed closed convex cone that

$$\lim_{m \to \infty} h_m(\hat{y}_m - y) \in K,$$

contradicting (3.2). □

4. The τ^w-contingent derivative of the Borwein perturbation map and the Borwein efficient solution map with constraints

Now, we derive the formulas for computing the τ^w-contingent derivative of the Borwein proper frontier map via the Hadamard derivative of the objective function and the τ^w-contingent derivative of the constraint map.

Proposition 4.1 *Let $\bar{p}$ be in P, $\bar{x} \in \mathcal{S}(\bar{p})$ and $\bar{y} = f(\bar{p}, \bar{x})$. If f is Hadamard differentiable at the point $(\bar{p}, \bar{x})$ with its derivative $f'_H(\bar{p}, \bar{x})$ and the weak directionally compactness of C at $(\bar{p}, \bar{x})$ holds, then, one obtains*

$$D^w \mathcal{S}(\bar{p}, \bar{x})(p) = \{x \in X \mid x \in D^w C(\bar{p}, \bar{x})(p) : f'_H(\bar{p}, \bar{x})(p, x) \in D^w \mathcal{B}(\bar{p}, \bar{y})(p)\}, \forall p \in P. \tag{4.1}$$

Proof Firstly, we will justify that

$$\{x \in X \mid x \in D^w C(\bar{p}, \bar{x})(p) : f'_H(\bar{p}, \bar{x})(p, x) \in D^w \mathcal{B}(\bar{p}, \bar{y})(p)\} \subset D^w \mathcal{S}(\bar{p}, \bar{x})(p), \forall p \in P.$$

Let x be in $D^wC(\bar{p},\bar{x})(p)$ such that $y := f'_H(\bar{p},\bar{x})(p,x) \in D^w\mathcal{B}(\bar{p},\bar{y})(p)$. Thus, one yields the existence of the sequence $t_n \downarrow 0$ and the sequence $(p_n,y_n) \subset \mathrm{gr}\mathcal{B}$ in order that $(p_n,y_n) \xrightarrow[s,w]{} (p,y)$ and

$$\bar{y} + t_n y_n \in \mathcal{B}(\bar{p} + t_n p_n). \tag{4.2}$$

This leads the existence of sequence x_n in X such that $x_n \in C(\bar{p} + t_n p_n)$ and $\bar{y} + t_n p_n = f(\bar{p} + t_n p_n, x_n)$. Setting $\hat{x}_n := \dfrac{x_n - \bar{x}}{t_n}$, we get

$$x_n = \bar{x} + t_n \hat{x}_n \in C(\bar{p} + t_n p_n), \tag{4.3}$$

and

$$\bar{y} + t_n y_n = f(\bar{p} + t_n p_n, \bar{x} + t_n \widetilde{x}_n). \tag{4.4}$$

We derive from (4.3) and the weak directionally compactness of C at $(\bar{p},\bar{x})$ that the sequence $\bar{x}_n$ contains a weak convergent subsequence. We can assume $\hat{x}_n \xrightarrow[w]{} \hat{x}$ with no loss of generality. Then, one has $\bar{x} \in D^wC(\bar{p},\bar{x})(p)$. Moreover, we can infer from (i) and (4.4) that

$$y_n = \frac{f(\bar{p} + t_n p_n, \bar{x} + t_n \hat{x}_n) - f(\bar{p},\bar{x})}{t_n} \to y.$$

Taking (4.2), (4.3), and (4.4) into account, one ensures the existence of the sequence $t_n \downarrow 0$ and the sequence (p_n, x_n) in $\mathrm{gr}\mathcal{S}$ such that $(p_n,x_n) \xrightarrow[s,w]{} (p,x)$ and

$$\bar{x} + t_n x_n \in \mathcal{S}(\bar{p} + t_n p_n), \forall n,$$

leading to x is in $D^w\mathcal{S}(\bar{p},\bar{x})(p)$.

Now, we prove that

$$D^w\mathcal{S}(\bar{p},\bar{x})(p) \subset \{x \in X \mid x \in D^wC(\bar{p},\bar{x})(p) : f'_H(\bar{p},\bar{x})(p,x) \in D^w\mathcal{B}(\bar{p},\bar{y})(p)\}.$$

Let $x \in D^w\mathcal{S}(\bar{p},\bar{x})(p)$. Then, there exist sequence $t_n \downarrow 0$ and the sequence (p_n,x_n) in $P \times X$ such that $(p_n,x_n) \xrightarrow[s,w]{} (p,x)$ and

$$\bar{x} + t_n x_n \in \mathcal{S}(\bar{p} + t_n p_n).$$

This yields that $\bar{x} + t_n x_n \in C(\bar{p} + t_n p_n)$ and

$$f\left(\bar{p} + t_n p_n, \bar{x} + t_n x_n\right) \in \mathcal{B}(\bar{p} + t_n p_n).$$

Hence, we obtain that $x \in D^wC(\bar{p},\bar{x})(p)$. Setting

$$y_n := \frac{f(\bar{p} + t_n p_n, \bar{x} + t_n x_n) - f(\bar{p},\bar{x})}{t_n}, \tag{4.5}$$

one has

$$\bar{y} + t_n y_n \in \mathcal{B}(\bar{p} + t_n p_n).$$

Moreover, we deduce from the Hadamard differentiability of f at $(\bar{p},\bar{x})$ and (4.5) that

$$y_n \to f'_H(\bar{p},\bar{x})(p,x).$$

Therefore, there exist $t_n \downarrow 0$ and $(p_n, y_n) \to (p, f'_H(\bar{p}, \bar{x})(p, x))$ such that

$$\bar{y} + t_n y_n \in \mathcal{B}(\bar{p} + t_n p_n),$$

which implies that $f'_H(\bar{p}, \bar{x})(p, x) \in D\mathcal{B}(\bar{p} + t_n p_n) \subset D^w\mathcal{B}(\bar{p} + t_n p_n)$.

The proof is complete. $\square$

Proposition 4.2 *Let $\bar{p}$ be a point in P, $\bar{x} \in \mathcal{S}(\bar{p})$ and $\bar{y} = f(\bar{p}, \bar{x})$. If the weak directionally compactness of C at $(\bar{p}, \bar{x})$ is satisfied and the Hadarmad derivative $f'_H(\bar{p}, \bar{x})$ exists, then,*

$$D^w F(\bar{p}, \bar{y})(p) = \{y \in Y \mid \exists x \in D^w C(\bar{p}, \bar{x}))(p), y = f'_H(\bar{p}, \bar{x})(p, x)\}, \forall p \in P. \tag{4.6}$$

Proof We firstly check that

$$\{y \in Y \mid \exists x \in D^w C(\bar{p}, \bar{x}))(p), y = f'_H(\bar{p}, \bar{x})(p, x)\} \subset D^w F(\bar{p}, \bar{y})(p).$$

Let $y \in Y$ such that there exist $p \in P$ and $x \in D^w C(\bar{x}, \bar{p})(p)$ and $y = f'_H(\bar{p}, \bar{x})(p, x)$. Since $x \in D^w C(\bar{p}, \bar{x})(p)$, there exist the sequences $t_n \downarrow 0$ and $(p_n, x_n) \xrightarrow[s,w]{} (p, x)$ in order that, for all n, $\bar{x} + t_n x_n \in C(\bar{p} + t_n p_n)$. Then,

$$f((\bar{p}, \bar{x}) + t_n(x_n, p_n)) = f(\bar{p} + t_n p_n, \bar{x} + t_n x_n) \in F(\bar{p} + t_n p_n), \forall n. \tag{4.7}$$

Setting $v_n := \frac{1}{t_n}(f((\bar{p}, \bar{x}) + t_n(p_n, x_n)) - f(\bar{p}, \bar{x}))$, then we derive from (4.7) and the fact that f is Hadamard differentiable f at $(\bar{p}, \bar{x})$ that

$$\bar{y} + t_n v_n \in F(\bar{p} + t_n u_n) \text{ and } v_n \to f'_H f(\bar{p}, \bar{x})(p, x).$$

Hence, $y = f'_H(\bar{p}, \bar{x})(p, x) \in DF(\bar{p}, \bar{y})(p) \subset D^w F(\bar{p}, \bar{y})(p)$.

Conversely, let $y \in D^w F(\bar{p}, \bar{y})(p)$. Then, there exist $t_n \downarrow 0$ and $(p_n, y_n) \xrightarrow[s,w]{} (p, y)$ with the property that $\bar{y} + t_n y_n \in F(\bar{p} + t_n p_n)$, for all n. Hence, we can find the sequence $x_n \in C(\bar{p} + t_n p_n)$ in order that

$$\bar{y} + t_n y_n = f(x_n, \bar{p} + t_n p_n), \forall n.$$

Setting $\widetilde{x}_n := \frac{x_n - \bar{x}}{t_n}$, we have

$$\bar{x} + t_n \widetilde{x}_n \in C(\bar{p} + t_n p_n)$$

and

$$\bar{y} + t_n y_n = f(\bar{p} + t_n p_n, \bar{x} + t_n \widetilde{x}_n), \forall n. \tag{4.8}$$

As the weak directionally compactness of C at $(\bar{p}, \bar{x})$ holds, for preceding t_n, p_n, and $\widetilde{x}_n$, we imply the existence of a subsequence, denoted also by $\widetilde{x}_n$, satisfying $\widetilde{x}_n \xrightarrow[w]{} \widetilde{x} \in D^w C(\bar{p}, \bar{x})(p)$. It follows from (4.8) and the existence of the Hadamard derivative of $f'_H(\bar{p}, \bar{x})$ that one has

$$y_n = \frac{f(\bar{p} + t_n p_n, \bar{x} + t_n \widetilde{x}_n) - \bar{y}}{t_n} \to f'_H(\bar{p}, \bar{x})(p, x).$$

Hence, $y = f'_H(\bar{p}, \bar{x})(p, x)$, which justifies the conclusion. $\square$

By employing Propositions 3.4, 3.5, and 4.1, we obtain the following result:

Proposition 4.3 *Let $\bar{p}$ be a point in P, $\bar{x} \in \mathcal{S}(\bar{p})$, and $\bar{y} = f(\bar{p}, \bar{x})$. Assume that all of the following conditions hold:*

(i) *the weak directionally compactness of F at $(\bar{p}, \bar{y})$ is satisfied;*

(ii) *F is K-minicomplete by $\mathcal{B}$ around $\bar{p}$;*

(iii) *F has the local Lipschitzness at $\bar{p}$;*

(iv) *F has a weak contingent proto-derivative at $(\bar{p}, \bar{y})$;*

(v) *there exists a neighborhood U of $\bar{p}$ in order that for any $p \in U, \mathcal{B}(p)$ contains only one point;*

(vi) *f has the Hadamard derivative $f'_H(\bar{p}, \bar{x})$;*

(vii) *C has the weak directionally compact property at $(\bar{p}, \bar{x})$.*

Then, for any $p \in P$,

$$\begin{aligned} D^w\mathcal{B}(\bar{p}, \bar{y})(p) &= \mathrm{BoMin}_K D^w F(\bar{p}, \bar{y})(p) \\ &= \mathrm{BoMin}_K \{y \in Y \mid \exists x \in D^w C(\bar{p}, \bar{x}))(p), y = f'_H(\bar{p}, \bar{x})(p, x)\}. \end{aligned}$$

The obtained results in Section 4 is illustrated in the following example.

Example 4.4 *Let $P = X = Y = l^2$, $K = l^2_+$, $f(p, x) = p + x$, and $C : l^2 \rightrightarrows l^2$ be defined by*

$$C(p) = \begin{cases} \{x \in X \mid x \in p + K, \|x\| \leq 2\|p\|\} \cup \{2p + p^2\}, & \text{if } p \in l^2_+ \cap B(0, 1), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then,

$$F(p) = \begin{cases} \{y \in Y \mid y \in 2p + K, \|y\| \leq 3\|p\|\} \cup \{3p + p^2\}, & \text{if } p \in l^2_+ \cap B(0, 1), \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$\mathcal{B}(p) = \begin{cases} \{2p\}, & \text{if } p \in l^2_+ \cap B(0, 1), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Taking $(\bar{p}, \bar{x}) = (0, 0)$, one has $\bar{y} = f(\bar{p}, \bar{x}) = 0$. We can check easily that the assumptions (ii), (iii), and (v) in Proposition 4.3 are fulfilled.

Now we will justify that the assumptions (i) and (vii) in Proposition 4.3 hold. Let $t_n \downarrow 0$, $p_n \to p \in P$ and $y_n \in Y$ satisfying

$$\bar{y} + t_n y_n \in F(\bar{p} + t_n p_n).$$

Then, there are only two cases.

∗ *Case 1. If $t_n y_n \in 2t_n p_n + K$ and $\|t_n y_n\| \leq 3\|t_n p_n\|$, then one has $y_n \in 2p_n + K$ and $\|y_n\| \leq 3\|p_n\|$. Since $p_n \to p$, there exists $M > 0$ such that $\|p_n\| < M$ for all n. Hence, $\|y_n\| \leq 3\|p_n\| < M$, which ensures the existence of a weak convergent subsequence of y_n.*

∗ *Case 2. If $t_n y_n = 3t_n p_n + t_n^2 p_n$, then $y_n = 3p_n + t_n p_n^2 \to 3p$.*

Hence, (i) is fulfilled and (vii) can be checked similarly.

Moreover, one has, for every $p, x \in l^2$ *,*

$$f'_H(\bar{p}, \bar{x})(p, x) = p + x,$$

i.e. the assumption (vi) in Proposition 4.3 holds.

Straightforward calculations show that

$$D^w C(\bar{p}, \bar{y})(p) = \begin{cases} \{x \in X \mid x \in p + K, \|x\| \le 2\|p\|\}, & \text{if } p \in l^2_+, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{4.9}$$

Indeed, let $x \in D^w C(\bar{p}, \bar{y})(p)$ *. Then, we can find the sequences* $t_n \downarrow 0$ *and* $(p_n, x_n) \xrightarrow[s,w]{} (p, x)$ *in order that* $\bar{x} + t_n x_n \in C(\bar{p} + t_n p_n)$ *for all* n *, i.e.*

$$t_n p_n \in l^2_+ \cap B(0,1), t_n x_n \in t_n p_n + K, \ \|t_n x_n\| \le 2\|t_n p_n\| \text{ or } t_n x_n = 2t_n p_n + t_n^2 p_n^2.$$

Consequently,

$$p_n \in l^2_+, x_n \in p_n + K, \ \|x_n\| \le 2\|p_n\| \text{or } x_n = 2p_n + t_n p_n^2.$$

Letting $n \to \infty$ *, one gets*

$$p \in l^2_+, x \in p + K, \|x\| \le 2\|p\|.$$

Therefore, x *is in the right hand side of (4.9).*

Conversely, let $p \in l^2_+$ *and* $x \in p + K$ *with* $\|x\| \le 2\|p\|$ *. Then, by taking* $t_n = \frac{1}{n}$ *,* $p_n = p$ *and* $x_n = x$ *, one verifies that the sequence* $t_n \downarrow 0$ *and the sequence* $(p_n, x_n) \to (p, x)$ *satisfying* $\bar{x} + t_n x_n \in C(\bar{p} + t_n p_n)$ *for all* n *. Hence,* $x \in D^w C(\bar{p}, \bar{y})(p)$ *.*

Furthermore, we can check that

$$D^w F(\bar{p}, \bar{y})(p) = \underline{D}^w F(\bar{p}, \bar{y})(p) = \begin{cases} \{y \in Y \mid y \in 2p + K, \|y\| \le 3\|p\|\}, & \text{if } p \in l^2_+, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$D^w \mathcal{B}(\bar{p}, \bar{y})(p) = \begin{cases} \{2p\}, & \text{if } p \in l^2_+, \\ \emptyset, & \text{otherwise}. \end{cases}$$

Thus, all the assumptions in Proposition 4.3 are satisfied. Thus, for any $p \in l^2$ *,*

$$\begin{aligned} D^w \mathcal{B}(\bar{p}, \bar{y})(p) &= \mathrm{BoMin}_K D^w F(\bar{p}, \bar{y})(p) \\ &= \mathrm{BoMin}_K \{y \in Y \mid \exists x \in D^w C(\bar{p}, \bar{x}))(p), y = f'_H(\bar{p}, \bar{x})(p, x)\}. \end{aligned}$$

Remark 4.5 *In the case that* P, X *, and* Y *are Euclidean spaces, i.e.* τ^w *-contingent derivatives coincide with contingent derivatives, the results in Sections 3 and 4 also may be new.*

Acknowledgment

The authors would like to thank the editors for the help in the processing of the article. The authors are very grateful to the anonymous referees for the very valuable remarks and suggestions, which helped to improve the paper. The first author is partially supported by Can Tho University.

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