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Approximation by matrix transforms in weighted Orlicz spaces

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Abstract: In this work the approximation problems of the functions by matrix transforms in weighted Orlicz spaces with Muckenhoupt weights are studied. We obtain the degree of approximation of functions belonging to Lipschitz class $Lip(\alpha, M, \omega)$ through matrix transforms $T_n^{(A)}(x, f)$, and Nörlund means $N_n(x, f)$ of their trigonometric Fourier series.

Key words: Matrix transforms, trigonometric approximation, Orlicz space, weighted Orlicz space, Boyd indices, Muckenhoupt weight, modulus of smoothness, weighted Lipschitz classes

1. Introduction

Let $\mathbb{T}$ denote the interval $[-\pi, \pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \le p \le \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$. A convex and continuous function $M : [0, \infty) \to [0, \infty)$ which satisfies the conditions

$$\begin{aligned} M(0) &= 0, \ M(x) > 0 \ \text{ for } x > 0 \\ \lim_{x \to 0} (M(x)/x) &= 0; \quad \lim_{x \to \infty} (M(x)/x) = \infty, \end{aligned}$$

is called a Young function. We will say that M satisfies the Δ_2-condition if $M(2u) \le cM(u)$ for any $u \ge u_0 \ge 0$ with some constant c, independent of u.

For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M(|f(x)|)\, dx < \infty.$$

The $N-$*function complementary to* M is defined by

$$N(y) := \max_{x \ge 0} (xy - M(x)), \text{ for } y \ge 0.$$

Let N be the complementary Young function of M. It is well known [32, p. 69], [45, pp. 52-68] that

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the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the Orlicz norm

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{\int_{\mathbb{T}} |f(x)g(x)|\,dx : g \in \widetilde{L}_N(\mathbb{T}),\ \int_{\mathbb{T}} N\left(|g(x)|\right)dx \le 1\right\},$$

or with the Luxemburg norm

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right)dx \le 1\right\}$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orliczspace* [32, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T})$, $1 < p < \infty$. If $M(x) = M(x,p) := x^p$, $1 < p < \infty$, then Orlicz spaces $L_M(\mathbb{T})$ coincides with the usual Lebesgue spaces $L_p(\mathbb{T})$, $1 < p < \infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularity theory, fluid dynamics, and statistical physics (e.g., [5,14,39,46]). Therefore, investigation of approximation of functions by means of Fourier trigonometric series in Orlicz spaces is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm. The inequalities

$$\|f\|^*_{L_M(\mathbb{T})} \le \|f\|_{L_M(\mathbb{T})} \le 2\,\|f\|^*_{L_M(\mathbb{T})}, \quad f \in L_M(\mathbb{T})$$

hold [32, p. 80].

If we choose $M(u) = u^p/p$, $1 < p < \infty$ then the complementary function is $N(u) = u^q/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p}\,\|u\|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \le \|u\|_{L_M(\mathbb{T})} \le q^{1/q}\,\|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int_{\mathbb{T}} |u(x)|^p\,dx\right)^{1/p}$ stands for the usual norm of the $L_p(\mathbb{T})$ space.

If N is complementary to M in Young's sense and $f \in L_M(\mathbb{T})$, $g \in L_N(\mathbb{T})$ then the so-called strong Hölder inequalities [32, p. 80]

$$\int_{\mathbb{T}} |f(x)g(x)|\,dx \le \|f\|_{L_M(\mathbb{T})}\,\|g\|^*_{L_N(\mathbb{T})},$$

$$\int_{\mathbb{T}} |f(x)g(x)|\,dx \le \|f\|^*_{L_M(\mathbb{T})}\,\|g\|_{L_N(\mathbb{T})}$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [45, p. 113].

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse function of the $N-$function M. The lower and upper indices [6, p. 350]

$$\alpha_M := \lim_{t\to+\infty} -\frac{\log h(t)}{\log t},\ \beta_M := \lim_{t\to o^+} -\frac{\log h(t)}{\log t}$$

of the function

$$h:(0,\infty)\to(0,\infty],\quad h(t):=\lim_{y\to\infty}\sup\frac{M^{-1}(y)}{M^{-1}(ty)},\ \ t>0,$$

first considered by Matuszewska and Orlicz [37], are called the *Boyd indices* of the Orlicz spaces $L_M(T)$.

It is known that the indices α_M and β_M satisfy $0\le\alpha_M\le\beta_M\le1,\ \alpha_N+\beta_M=1,\ \alpha_M+\beta_N=1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0<\alpha_M\le\beta_M<1$. The detailed information about the Boyd indices can be found in [7–10,38].

A measurable function $\omega:\mathbb{T}\to[0,\infty]$ is called a weight function if the set $\omega^{-1}(\{0,\infty\})$ has Lebesgue measure zero. With any given weight ω we associate the *ω-weighted Orlicz space* $L_M(\mathbb{T},\omega)$ consisting of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L_M(\mathbb{T},\omega)}:=\|f\omega\|_{L_M(\mathbb{T})}.$$

Let $1<p<\infty,\ 1/p+1/p'=1$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy Muckenhoupt's A_p*-condition* on $\mathbb{T}$ if

$$\sup_J\left(\frac{1}{|J|}\int_J\omega^p(t)\,dt\right)^{1/p}\left(\frac{1}{|J|}\int_J\omega^{-p'}(t)\,dt\right)^{1/p'}<\infty\ ,$$

where J is any subinterval of $\mathbb{T}$, and $|J|$ denotes its length [40].

Let us indicate by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on $\mathbb{T}$.

Let further $t_1,t_2,...,t_n$ be distinct points on $\mathbb{T}$ and let $\lambda_1,\ ...,\lambda_n$ be real numbers. If $1<p<\infty,\ \frac{1}{p}+\frac{1}{q}=1$ and $-\frac{1}{p}<\lambda_j<\frac{1}{q}\ ,\ j=1,...n$ then the weight function

$$\omega(\tau):=\prod_{j=1}^{n}|\tau-t_j|^{\lambda_j},(\tau\in\mathbb{T})$$

belongs to $A_p(\mathbb{T})$.

According to [35], [36, Lemma 3.3], and [36, Section 2.3], if $L_M(\mathbb{T})$ is reflexive and the weighted function ω satisfies the condition $\omega\in A_{1/\alpha_M}(\mathbb{T})\cap A_{1/\beta_M}(\mathbb{T})$, then the space $L_M(\mathbb{T},\omega)$ is also reflexive.

Let $L_M(\mathbb{T},\omega)$ be a weighted Orlicz space, let $0<\alpha_M\le\beta_M<1$ and let $\omega\in A_{\frac{1}{\alpha_M}}(\mathbb{T})\cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. For $f\in L_M(\mathbb{T},\omega)$ we set

$$(\nu_hf)(x):=\frac{1}{2h}\int_{-h}^{h}f(x+t)\,dt,\ 0<h<\pi,\ x\in\mathbb{T}.$$

By reference [24, Lemma 1], the shift operator ν_h is a bounded linear operator on $L_M(\mathbb{T},\omega)$:

$$\|\nu_h(f)\|_{L_M(\mathbb{T},\,\omega)}\le c\,\|f\|_{L_M(\mathbb{T},\,\omega)}.$$

The function

$$\Omega_{M,\omega}(\delta,\ f):=\sup_{0<h\le\delta}\|f-(\nu_hf)\|_{L_M(\mathbb{T},\omega)},\ \delta>0$$

is called the *modulus of continuity* of $f \in L_M(\mathbb{T}, \omega)$.

It can easily be shown that $\Omega_{M,\ \omega}(\cdot,\ f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta \to 0} \Omega_{M,\omega}(\delta,\ f) = 0, \quad \Omega_{M,\omega}(\delta,\ f+g) \leq \Omega_{M,\omega}(\delta,\ f) + \Omega_{M,\omega}(\delta, g)$$

for $f,\ g \in L_M(\mathbb{T},\ \omega)$.

Let $0 < \alpha \leq 1$. The set of functions $f \in L_M(\mathbb{T}, \omega)$ such that

$$\Omega_{M,\omega}(f, \delta) = O(\delta^{\alpha}), \quad \delta > 0$$

is called the *Lipschitzclass* $Lip(\alpha, M, \omega)$. Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \tag{1.1}$$

be the Fourier series of the function $f \in L^1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ are the Fourier coefficients of the function f. The *n-th partial sums, Cesaro means* of the series (1.1) are defined, respectively, as

$$\begin{aligned} S_n(x, f) &= \frac{a_0}{2} + \sum_{k=1}^{n} (a_k(f) \cos kx + b_k(f) \sin kx), \\ &= \frac{a_0}{2} + \sum_{k=1}^{n} B_k(x, f) = \sum_{k=0}^{n} B_k(x, f)\ ,\ B_0(x, f) := \frac{a_0}{2} \text{ and } B_k(x, f) := (a_k(f) \cos kx + b_k(f) \sin kx), \\ \sigma_n(x, f) &= \frac{1}{n+1} \sum_{m=0}^{n} S_m(x, f). \end{aligned}$$

Let $\{p_n\}$ be a real sequence of positive numbers and let $P_n. = \sum\limits_{k=0}^{n} p_k$. As in [26] we define Nörlund means of the Fourier series of f with respect to the sequence $\{p_n\}$ the following form:

$$N_n(x, f) := \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(x, f)$$

It is known that if $p_n = 1$ $(n = 0, 1, 2, ...)$, then $N_n(x, f)$ Nörlund means coincide with the Cesaro mean $\sigma_n(x, f)$.

We suppose that $A = (a_{n,k})$ is the infinite lower triangular matrix with nonnegative entries. Let

$$s_n^{(A)} = \sum_{k=0}^{n} a_{n,k}\ ,\ n = 0, 1, ...$$

We define the matrix transform of Fourier series of f, by

$$T_n^{(A)}(x, f) = \sum_{k=0}^{n} a_{n,k} S_k(x, f).$$

It is clear that if $a_{n,k} = \frac{p_{n-k}}{Pn}$, then the matrix transform $T_n^{(A)}(\cdot, f)$ coincides with Nörlund means $N_n(\cdot, f)$.

Let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. If there exists a constant c, depending on the sequence $\{p_n\}_0^\infty$, such that, for all $n \geq m$ the inequality

$$p_n \leq cp_m \quad (p_n \geq cp_m)$$

satisfies, then sequence $\{p_n\}_0^\infty$ is called almost monotone decreasing (increasing).

Let $W_M^1(\mathbb{T}, \omega)$ be the linear space of functions for which f is absolutely continuous on $\mathbb{T}$ and $f' \in L_M(\mathbb{T}, \omega)$.

We use $c, c_1, c_2, ...$ (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest. We also will use the relation $f = O(g)$ which means that $f \leq cg$ for a constant c independent of f and g.

The approximation properties of the partial sums of the Fourier series were studied, and some direct and inverse theorems for approximation by polynomials in weighted Orlicz spaces were proved in [24]. Using the $k-th$ modulus of smoothness, the generalized Lipschitz class of functions in weighted Orlicz spaces was also defined in [24]. In particular, a constructive characterization of the generalized Lipschitz classes in these spaces was obtained in [24]. Approximation properties of the matrix transforms of functions in the weighted variable exponent Lebesgue spaces were investigated in [26]. In this work using modulus of continuity in weighted Orlicz spaces we define Lipschitz class $Lip(\alpha, M, \omega)$. In this study, we investigate the degree of approximation of functions belonging to Lipschitz class $Lip(\alpha, M, \omega)$ through matrix transforms $T_n^{(A)}(x, f)$ and Nörlund means $N_n(\cdot, f)$ of their trigonometric Fourier series. In this work, we give the weighted Orlicz space versions of the results obtained in [26] in the case of weighted Lebesgue spaces with variable exponent. Similar problems about approximation properties of the different sums, constructed according to the Fourier series of given functions in the different spaces have been investigated by several authors (see, for example, [1–4,11–13,15–31,33,34,41–44,47]).

Our main results are as follows:

Theorem 1.1 *Let* $f \in Lip(\alpha, M, \omega)$, $0 < \alpha < 1$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$, *let* $A = (a_{n,k})$ *be a lower triangular matrix with* $\left|s_n^{(A)} - 1\right| = O(n^{-\alpha})$ *and one of the following conditions holds:*

(i) A *has almost monotone decreasing rows and* $(n+1)a_{n,0} = O(1)$,

(ii) A *has almost monotone increasing rows and* $(n+1)a_{n,r} = O(1)$, *where* r *is the integer part of* $\frac{n}{2}$. *Then the estimate*

$$\left\| f - T_n^{(A)}(\cdot, f) \right\|_{L_M(\mathbb{T}, \omega)} = O(n^{-\alpha})$$

holds.

From Theorem 1.1 we have the following Corollary:

Corollary 1.2 *Let* $f \in Lip(\alpha, M, \omega)$, $0 < \alpha < 1$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$, *let* $\{p_n\}$ *be a real sequence of positive numbers and one of the following conditions holds:*

(i) $\{p_n\}$ *is almost monotone increasing and* $(n+1)p_n = O(P_n)$

(ii) $\{p_n\}$ *is almost monotone decreasing. Then the relation*

$$\|f - N_n(\cdot, f)\|_{L_M(\mathbb{T},\omega)} = O(n^{-\alpha})$$

holds.

Theorem 1.3 *Let* $f \in Lip(1, M, \omega)$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$ *, let* $A = (a_{n,k})$ *be a lower triangular matrix with* $\left|s_n^{(A)} - 1\right| = O\left(n^{-1}\right)$ *and one of the following conditions holds:*

(i) $\sum\limits_{k=1}^{n-1} |a_{n,k-1} - a_{n,k}| = O(n^{-1})$,

(ii) $\sum\limits_{k=1}^{n-1} (n-k)\,|a_{n,k-1} - a_{n,k}| = O(1)$.

Then the estimate

$$\left\|f - T_n^{(A)}(\cdot, f)\right\|_{L_M(\mathbb{T},\omega)} = O(n^{-1}) \tag{1.2}$$

holds.

From Theorem 1.3 we get the following Corollary:

Corollary 1.4 *Let* $f \in Lip(1, M, \omega)$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$, *let* $\{p_n\}$ *be a real sequence of positive numbers and the inequality*

$$\sum_{k=1}^{n-1} |p_k - p_{k+1}| = O(\frac{P_n}{n}),$$

holds. Then the following relation holds:

$$\|f - N_n(\cdot, f)\|_{L_M(\mathbb{T},\omega)} = O(n^{-1}).$$

2. Auxiliary results

In the proof of the main results we need the following Lemmas.

Lemma 2.1 *Let* $f \in Lip(\alpha, M, \omega)$, $0 < \alpha \leq 1$ *and* $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$. *Then the relation*

$$\|f - S_n(\cdot, f)\|_{L_M(\mathbb{T},\omega)} = O(n^{-\alpha}),\ n = 1, 2, 3..$$

holds.

Proof Let $f \in Lip(\alpha, M, \omega), 0 < \alpha \leq 1$ and $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}$ and let T_n be the polyonomial of best approximation to f. By [24, Theorem 2] the following relation holds:

$$\|f - T_n\|_{L_M(\mathbb{T},\omega)} = O\left(\Omega_{M,\omega}(f, \frac{1}{n})\right) = O\left(n^{-\alpha}\right). \tag{2.1}$$

On the other hand, if $f \in L_M(T,\ \omega)$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$, then according to [24, p.155, relation (15)] there exists the positive constant c_1 such that ineguality

$$\|S_n(\cdot, f)\|_{L_M(\mathbb{T},\omega)} \leq c_1 \|f\|_{L_M(\mathbb{T},\omega)} \tag{2.2}$$

holds. Then from (2.1) and (2.2) we conclude the required result:

$$\begin{aligned}\|f - S_n(\cdot, f)\|_{L_M(\mathbb{T},\omega)} &\leq \|f - T_n\|_{L_M(\mathbb{T},\omega)} + \|T_n - S_n(\cdot, f)\|_{L_M(\mathbb{T},\omega)} \\ &\leq \|f - T_n\|_{L_M(\mathbb{T},\omega)} + \|S_n(\cdot, T_n - f)\|_{L_M(\mathbb{T},\omega)} \\ &= O\left(\|f - T_n\|_{L_M(\mathbb{T},\omega)}\right) = O\left(n^{-\alpha}\right)\end{aligned}$$

the required result. □

Lemma 2.2 *Let* $f \in Lip(1, M, \omega)$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$. *Then the estimate*

$$\|S_n(f) - \sigma_n(f)\|_{L_M(\mathbb{T},\omega)} = O(n^{-1}),\ n = 1, 2, 3...$$

holds.

Proof We suppose that $f \in Lip(1, M, \omega)$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$. We consider a Steklov mean operator given by

$$f_\delta(x) := \frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta}\left(\frac{1}{h}\int_0^h f(x+t)\,dt\right)dh,\ \delta > 0.$$

Using the method in the proof of [26, relation (29)] we can show that in the weighted Orlicz spaces $L_M(\mathbb{T}, \omega)$ the inequality

$$\|f'_\delta\| \leq c_2\delta^{-1}\Omega_{M,\omega}(f, \delta) \leq c_3$$

holds. Then applying Fatou's Lemma we have

$$\|f'\|_{L_M(T,\ \omega)} \leq \left\|\lim_{\delta\to 0^+} f'_\delta\right\|_{L_M(T,\ \omega)} \leq \lim\inf_{\delta\to 0}\|f'_\delta\|_{L_M(T,\ \omega)} \leq c_4.$$

That is $f' \in L_M(\mathbb{T}, \omega)$. Then according to definition of $W^1_M(\mathbb{T}, \omega)$ we obtain $f \in W^1_M(\mathbb{T}, \omega)$. If the function f has the Fourier series $\sum_{k=0}^{\infty} B_k(x, f)$, then conjugate function $\widetilde{f'}$ has the Fourier series $\sum_{k=1}^{\infty} kB_k(x, f)$. If $f \in L_M(T,\ \omega)$, $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}$, then according to [24, relation (15)] there exists the positive constant c_5 such that the following ineguality holds:

$$\left\|\widetilde{f}\right\|_{L_M(\mathbb{T},\omega)} \leq c_5\|f\|_{L_M(\mathbb{T},\omega)}. \tag{2.3}$$

.

Taking into account that

$$s_n(x, f) - \sigma_n(x, f) = \sum_{k=1}^{n}\frac{k}{n+1}B_k(x, f) \tag{2.4}$$

by (2.2) and (2.3) we have

$$\begin{aligned}\|s_n(\cdot,f)-\sigma_n(\cdot,f)\|_{L_M(\mathbb{T},\omega)} &= \left\|\sum_{k=1}^{n}\frac{k}{n+1}B_k(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} \\ &= \frac{1}{n+1}\left\|s_n\left(\cdot,\widetilde{f'}\right)\right\|_{L_M(\mathbb{T},\omega)} \\ &\leq \frac{c_6}{n}\|f'\|_{L_M(\mathbb{T},\omega)} = O\left(n^{-1}\right).\end{aligned}$$

□

Lemma 2.3 *[16] Let* $A=(a_{n,k})$ *be an infinite lower triangular matrix with* $\left|s_n^{(A)}-1\right| = O\left(n^{-\alpha}\right)$, $0<\alpha<1$ *and one of the following conditions holds: (i)* A *has almost monotone decreasing rows and* $(n+1)\,a_{n,0}= O(1)$, *(ii)* A *has almost monotone increasing rows and* $(n+1)\,a_{n,r}=O(1)$, *where* r *is the integer part of* $\frac{n}{2}$. *Then the estimate*

$$\sum_{k=1}^{n}k^{-\alpha}a_{n,k}=O\left(n^{-\alpha}\right)$$

holds.

Lemma 2.4 *[21] The following relation holds:*

$$\left|\sum_{m=0}^{k}a_{n,m}-(k+1)\,a_{n,k}\right|\leq\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|. \quad k=1,2,...,n.$$

3. Proofs of the main results

1.1. Let $f\in Lip(\alpha,M,\omega)$, $0<\alpha<1$, $\omega\in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}$, let $A=(a_{n,k})$ be a lower triangular matrix with $\left|s_n^{(A)}-1\right|=O\left(n^{-\alpha}\right)$ and one of the conditions (i) and (ii) of the theorem be satisfied. By definitions of $T_n^{(A)}(f)(x)$ and $s_n^{(A)}$ we have

$$\begin{aligned}T_n^{(A)}(x,f)-f(x) &= \sum_{k=0}^{n}a_{n,k}S_k(x,f)-f(x) \\ &= \sum_{k=0}^{m}a_{n,k}S_k(x,f)-f(x)+S_n^{(A)}(x,f)-S_n^{(A)}(x,f) \\ &= \sum_{k=0}^{n}a_{n,k}\left[S_k(x,f)-f(x]+S_n^{(A)}(x,f)-f(x).\right.\end{aligned}$$

Taking into account that $\left|s_n^{(A)}-1\right|=O\left(n^{-\alpha}\right)$, by Lemma 2.1 and 2.3 the last equality yields

$$\begin{aligned}
&\left\| f - T_n^{(A)}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} \\
= \;& a_{n,o} \left\| S_0(\cdot, f) - f \right\|_{L_M(\mathbb{T},\omega)} \\
&+ \sum_{k=1}^{n} a_{n,k} \left\| S_k(\cdot, f) - f \right\|_{L_M(\mathbb{T},\omega)} + \left| S_n^{(A)} - 1 \right| \|f\|_{L_M(\mathbb{T},\omega)} \\
= \;& O\left(\frac{1}{n+1}\right) + O(1) \sum_{k=1}^{n} a_{n,k} k^{-\alpha} + O\left(n^{-\alpha}\right) = O\left(n^{-\alpha}\right),
\end{aligned}$$

which completes the proof of Theorem 1.1.

1.3. Let $f \in Lip(1, M, \omega)$, $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}$ and let $A = (a_{n,k})$ be a lower triangular matrix with $\left| s_n^{(A)} - 1 \right| = O\left(n^{-1}\right)$. Using Lemma 2.1 for $\alpha = 1$ we obtain

$$\begin{aligned}
\left\| f - T_n^{(A)}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} &\leq \left\| S_n(\cdot, f) - T_n^{(A)}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} + \left\| f - S_n(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} \\
&= \left\| S_n(\cdot, f) - T_n^{(A)}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} + O\left(n^{-1}\right). \qquad (3.1)
\end{aligned}$$

We put $A_{n,k} := \sum\limits_{m=k}^{n} a_{n,m}$. Then the following equality holds:

$$\begin{aligned}
T_n^{(A)}(x, f) &= \sum_{k=0}^{n} a_{n,k} S_k(x, f) = \sum_{k=0}^{n} a_{n,k} \left(\sum_{m=0}^{k} B_m(x, f) \right) \\
&= \sum_{k=0}^{n} \left(\sum_{m=k}^{n} a_{n,m} \right) B_k(x, f) = \sum_{k=0}^{n} A_{n,k} B_k(x, f). \qquad (3.2)
\end{aligned}$$

Using $s_n^{(A)} = \sum_{k=0}^{n} a_{n,k}$ we have

$$\begin{aligned}
S_n(x, f) &= \sum_{m=0}^{n} B_m(x, f) = A_{n,0} \sum_{k=0}^{n} B_k(x, f) + (1 - A_{n,0}) \sum_{k=0}^{n} B_k(x, f) \\
&= \sum_{k=0}^{n} A_{n,0} B_k(x, f) + \left(1 - s_n^{(A)}\right) S_k(x, f). \qquad (3.3)
\end{aligned}$$

Now combining (3.2) and (3.3) we find

$$T_n^{(A)}(x, f) - S_n(x, f) = \sum_{k=1}^{n} (A_{n,k} - A_{n,0}) B_k(x, f) + \left(s_n^{(A)} - 1\right) S_n(x, f).$$

From the last equality we get

$$\begin{aligned}\left\|S_n(\cdot,f)-T_n^{(A)}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} &\leq \left\|\sum_{k=1}^{n}\left(A_{n,k}-A_{n,0}\right)B_k\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)}\\ &+c_7\left|s_n^{(A)}-1\right|\|f\|_{L_M(\mathbb{T},\omega)}\\ &=\left\|\sum_{k=1}^{n}\left(A_{n,k}-A_{n,0}\right)B_k\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)}\\ &+O\left(n^{-1}\right) \qquad (3.4)\end{aligned}$$

We denote

$$a_{n,k}^{*}=\frac{A_{n,k}-A_{n,0}}{k},\ k=1,2,...,n.$$

Using Abel's transformation (see, for example: [26]) we get

$$\begin{aligned}\sum_{k=1}^{n}\left(A_{n,k}-A_{n,0}\right)B_k\left(x,f\right) &= \sum_{k=1}^{n}B_k\left(x,f\right)a_{n,k}^{*}k=a_{n,n}^{*}\sum_{m=1}^{n}mB_m\left(x,f\right)\\ &+\sum_{k=1}^{n-1}\left(a_{n,k}^{*}-a_{n,k+1}^{*}\right)\left(\sum_{m=1}^{k}mB_m\left(x,f\right)\right). \qquad (3.5)\end{aligned}$$

From the last equality we obtain

$$\begin{aligned}\left\|\sum_{k=1}^{n}\left(A_{n,k}-A_{n,0}\right)B_k\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)} &\leq \left|a_{n,n}^{*}\right|\left\|\sum_{m=1}^{n}mB_m\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)}\\ &+\sum_{k=1}^{n-1}\left|a_{n,k}^{*}-a_{n,k+1}^{*}\right|\\ &\times\left(\left\|\sum_{m=1}^{k}mB_m\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)}\right). \qquad (3.6)\end{aligned}$$

Now we estimate

$$\left\|\sum_{m=1}^{n}mB_m\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)}.$$

According to (2.4) and Lemma 2.2 the estimation

$$\begin{aligned}\left\|\sum_{m=1}^{n}mB_m\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)} &= (n+1)\left\|S_n\left(\cdot,f\right)-\sigma_n\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)}\\ &=(n+1)\,O\left(n^{-1}\right)=O\left(1\right) \qquad (3.7)\end{aligned}$$

holds.

On the other hand, $\left|s_n^{(A)}-1\right|=O\left(n^{-1}\right)$, which implies that

$$\begin{aligned}\left|a_{n,n}^{*}\right| &= \frac{\left|A_{n,n}-A_{n,0}\right|}{n}=\frac{\left|a_{n,n}-s_n^{(A)}\right|}{n}=\frac{1}{n}\left(s_n^{(A)}-a_{n,n}\right)\\ &\leq \frac{1}{n}s_n^{(A)}=\frac{1}{n}O\left(1\right)=O\left(\frac{1}{n}\right). \qquad (3.8)\end{aligned}$$

Using the relations (3.4), (3.6)–(3.8) we get

$$\begin{aligned}\left\|s_n\left(\cdot,f\right)-T_n^{(A)}\left(\cdot,f\right)\right\|_{L_M(\mathbb{T},\omega)} &= O\left(1\right)O\left(\frac{1}{n}\right)+O\left(1\right)\sum_{k=1}^{n-1}\left|a_{n,k}^{*}-a_{n,k+1}^{*}\right|\\ &= O\left(\frac{1}{n}\right)+\sum_{k=1}^{n-1}\left|a_{n,k}^{*}-a_{n,k+1}^{*}\right|. \qquad (3.9)\end{aligned}$$

Now we estimate

$$\sum_{k=1}^{n-1}\left|a_{n,k}^{*}-a_{n,k+1}^{*}\right|.$$

We can show that the following equality holds:

$$a_{n,k}^{*}-a_{n,k+1}^{*}=\frac{1}{k\left(k+1\right)}\left\{\left(k+1\right)a_{n,k}-\sum_{m=0}^{k}a_{n,m}\right\}.$$

We suppose that condition (i) of Theorem 1.3 is satisfied. Then by Lemma 2.4 the estimate

$$\begin{aligned}\sum_{k=1}^{n-1}\left|a_{n,k}^{*}-a_{n,k+1}^{*}\right| &= \sum_{k=1}^{n-1}\frac{1}{k\left(k+1\right)}\left|\sum_{m=0}^{k}a_{n,m}-\left(k+1\right)a_{n,k}\right|\\ &\leq \sum_{k=1}^{n-1}\frac{1}{k\left(k+1\right)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|\\ &= \sum_{m=1}^{n-1}m\left|a_{n,m-1}-a_{n,m}\right|\sum_{k=m}^{n-1}\frac{1}{k\left(k+1\right)}\\ &\leq \sum_{m=1}^{n-1}m\left|a_{n,m-1}-a_{n,m}\right|\sum_{k=m}^{\infty}\frac{1}{k\left(k+1\right)}\\ &= \sum_{m=1}^{n-1}\left|a_{n,m-1}-a_{n,m}\right|=O\left(\frac{1}{n}\right) \qquad (3.10)\end{aligned}$$

holds. Now combininig (3.1), (3.9), and (3.10) we obtain the relation (1.2) of Theorem 1.3.

Now under the condition (ii) we prove relation (1.2). Let $r := \left[\frac{n}{2}\right]$. By Lemma 2.4 the following inequality holds:

$$\begin{aligned}\sum_{k=1}^{n-1}\left|a_{n,k}^{*}-a_{n,k+1}^{*}\right| &\leq \sum_{k=1}^{n-1}\frac{1}{k\,(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right| \\ &\leq \sum_{k=1}^{r}\frac{1}{k\,(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right| \\ &\quad +\sum_{k=r}^{n-1}\frac{1}{k\,(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|, \end{aligned} \tag{3.11}$$

We estimate

$$\sum_{k=1}^{r}\frac{1}{k\,(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|.$$

Since the condition (ii) satisfies, by Abel's transformation we get

$$\begin{aligned}\sum_{k=1}^{r}\frac{1}{k\,(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right| &\leq \sum_{k=1}^{r}\left|a_{n,k-1}-a_{n,k}\right| \\ &= \sum_{k=1}^{r}\frac{1}{(n-k)}(n-k)\left|a_{n,k-1}-a_{n,k}\right| \\ &\leq \frac{1}{(n-r)}O\left(1\right)=O\left(\frac{1}{n}\right). \end{aligned} \tag{3.12}$$

Now we estimate the second term on the right side of (3.11). The inequality

$$\begin{aligned} & \sum_{k=r}^{n-1}\frac{1}{k\,(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right| \\ \leq\; & \sum_{k=r}^{n-1}\frac{1}{k\,(k+1)}\left\{\sum_{m=1}^{r}m\left|a_{n,m-1}-a_{n,m}\right|+\sum_{m=r}^{k}m\left|a_{n,m-1}-a_{n,m}\right|\right\} \\ =\; & \sum_{k=r}^{n-1}\frac{1}{k\,(k+1)}\sum_{m=1}^{r}m\left|a_{n,m-1}-a_{n,m}\right|+\sum_{k=r}^{n-1}\frac{1}{k\,(k+1)}\sum_{m=r}^{k}m\left|a_{n,m-1}-a_{n,m}\right| \\ :\; & =J_{n_1}+J_{n_2} \end{aligned} \tag{3.13}$$

holds.

Taking into account $\sum\limits_{k=1}^{r}\left|a_{n,k-1}-a_{n,k}\right|=O\left(\frac{1}{n}\right)$ and by (3.11) we have

$$
\begin{aligned}
J_{n_1} &\leq \sum_{k=r}^{n-1} \frac{1}{k(k+1)} \sum_{m=1}^{r} |a_{n,m-1} - a_{n,m}| = O\left(n^{-1}\right) \sum_{k=r}^{n-1} \frac{1}{k+1} \\
&= O\left(n^{-1}\right)(n-r)\frac{1}{r+1} = O\left(n^{-1}\right). \qquad (3.14)
\end{aligned}
$$

Now we estimate the expression J_{n_2} [26] :

$$
\begin{aligned}
J_{n_2} &\leq \sum_{k=r}^{n-1} \frac{1}{k(k+1)} \sum_{m=r}^{k} |a_{n,m-1} - a_{n,m}| \leq \frac{1}{r+1} \sum_{k=r}^{n-1} \left(\sum_{m=r}^{k} |a_{n,m-1} - a_{n,m}| \right) \\
&\leq \frac{2}{n} \sum_{k=r}^{n-1} \left(\sum_{m=r}^{k} |a_{n,m-1} - a_{n,m}| \right) \leq \frac{2}{n} \sum_{k=r}^{n-1} (n-k) |a_{n,k-1} - a_{n,k}| \\
&\leq \frac{2}{n} \sum_{k=1}^{n-1} (n-k) |a_{n,k-1} - a_{n,k}| = \frac{2}{n} O(1) = O\left(n^{-1}\right). \qquad (3.15)
\end{aligned}
$$

Using the relations (3.13)–(3.15), we have

$$
\sum_{k=r}^{n-1} \frac{1}{k(k+1)} \sum_{m=1}^{k} m\, |a_{n,m-1} - a_{n,m}| \leq O\left(n^{-1}\right).
$$

The last relation, (3.11) and (3.12) imply that

$$
\sum_{k=1}^{n-1} \left| a_{n,k}^{*} - a_{n,k+1}^{*} \right| = O\left(n^{-1}\right). \qquad (3.16)
$$

Therefore, taking into account the relations (3.1), (3.9), and (3.16) we get

$$
\left\| f - T_n^{(A)}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} = O\left(n^{-1}\right).
$$

The proof of Theorem 1.3 is completed.

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