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Geometric properties of partial sums of generalized Koebe function

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Abstract: The aim of the present paper is to investigate the starlikeness, convexity, and close-to-convexity of some partial sums of the generalized Koebe function. Furthermore, we give some special results related with special cases of c constant. The results, which are presented in this paper, would generalize those in related works of several earlier authors.

Key words: Analytic functions, starlike functions, convex functions, partial sums, radius of starlike and convex functions, generalized Koebe function

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the class of functions f that are analytic and univalent in $\mathbb{D}$ and are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Three most famous subclasses of univalent functions are the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, the class $\mathcal{K}(\alpha)$ of convex functions of order α and the class $\mathcal{R}(\alpha)$ of close-to-convex functions of order α. By definition, we have

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha;\ z \in \mathbb{D};\ 0 \le \alpha < 1 \right\},$$

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha;\ z \in \mathbb{D};\ 0 \le \alpha < 1 \right\}$$

and

$$\mathcal{R}(\alpha) = \left\{ f \in \mathcal{S} : \Re\left(f'(z)\right) > \alpha;\ z \in \mathbb{D};\ 0 \le \alpha < 1 \right\}.$$

The classes consisting of starlike, convex, and close-to-convex functions are usually denoted by $\mathcal{S}^* = \mathcal{S}^*(0)$, $\mathcal{K} = \mathcal{K}(0)$ and $\mathcal{R} = \mathcal{R}(0)$, respectively (see Duren [1]). It is well known that the familiar Koebe function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = z + \sum_{k=2}^{\infty} kz^k \tag{1.2}$$

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is the extremal function for the classes $\mathcal{S}$ and $\mathcal{S}^*$. Furthermore, the functions $g(z)$ and $h(z)$ given by

$$g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \tag{1.3}$$

and

$$h(z) = -z - 2\log(1-z) = z + \sum_{k=2}^{\infty} \frac{2}{k} z^k \tag{1.4}$$

are the extremal functions for the class $\mathcal{K}$ and $\mathcal{R}$, respectively.

For a function $f(z)$ given by (1.1), we introduce the partial sums of $f(z)$ by

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k. \tag{1.5}$$

Note that generally, the partial sum cannot preserve the same character as the initial function. For example, $f_n(z) \notin \mathcal{S}^*$ for $f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k \in \mathcal{S}^*$ and $g_n(z) \notin \mathcal{K}$ for $g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in \mathcal{K}$. Szegö [4] proved that if $f(z) \in \mathcal{S}^*$, then $f_n(z) \in \mathcal{S}^*$ for $|z| < 1/4$ and $f_n(z) \in \mathcal{K}$ for $|z| < 1/8$.

2. Main results

The generalized Koebe function is defined by

$$f_c(z) = \frac{1}{2c}\left[\left(\frac{1+z}{1-z}\right)^c - 1\right] = z + cz^2 + \frac{2c^2+1}{3}z^3 + \frac{2c+c^3}{3}z^4 + ... \tag{2.1}$$

where c a nonzero complex constant. In special cases of c, we obtain some familiar functions. For example, $f_2(z) = f(z)$ and $f_1(z) = g(z)$ are given by (1.2) and (1.3), respectively. Hille [3] proved that f_c is not univalent in $\mathbb{D}$ if and only if c is not in the union A of the closed disks $\{|z+1| \le 1\}$ and $\{|z-1| \le 1\}$. Unlike, Yamashita [6] gave radius of univalency of f_c in the non-Euclidean disk $\Delta(z,w) = \left\{w : |w-z| \diagup \left|1-\bar{z}w\right| < r,\ z \in \mathbb{D}\right\}$. In this paper, for the generalized Koebe function $f_c(z)$ given by (2.1), we consider the partial sums

$$f_{c,3}(z) = z + cz^2 + \frac{2c^2+1}{3}z^3 \tag{2.2}$$

and

$$f_{c,4}(z) = z + cz^2 + \frac{2c^2+1}{3}z^3 + \frac{2c+c^3}{3}z^4. \tag{2.3}$$

Owa et al. [4] studied radii of starlikeness and convexity of given order for partial sums $f_{1,3}(z)$, $f_{2,3}(z)$, $f_{1,4}(z)$, and $f_{2,4}(z)$. In 2012, Hayami et al. [2] considered several special partial sums for the extremal functions $f(z)$, $g(z)$, and $h(z)$ of the classes $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{R}$ given by Eqs. (1.2)–(1.4), respectively. In our paper, we investigated the starlikeness, convexity, and close-to-convexity of the partial sums $f_{c,3}(z)$ and $f_{c,4}(z)$ defined by Eqs. (2.2) and (2.3), respectively.

2.1. Radii for starlikeness of functions $f_{c,3}$ and $f_{c,4}$

For the partial sums $f_{c,k}$ $(k=3,4)$ of the function $f(z)$ given by (2.1), we begin with considering the radius r for $f_{c,k}\in\mathcal{S}^*(\alpha)$ $(k=3,4)$ which means that $f_{c,k}$ belongs to the class $\mathcal{S}^*(\alpha)$ in $\mathbb{D}_r=\{z\in\mathbb{C}: |z|<r, 0<r\le 1\}$. First, for the function $f_{c,3}(z)$ given by (2.2), we obtain radius of starlikeness.

Theorem 2.1 *Let $c>0$. Then, $f_{c,3}(z)\in\mathcal{S}^*(\alpha)$, where*

$$\alpha:=\alpha(c,r)=\begin{cases} 3-\frac{2-cr}{1-cr+\frac{2c^2+1}{3}r^2} & \text{if } 0<r\le\frac{8+13c^2-\sqrt{64+196c^2+145c^4}}{2c(2c^2+1)} \\ 2-\frac{12+15c^2+(4+(13+10c^2)c^2)r^2+2(1+2c^2)R(r,c)}{2(4+5c^2)(3-(1+2c^2)r^2)} & \text{if } \frac{8+13c^2-\sqrt{64+196c^2+145c^4}}{2c(2c^2+1)}<r\le\sqrt{\frac{32+37c^2}{32+103c^2+78c^4}} \end{cases}.$$

Here,

$$R(r,c)=\sqrt{((4+5c^2)(9+3(2+c^2)r^2+(1+2c^2)^2r^4))/(1+2c^2)}.$$

Proof We consider $\alpha:=\alpha(c,r)$ such that

$$\Re\left(\frac{zf'_{c,3}(z)}{f_{c,3}(z)}\right)=\Re\left(3-\frac{cz+2}{1+cz+\frac{2c^2+1}{3}z^2}\right)>\alpha \tag{2.4}$$

for

$$0\le r<\frac{8+13c^2-\sqrt{64+196c^2+145c^4}}{2c\,(2c^2+1)}.$$

It follows from (2.4) that for $z=re^{i\theta}$,

$$\begin{aligned}&\Re\left(\frac{cz+2}{1+cz+\frac{2c^2+1}{3}z^2}\right)\\ =\quad& 1+\frac{\left(1-\frac{2c^2+1}{3}r^2\right)\left(1+\frac{2c^2+1}{3}r^2+cr\cos\theta\right)}{1-\frac{c^2+2}{3}r^2+\left(\frac{2c^2+1}{3}\right)^2r^4+2cr(1+\frac{2c^2+1}{3}r^2)\cos\theta+4r^2\frac{2c^2+1}{3}\cos^2\theta}.\end{aligned}$$

That is,

$$\frac{\left(1-\frac{2c^2+1}{3}r^2\right)\left(1+\frac{2c^2+1}{3}r^2+cr\cos\theta\right)}{1-\frac{c^2+2}{3}r^2+\left(\frac{2c^2+1}{3}\right)^2r^4+2cr(1+\frac{2c^2+1}{3}r^2)\cos\theta+4r^2\frac{2c^2+1}{3}\cos^2\theta}<2-\alpha.$$

Now, let the function $g_c(t)$ be given by

$$g_c(t)=\frac{\left(1-\frac{2c^2+1}{3}r^2\right)\left(1+\frac{2c^2+1}{3}r^2+crt\right)}{1-\frac{c^2+2}{3}r^2+\left(\frac{2c^2+1}{3}\right)^2r^4+2cr(1+\frac{2c^2+1}{3}r^2)t+4r^2\frac{2c^2+1}{3}t^2},\quad (t=\cos\theta),$$

so that we have

$$\begin{aligned}g'_c(t)\quad&=\quad -cr\left[1-\frac{2c^2+1}{3}r^2\right]\\ &\times\frac{\left[1+\left(3c^2+2\right)r^2+\left(\frac{2c^2+1}{3}\right)^2r^4+\frac{8}{c}\frac{2c^2+1}{3}r\left(1+\frac{2c^2+1}{3}r^2\right)t+4r^2\frac{2c^2+1}{3}t^2\right]}{\left[1-\frac{c^2+2}{3}r^2+\left(\frac{2c^2+1}{3}\right)^2r^4+2cr(1+\frac{2c^2+1}{3}r^2)t+4r^2\frac{2c^2+1}{3}t^2\right]^2}.\end{aligned}$$

By setting

$$h_c(t) = 1 + \left(3c^2+2\right)r^2 + \left(\frac{2c^2+1}{3}\right)^2 r^4 + \frac{8}{c}\frac{2c^2+1}{3} r\left(1+\frac{2c^2+1}{3}r^2\right)t + 4r^2\frac{2c^2+1}{3}t^2,$$

we see that
(i) $h_c(t) < 0 \Longrightarrow g_c'(t) > 0$,
(ii) $h_c(t) > 0 \Longrightarrow g_c'(t) < 0$,
(iii) $h_c(t) = 0$ for $t = t_1$ and $t = t_2$ $(t_1 > t_2)$. We can easily see that $t_2 < -1$. Since

$$t_1 = \frac{-2\left(1+\frac{2c^2+1}{3}r^2\right) + \sqrt{\frac{5c^2+4}{2c^2+1}\left(1+\frac{c^2+2}{3}r^2+\left(\frac{2c^2+1}{3}\right)^2 r^4\right)}}{2cr} < 0,$$

we discuss the following two cases $t_1 \leq -1$ and $-1 < t_1 < 1$. If

$$0 < r \leq \frac{8+13c^2-\sqrt{64+196c^2+145c^4}}{2c\left(2c^2+1\right)},$$

then $t_1 \leq -1$ holds true, so that $h(t) \geq 0$. Consequently, we conclude that

$$g_c(t) \leq g_c(-1) = \frac{1-\frac{2c^2+1}{3}r^2}{1-cr+\frac{2c^2+1}{3}r^2} \leq 2-\alpha.$$

That is,

$$\alpha = 2 - \frac{1-\frac{2c^2+1}{3}r^2}{1-cr+\frac{2c^2+1}{3}r^2} = 3 - \frac{2-cr}{1-cr+\frac{2c^2+1}{3}r^2}.$$

Thus, we have

$$\Re\left(1+\frac{zf_{c,3}'(z)}{f_{c,3}(z)}\right) > \alpha$$

and

$$\alpha = 3 - \frac{2-cr}{1-cr+\frac{2c^2+1}{3}r^2} \qquad \left(0 < r \leq \frac{8+13c^2-\sqrt{64+196c^2+145c^4}}{2c\left(2c^2+1\right)}\right).$$

Similarly, if

$$\frac{8+13c^2-\sqrt{64+196c^2+145c^4}}{2c\left(2c^2+1\right)} < r \leq \sqrt{\frac{32+37c^2}{32+103c^2+78c^4}},$$

then the case $-1 < t_1 < 1$ holds true, so that

$$g_c(t) \leq g_c(t_1) = \frac{12+15c^2+(4+(13+10c^2)c^2)r^2+2(1+2c^2)R(r,c)}{2(4+5c^2)(3-(1+2c^2)r^2)}.$$

Therefore, we obtain that

$$\alpha = 2 - \frac{12+15c^2+(4+(13+10c^2)c^2)r^2+2(1+2c^2)R(r,c)}{2(4+5c^2)(3-(1+2c^2)r^2)}$$

for $\frac{8+13c^2-\sqrt{64+196c^2+145c^4}}{2c(2c^2+1)} < r \leq \sqrt{\frac{32+37c^2}{32+103c^2+78c^4}}$, which evidently complete the proof of Theorem 2.1. □

Next, for the function $f_{c,4}(z)$ given by (2.3), we obtain radius of starlikeness.

Theorem 2.2 *Let $c > 0$. Then, the partial sum $f_{c,4}(z) \in \mathcal{S}^*(\alpha)$, where*

$$\alpha := \alpha(c,r) = 4 - \frac{3 - 2cr + \frac{(2c^2+1)}{3}r^2}{1 - cr + \frac{2c^2+1}{3}r^2 - \frac{2c+c^3}{3}r^3},$$

for

$$0 < r \leq \frac{1 + 2c^2 + T(c) + \frac{1-4c^2(3+c^2)}{T(c)}}{4c(2+c^2)}.$$

Here,

$$T(c) = \sqrt[3]{1 + 78c^2 + 48c^4 + 8c^6 + 4c(2+c^2)\sqrt{8c^6 + 52c^4 + 87c^2 + 3}}.$$

Proof For $f_{c,4}(z) = z + cz^2 + \frac{2c^2+1}{3}z^3 + \frac{2c+c^3}{3}z^4$, we have

$$\begin{aligned} \Re\left(\frac{zf'_{c,4}(z)}{f_{c,4}(z)}\right) &= \Re\left(\frac{1 + 2cz + (2c^2+1)z^2 + \frac{4(2c+c^3)}{3}z^3}{1 + cz + \frac{2c^2+1}{3}z^2 + \frac{2c+c^3}{3}z^3}\right) \qquad (2.5) \\ &= 4 - \Re\left(\frac{3 + 2cz + \frac{(2c^2+1)}{3}z^2}{1 + cz + \frac{2c^2+1}{3}z^2 + \frac{2c+c^3}{3}z^3}\right) \\ &= 4 - \Re\left(\frac{3 + 2cre^{i\theta} + \frac{(2c^2+1)}{3}r^2e^{2i\theta}}{1 + cre^{i\theta} + \frac{2c^2+1}{3}r^2e^{2i\theta} + \frac{2c+c^3}{3}r^3e^{3i\theta}}\right). \end{aligned}$$

By using *Mathematica (version 8.0)*, we find that the expression in (2.5) takes on its minimum value for $\theta = \pi$. This yields

$$\Re\left(\frac{zf'_{c,4}(z)}{f_{c,4}(z)}\right) \geq 4 - \frac{3 - 2cr + \frac{(2c^2+1)}{3}r^2}{1 - cr + \frac{2c^2+1}{3}r^2 - \frac{2c+c^3}{3}r^3} \quad (0 < r \leq r_0).$$

Now, let the function $h(r)$ be given by

$$h(r) = 4 - \frac{3 - 2cr + \frac{(2c^2+1)}{3}r^2}{1 - cr + \frac{2c^2+1}{3}r^2 - \frac{2c+c^3}{3}r^3} \quad (0 < r \leq r_0).$$

Since $0 = h(r_0) \leq h(r) \leq 1$ for

$$r_0 = \frac{1 + 2c^2 + T(c) + \frac{1-4c^2(3+c^2)}{T(c)}}{4c(2+c^2)}$$

where

$$T(c) = \sqrt[3]{1 + 78c^2 + 48c^4 + 8c^6 + 4c(2+c^2)\sqrt{8c^6 + 52c^4 + 87c^2 + 3}},$$

we readily have

$$\Re\left(\frac{zf'_{c,4}(z)}{f_{c,4}(z)}\right) > \alpha\left(c,r\right) = 4 - \frac{3-2cr+\frac{(2c^2+1)}{3}r}{1-cr+\frac{2c^2+1}{3}r-\frac{2c+c^3}{3}r}$$

which completes the proof of Theorem 2.2. □

Corollary 2.3 *Radius of starlikeness of order α of the special cases of function $f_{c,3}$ is as follows:*

i $f_{\frac{1}{3},3}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{\frac{1}{3}r-2}{\frac{11}{27}r^2-\frac{1}{3}r+1}+3$ $(0 < r \le 0.1063)$

or $\alpha = \frac{3321-2255r^2-2\sqrt{451}\sqrt{729+513r^2+121r^4}}{2214-902r^2}$ $(0.106\,3 < r \le 0.9017)$

ii $f_{1,3}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{r-2}{r^2-r+1}+3$ $(0 < r \le 0.1459)$

or $\alpha = \frac{9-15r^2-2\sqrt{3}\sqrt{1+r^2+r^4}}{6(1-r^2)}$ $(0.145\,9 < r \le 0.569\,16)$

iii $f_{\sqrt{2},3}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{\sqrt{2}r-2}{\frac{5}{3}r^2-\sqrt{2}r+1}+3$ $(0 < r \le 0.1282)$

or $\alpha = \frac{63-175r^2-\sqrt{70}\sqrt{9+12r^2+25r^4}}{42-70r^2}$ $(0.128\,2 < r \le 0.439\,01)$

iv $f_{2,3}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{2r-2}{3r^2-2r+1}+3$ $(0 < r \le 0.1031)$

or $\alpha = \frac{6-30r^2-\sqrt{6}\sqrt{1+2r^2+9r^4}}{4(1-3r^2)}$ $(0.103\,1 < r \le 0.326\,1)$

v $f_{3,3}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{3r-2}{\frac{19}{3}r^2-3r+1}+3$ $(0 < r \le 0.07453)$

or $\alpha = \frac{63-665r^2-2\sqrt{19}\sqrt{9+33r^2+361r^4}}{42-266r^2}$ $(0.0745 < r \le 0.2239)$

vi $f_{\pi,3}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{\pi r-2}{\left(\frac{2\pi^2}{3}+\frac{1}{3}\right)r^2-\pi r+1}+3$ $(0 < r \le 0.07159)$

or $\alpha = \frac{5}{2}+\frac{3}{(1+2\pi^2)r^2-3}+\frac{(1+2\pi^2)\sqrt{9+3(2+\pi^2)r^2+(1+2\pi^2)^2r^4}}{((1+2\pi^2)r^2-3)\sqrt{4+13\pi^2+10\pi^4}}$ $(0.07159 < r \le 0.214\,32)$

vii $f_{4,3}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{4r-2}{11r^2-4r+1}+3$ $(0 < r \le 0.05758)$

or $\alpha = \frac{21-385r^2-\sqrt{77}\sqrt{1+6r^2+121r^4}}{14-154r^2}$ $(0.05758 < r \le 0.169\,78)$.

Corollary 2.4 *Radius of starlikeness of order α of the special cases of function $f_{c,4}$ is as follows:*

i $f_{\frac{1}{3},4}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = 4-\frac{\frac{11}{27}r^2-\frac{2}{3}r+3}{-\frac{19}{81}r^3+\frac{11}{27}r^2-\frac{1}{3}r+1}$ $(0 < r \le 1.357\,6)$

ii $f_{1,4}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{r^2-2r+3}{r^3-r^2+r-1}+4$ $(0 < r \le 0.605\,8)$

iii $f_{\sqrt{2},4}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{\frac{5}{3}r^2-2\sqrt{2}r+3}{\frac{4}{3}\sqrt{2}r^3-\frac{5}{3}r^2+\sqrt{2}r-1}+4$ $(0 < r \le 0.467\,4)$

iv $f_{2,4}\left(z\right) \in \mathcal{S}^*(\alpha)$ for $\alpha = 4-\frac{3r^2-4r+3}{-4r^3+3r^2-2r+1}$ $(0 < r \le 0.354\,5)$

v $f_{3,4}(z) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{\frac{19}{3}r^2-6r+3}{11r^3-\frac{19}{3}r^2+3r-1}+4$ $(0 < r \leq 0.25)$

vi $f_{\pi,4}(z) \in \mathcal{S}^*(\alpha)$ for $\alpha = \frac{\left(\frac{\pi^2+1}{3}\right)r^2-2\pi r+3}{\left(\frac{\pi^2+1}{3}\right)(\pi r^3+r^2)+\pi r-1}+4$ $(0 < r \leq 0.2398)$

vii $f_{4,4}(z) \in \mathcal{S}^*(\alpha)$ for $\alpha = 4 - \frac{11r^2-8r+3}{-24r^3+11r^2-4r+1}$ $(0 < r \leq 0.1921)$.

2.2. Radii for convexity of functions $f_{c,3}$ and $f_{c,4}$

In this subsection, we discuss radius problems for the partial sums $f_{c,k}$ $(k = 3, 4)$ to be in the class $\mathcal{C}(\alpha)$.

Theorem 2.5 *Let $c > 0$. Then, $f_{c,3}(z) \in \mathcal{C}(\alpha)$, where*

$$\alpha := \alpha(c,r) = \begin{cases} 3 - \frac{2(1-cr)}{1-2cr+(2c^2+1)r^2} & \text{if } 0 < r \leq \frac{2+3c^2-\sqrt{4+11c^2+7c^4}}{c(2c^2+1)} \\ 2 - \frac{1+c^2+(1+(3+2c^2)c^2)r^2+(1+2c^2)P(r,c)}{2(1+c^2)(1-(1+2c^2)r^2)} & \text{if } \frac{2+3c^2-\sqrt{4+11c^2+7c^4}}{c(2c^2+1)} < r \leq \sqrt{\frac{8+7c^2}{24+71c^2+46c^4}} \end{cases}. \tag{2.6}$$

Here,

$$P(r,c) = \sqrt{((1+c^2)(1+2r^2+(1+2c^2)^2r^4))/(1+2c^2)}.$$

Proof We consider $\alpha := \alpha(c,r)$ such that

$$\Re\left(1+\frac{zf''_{c,3}(z)}{f'_{c,3}(z)}\right) = \Re\left(3-\frac{2cz+2}{1+2cz+(2c^2+1)z^2}\right) > \alpha \tag{2.7}$$

for

$$0 < r \leq \frac{2+3c^2}{c(2c^2+1)} - \sqrt{\frac{4+11c^2+7c^4}{c^2(2c^2+1)^2}}.$$

Following from (2.7), we obtain

$$\begin{aligned} & \Re\left(\frac{cz+1}{1+2cz+(2c^2+1)z^2}\right) \\ = \quad & \frac{1}{2}+\frac{\left(1-\left(2c^2+1\right)r^2\right)\left(1+\left(2c^2+1\right)r^2+2cr\cos\theta\right)}{2\left[1-2r^2+(2c^2+1)^2r^4+4cr\cos\theta+4cr^3\left(2c^2+1\right)\cos\theta+4r^2(2c^2+1)\cos^2\theta\right]} \\ < \quad & \frac{3-\alpha}{2} \quad \left(z = re^{i\theta}\right) \end{aligned}$$

and consequently

$$\frac{\left(1-\left(2c^2+1\right)r^2\right)\left(1+\left(2c^2+1\right)r^2+2cr\cos\theta\right)}{1-2r^2+(2c^2+1)^2r^4+4cr\cos\theta+4cr^3\left(2c^2+1\right)\cos\theta+4r^2(2c^2+1)\cos^2\theta} < 2-\alpha. \tag{2.8}$$

Now, we let the function $g_c(t)$ be given by

$$g_c(t) = \frac{\left(1-\left(2c^2+1\right)r^2\right)\left(1+\left(2c^2+1\right)r^2+2crt\right)}{1-2r^2+(2c^2+1)^2r^4+4cr\left(1+(2c^2+1)r^2\right)t+4r^2(2c^2+1)t^2}, \quad (t = \cos\theta), \tag{2.9}$$

so that we have

$$g_c'(t) = -2r\left[1-\left(2c^2+1\right)r^2\right] \times \frac{\left[c+2c\left(4c^2+3\right)r^2+c\left(2c^2+1\right)^2r^4+4r\left(2c^2+1\right)\left(1+\left(2c^2+1\right)r^2\right)t+4cr^2\left(2c^2+1\right)t^2\right]}{\left[1-2r^2+(2c^2+1)^2r^4+4crt+4cr^3\left(2c^2+1\right)t+4r^2(2c^2+1)t^2\right]^2}. \tag{2.10}$$

By setting

$$h_c(t) = c+2c\left(4c^2+3\right)r^2+c\left(2c^2+1\right)^2r^4+4r\left(2c^2+1\right)\left(1+\left(2c^2+1\right)r^2\right)t+4cr^2\left(2c^2+1\right)t^2, \tag{2.11}$$

we see that

(i) $h_c(t)<0 \Longrightarrow g_c'(t)>0$,

(ii) $h_c(t)>0 \Longrightarrow g_c'(t)<0$,

(iii) $h_c(t)=0$ for $t=t_1$ and $t=t_2$ $(t_1>t_2)$ where $t_2<-1$. Furthermore, since

$$t_1 = \frac{-\left(1+\left(2c^2+1\right)r^2\right)+\sqrt{\frac{1+c^2}{1+2c^2}\left(1+2r^2+(2c^2+1)^2r^4\right)}}{2cr} < 0, \tag{2.12}$$

we consider the radius r in the two cases $t_1 \le -1$ and $-1<t_1<1$.

Case I: Taking

$$0<r\le \frac{2+3c^2-\sqrt{4+11c^2+7c^4}}{c\left(2c^2+1\right)}$$

implies that $t_1 \le -1$, so that $h(t)\ge 0$. Consequently, we conclude that

$$g_c(t) \le g_c(-1) = \frac{1-\left(2c^2+1\right)r^2}{1-2cr+\left(2c^2+1\right)r^2} \le 2-\alpha. \tag{2.13}$$

That is,

$$\alpha = 2-\frac{1-\left(2c^2+1\right)r^2}{1-2cr+\left(2c^2+1\right)r^2} = 3-\frac{2(1-cr)}{1-2cr+\left(2c^2+1\right)r^2}.$$

Thus, we have

$$\Re\left(1+\frac{zf_{c,3}''(z)}{f_{c,3}'(z)}\right) > \alpha$$

and

$$\alpha = 3-\frac{2(1-cr)}{1-2cr+\left(2c^2+1\right)r^2} \qquad \left(0<r\le\frac{2+3c^2-\sqrt{4+11c^2+7c^4}}{c\left(2c^2+1\right)}\right). \tag{2.14}$$

Case II: If we take

$$\frac{2+3c^2-\sqrt{4+11c^2+7c^4}}{c\left(2c^2+1\right)} < r \le \sqrt{\frac{8+7c^2}{24+71c^2+46c^4}}$$

then the case $-1 < t_1 < 1$ holds true, so that

$$g_c(t) \le g_c(t_1) = \frac{1+c^2+(1+(3+2c^2)c^2)r^2+(1+2c^2)P(r,c)}{2(1+c^2)(1-(1+2c^2)r^2)}.$$

Therefore, we obtain that

$$\alpha = 2 - \frac{1+c^2+(1+(3+2c^2)c^2)r^2+(1+2c^2)P(r,c)}{2(1+c^2)(1-(1+2c^2)r^2)}$$

for $\frac{2+3c^2-\sqrt{4+11c^2+7c^4}}{c(2c^2+1)} < r \le \sqrt{\frac{8+7c^2}{24+71c^2+46c^4}}$, which evidently completes the proof of Theorem 2.5. □

Theorem 2.6 *Let $c>0$. Then, the partial sum $f_{c,4}(z) \in \mathcal{C}(\alpha)$, where*

$$\alpha := \alpha(r) = 4 - \frac{3-4cr+(2c^2+1)r^2}{1-2cr+(2c^2+1)r^2-\frac{4(2c+c^3)}{3}r^3},$$

for

$$0 < r \le \frac{3+6c^2+M(c)+\frac{9-4c^2(23+7c^2)}{M(c)}}{16c(2+c^2)}.$$

Here,

$$M(c) = \sqrt[3]{27+6c^2(187+70c^2+4c^4)+16c(2+c^2)\sqrt{88c^6+572c^4+954c^2+81}}.$$

Proof For $f_{c,4}(z) = z+cz^2+\frac{2c^2+1}{3}z^3+\frac{2c+c^3}{3}z^4$, we have

$$\begin{aligned}\Re\left(1+\frac{zf''_{c,4}(z)}{f'_{c,4}(z)}\right) &= \Re\left(1+\frac{2cz+2(2c^2+1)z^2+4\left(2c+c^3\right)z^3}{1+2cz+(2c^2+1)z^2+\frac{4(2c+c^3)}{3}z^3}\right) \\ &= 4-\Re\left(\frac{3+4cz+(2c^2+1)z^2}{1+2cz+(2c^2+1)z^2+\frac{4(2c+c^3)}{3}z^3}\right) \\ &= 4-\Re\left(\frac{3+4cre^{i\theta}+(2c^2+1)r^2e^{2i\theta}}{1+2cre^{i\theta}+(2c^2+1)r^2e^{2i\theta}+\frac{4(2c+c^3)}{3}r^3e^{3i\theta}}\right).\end{aligned} \tag{2.15}$$

By using *Mathematica (version 8.0)*, we find that the expression in (2.15) takes on its minimum value for $\theta=\pi$. This yields

$$\Re\left(1+\frac{zf''_{c,4}(z)}{f'_{c,4}(z)}\right) \ge 4-\frac{3-2cr+(2c^2+1)r^2}{1-2cr+(2c^2+1)r^2-\frac{4(2c+c^3)}{3}r^3} \quad (0<r\le r_0)$$

Now, we let the function $h(r)$ be given by

$$h(r) = 4-\frac{3-2cr+(2c^2+1)r^2}{1-2cr+(2c^2+1)r^2-\frac{4(2c+c^3)}{3}r^3} \quad (0<r\le r_0).$$

Since $0 = h(r_0) \le h(r) \le 1$ for

$$r_0 = \frac{3 + 6c^2 + M(c) + \frac{9-4c^2(23+7c^2)}{M(c)}}{16c(2+c^2)}$$

where

$$M(c) = \sqrt[3]{27 + 6c^2(187 + 70c^2 + 4c^4) + 16c(2+c^2)\sqrt{88c^6 + 572c^4 + 954c^2 + 81}},$$

we readily have

$$\Re\left(1 + \frac{zf''_{c,4}(z)}{f'_{c,4}(z)}\right) > \alpha(r) = 4 - \frac{3 - 4cr + (2c^2+1)r^2}{1 - 2cr + (2c^2+1)r^2 - \frac{4(2c+c^3)}{3}r^3}$$

which completes the proof of Theorem 2.2. □

Corollary 2.7 *Radius of convexity of order α of the special cases of function $f_{c,3}$ is as follows:*

i $f_{\frac{1}{3},3}(z) \in \mathcal{C}(\alpha)$ for $\alpha = \frac{\frac{2}{3}r-2}{\frac{11}{9}r^2-\frac{2}{3}r+1} + 3$ $(0 < r \le 0.07188)$

or $\alpha = \frac{270-550r^2-\sqrt{110}\sqrt{81+162r^2+121r^4}}{20(9-11r^2)}$ $(0.07188 < r \le 0.52004)$

ii $f_{1,3}(z) \in \mathcal{C}(\alpha)$ for $\alpha = \frac{2r-2}{3r^2-2r+1} + 3$ $(0 < r \le 0.103\,19)$

or $\alpha = \frac{6-30r^2-\sqrt{6}\sqrt{1+2r^2+9r^4}}{4(1-3r^2)}$ $(0.103\,19 < r \le 0.32616)$

iii $f_{\sqrt{2},3}(z) \in \mathcal{C}(\alpha)$ for $\alpha = \frac{2\sqrt{2}r-2}{5r^2-2\sqrt{2}r+1} + 3$ $(0 < r \le 0.092\,14)$

or $\alpha = \frac{9-75r^2-\sqrt{15}\sqrt{1+2r^2+25r^4}}{6(1-5r^2)}$ $(0.092\,14 < r \le 0.25071)$

iv $f_{2,3}(z) \in \mathcal{C}(\alpha)$ for $\alpha = \frac{4r-2}{9r^2-4r+1} + 3$ $(0 < r \le 0.07504)$

or $\alpha = \frac{3\left(5-75r^2-\sqrt{5}\sqrt{1+2r^2+81r^4}\right)}{10(1-9r^2)}$ $(0.07504 < r \le 0.18569)$

v $f_{3,3}(z) \in \mathcal{C}(\alpha)$ for $\alpha = \frac{6r-2}{19r^2-6r+1} + 3$ $(0 < r \le 0.054\,66)$

or $\alpha = \frac{30-950r^2-\sqrt{190}\sqrt{1+2r^2+361r^4}}{20-380r^2}$ $(0.054\,66 < r \le 0.12718)$

vi $f_{\pi,3}(z) \in \mathcal{C}(\alpha)$ for $\alpha = \frac{2\pi r-2}{(2\pi^2+1)r^2-2\pi r+1} + 3$ $(0 < r \le 0.05254)$

or $\alpha = \frac{3-5(1+2\pi^2)r^2-(1+2\pi^2)\sqrt{\frac{1+2r^2+(1+2\pi^2)^2r^4}{1+3\pi^2+2\pi^4}}}{2(1-(1+2\pi^2)r^2)}$ $(0.05254 < r \le 0.12169)$

vii $f_{4,3}(z) \in \mathcal{C}(\alpha)$ for $\alpha = \frac{8r-2}{33r^2-8r+1} + 3$ $(0 < r \le 0.04237)$

or $\alpha = \frac{51-2805r^2-\sqrt{561}\sqrt{1+2r^2+1089r^4}}{34(1-33r^2)}$ $(0.04237 < r \le 0.09631)$.

Corollary 2.8 *Radius of convexity of order α of the special cases of function $f_{c,4}$ is as follows:*

i $f_{\frac{1}{3},4}(z) \in \mathcal{C}(\alpha)$ for $\alpha = 4 - \frac{\frac{11}{9}r^2-\frac{4}{3}r+3}{-\frac{76}{81}r^3+\frac{11}{9}r^2-\frac{2}{3}r+1}$ $(0 < r \le 0.908\,70)$

ii $f_{1,4}(z) \in \mathcal{C}(\alpha)$ for $\alpha = 4 - \frac{3r^2-4r+3}{-4r^3+3r^2-2r+1}$ $(0 < r \leq 0.354\,56)$

iii $f_{\sqrt{2},4}(z) \in \mathcal{C}(\alpha)$ for $\alpha = 4 - \frac{5r^2-4\sqrt{2}r+3}{-\frac{16}{3}\sqrt{2}r^3+5r^2-2\sqrt{2}r+1}$ $(0 < r \leq 0.263\,24)$

iv $f_{2,4}(z) \in \mathcal{C}(\alpha)$ for $\alpha = 4 - \frac{9r^2-8r+3}{-16r^3+9r^2-4r+1}$ $(0 < r \leq 0.193\,34)$

v $f_{3,4}(z) \in \mathcal{C}(\alpha)$ for $\alpha = 4 - \frac{19r^2-12r+3}{-44r^3+19r^2-6r+1}$ $(0 < r \leq 0.132\,71)$

vi $f_{\pi,4}(z) \in \mathcal{C}(\alpha)$ for $\alpha = \frac{(2\pi^2+1)r^2-4\pi r+3}{\frac{4\pi}{3}(2+\pi^2)r^3-(2\pi^2+1)r^2+2\pi r-1} + 4$ $(0 < r \leq 0.127\,03)$

vii $f_{4,4}(z) \in \mathcal{C}(\alpha)$ for $\alpha = 4 - \frac{33r^2-16r+3}{-96r^3+33r^2-8r+1}$ $(0 < r \leq 0.100\,78)$.

2.3. Radii for close-to-convexity of functions $f_{c,3}$ and $f_{c,4}$

In this subsection, we give radius problems for the partial sums $f_{c,k}$ $(k = 3, 4)$ to be in the class $\mathcal{R}(\alpha)$.

Theorem 2.9 *Let $c > 0$. Then, the partial sum $f_{c,3}(z) \in \mathcal{R}(\alpha)$, where*

$$\alpha := \alpha(c, r) = \begin{cases} 1 - 2cr + \left(2c^2+1\right) r^2 & \text{if } 0 < r \leq \frac{c}{2(2c^2+1)} \\ \frac{2+3c^2}{2(2c^2+1)} - \left(2c^2+1\right) r^2 & \text{if } \frac{c}{2(2c^2+1)} \leq r \leq \frac{\sqrt{2+3c^2}}{\sqrt{2}(2c^2+1)} \end{cases} . \tag{2.16}$$

Proof A simple computation gives us that

$$\begin{aligned} \Re(f'_{c,3}(z)) &= \Re\left(1 + 2cz + (2c^2+1)z^2\right) \\ &= 1 - (2c^2+1)r^2 + 2cr\cos\theta + 2(2c^2+1)r^2\cos^2\theta \end{aligned}$$

for $z = re^{i\theta}$. Letting

$$g_c(t) = 1 - (2c^2+1)r^2 + 2crt + 2(2c^2+1)r^2t^2, \quad (t = \cos\theta).$$

We can easily see that

$$g'_c(t) = 2cr + 4(2c^2+1)r^2t = 0$$

for $t_1 = \frac{-c}{2r(2c^2+1)} < 0$. Here, there are two cases according to t_1.

Case I: Let $0 < r \leq \frac{c}{2(2c^2+1)}$. Therefore, we have $t_1 \leq -1$. This implies that

$$\begin{aligned} g_c(t) &\geq g_c(-1) = 1 - 2cr + \left(2c^2+1\right) r^2 := \alpha(c, r) \\ &= \alpha\left(\frac{c}{2(2c^2+1)}\right) = \frac{5c^2+4}{2(2c^2+1)}. \end{aligned}$$

Case II: Let $\frac{c}{2(2c^2+1)} \leq r \leq \frac{\sqrt{2+3c^2}}{\sqrt{2}(2c^2+1)}$. Therefore, we have $-1 < t_1 \leq 1$. Consequently, we conclude that

$$\begin{aligned} g_c(t) &\geq g_c(t_1) = \frac{2+3c^2}{2(2c^2+1)} - \left(2c^2+1\right) r^2 := \alpha(c, r) \\ &= \alpha\left(\frac{\sqrt{2+3c^2}}{\sqrt{2}(2c^2+1)}\right) = 0. \end{aligned}$$

This completes the proof of the theorem. □

Reasoning along the same lines as in the proof of the Theorem 2.1 for the $f_{c,4}(z)$, we obtain the following theorem. We omit the details.

Theorem 2.10 *Let* $0 < c \leq \sqrt{\frac{3\sqrt{2}-4}{2}} = 0.348\,31$. *Then, the partial sum* $f_{c,4}(z) \in \mathcal{R}(\alpha)$, *where*

$$\alpha := \alpha(c,r) = 1 - (2c^2+1)r^2 - 2cr\left(1 - 2(2+c^2)r^2\right) + 2(2c^2+1)r^2 - \frac{16cr^3(2+c^2)}{3}$$

for

$$0 < r \leq \frac{1 + 2c^2 - \sqrt{1 - 8c^2 - 2c^4}}{6c\,(c^2+2)}.$$

Corollary 2.11 *Radius of close-to-convexity of order* α *of the special cases of function* $f_{c,3}$ *is as follows.*

i $f_{\frac{1}{3},3}(z) \in \mathcal{R}(\alpha)$ for $\alpha = \frac{11}{9}r^2 - \frac{2}{3}r + 1$ $(0 < r \leq 0.13636)$ or $\alpha = \frac{21}{22} - \frac{11}{9}r^2$ $(0.13636 \leq r \leq 0.88374)$

ii $f_{1,3}(z) \in \mathcal{R}(\alpha)$ for $\alpha = 3r^2 - 2r + 1$ $(0 < r \leq 0.16667)$ or $\alpha = \frac{5}{6} - 3r^2$ $(0.16667 \leq r \leq 0.52705)$

iii $f_{\sqrt{2},3}(z) \in \mathcal{R}(\alpha)$ for $\alpha = 5r^2 - 2\sqrt{2}r + 1$ $(0 < r \leq 0.14142)$ or $\alpha = \frac{4}{5} - 5r^2$ $(0.14142 \leq r \leq 0.4)$

iv $f_{2,3}(z) \in \mathcal{R}(\alpha)$ for $\alpha = 9r^2 - 4r + 1$ $(0 < r \leq 0.11111)$ or $\alpha = \frac{7}{9} - 9r^2$ $(0.11111 \leq r \leq 0.29397)$

v $f_{3,3}(z) \in \mathcal{R}(\alpha)$ for $\alpha = 19r^2 - 6r + 1$ $(0 < r \leq 0.078947)$ or $\alpha = \frac{29}{38} - 19r^2$ $(0.078947 \leq r \leq 0.20042)$

vi $f_{\pi,3}(z) \in \mathcal{R}(\alpha)$ for $\alpha = \left(2\pi^2+1\right)r^2 - 2\pi r + 1$ $(0 < r \leq 0.07574)$
or $\alpha = \frac{3\pi^2+2}{4\pi^2+2} - r^2\left(2\pi^2+1\right)$ $(0.07574 \leq r \leq 0.19169)$

vii $f_{4,3}(z) \in \mathcal{R}(\alpha)$ for $\alpha = 33r^2 - 8r + 1$ $(0 < r \leq 0.0606)$ or $\alpha = \frac{25}{33} - 33r^2$ $(0.0606 \leq r \leq 0.15152)$.

Corollary 2.12 *The partial sum* $f_{\frac{1}{3},4}(z) \in \mathcal{R}(\alpha)$, *where*

$$\alpha := \alpha(c,r) = \frac{2}{3}r\left(\frac{38}{9}r^2 - 1\right) + \frac{11}{9}r^2 - \frac{304}{81}r^3 + 1$$

for

$$0 < r \leq 0.219\,85.$$

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