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On H-curvature of (α,β)-metrics

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Abstract: The non-Riemannian quantity $\mathbf{H}$ was introduced by Akbar-Zadeh to characterization of Finsler metrics of constant flag curvature. In this paper, we study two important subclasses of Finsler metrics in the class of so-called (α,β)-metrics, which are defined by $F=\alpha\varphi(s)$, $s=\beta/\alpha$, where α is a Riemannian metric and β is a closed 1-form on a manifold. We prove that every polynomial metric of degree $k\geq 3$ and exponential metric has almost vanishing $\mathbf{H}$-curvature if and only if $\mathbf{H}=0$. In this case, F reduces to a Berwald metric. Then we prove that every Einstein polynomial metric of degree $k\geq 3$ and exponential metric satisfies $\mathbf{H}=0$. In this case, F is a Berwald metric.

Key words: Polynomial metrics, exponential metric, almost vanishing $\mathbf{H}$-curvature.

1. Introduction

Let (M,F) be a Finsler manifold. Then a global vector field $\mathbf{G}$ is induced by the Finsler metric F on slit tangent bundle TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by $\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i}$, where $G^i=G^i(x,y)$ are scalar functions on TM_0 [8].

In [1], Akbar-Zadeh considered a non-Riemannian quantity $\mathbf{H}$ obtained from the mean Berwald curvature $\mathbf{E}$ by the covariant horizontal differentiation along geodesics. He proved that for a Weyl metric, the flag curvature $\mathbf{K}$ is a scalar function on the manifold $\mathbf{K}=\mathbf{K}(x)$ if and only if $\mathbf{H}=\mathbf{0}$ [7]. The quantity $\mathbf{H}_y=H_{ij}dx^i\otimes dx^j$ is defined as the covariant derivative of mean Berwald curvature along geodesics. In local coordinates,

$$H_{ij}=\frac{1}{2}\Big[y^m\frac{\partial^4G^k}{\partial y^i\partial y^j\partial y^k\partial x^m}-2G^m\frac{\partial^4G^k}{\partial y^i\partial y^j\partial y^k\partial y^m}-\frac{\partial G^m}{\partial y^i}\frac{\partial^3G^k}{\partial y^j\partial y^k\partial y^m}-\frac{\partial G^m}{\partial y^j}\frac{\partial^4G^k}{\partial y^i\partial y^k\partial y^m}\Big].$$

A Finsler metric F on an n-dimensional manifold M is called of almost vanishing H-curvature if

$$\mathbf{H}=\frac{n+1}{2}F^{-1}\theta\mathbf{h}, \tag{1.1}$$

where $\theta:=\theta_i(x)y^i$ is a 1-form on M and $\mathbf{h}=h_{ij}dx^i\otimes dx^j$ is the angular tensor [7].

Najafi et al. [6] proved that every R-quadratic metric satisfies $\mathbf{H}=0$. Then, Najafi et al. [7] generalized the Akbar-Zadeh theorem and proved that a Finsler metric F has almost isotropic flag curvature $\mathbf{K}=3\theta/F+\sigma$ if and only if it has almost vanishing H-curvature, where $\theta=\theta_i(x)y^i$ is a 1-form and $\sigma=\sigma(x)$ is a scalar

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function on manifold. Mo [4] found a new equation between H-curvature and Riemannian curvature on a Finsler manifold. Tayebi and Najafi [13] showed that every m-th root metric with almost vanishing H-curvature satisfies $\mathbf{H}=0$. Moreover, Xia [23] proved that a Randers metric has almost isotropic S-curvature if and only if it is of almost vanishing H-curvature. Recently, Zohrehvand and Rezaii [26] have obtained necessary and sufficient conditions for a square metric to be of almost vanishing H-curvature.

Randers metric and square metric belong to the class of (α,β)-metrics. Therefore, in order to find explicit examples of Finsler metrics of almost vanishing H-curvature, we consider (α,β)-metrics. An (α,β)-metric is a Finsler metric of the form $F:=\alpha\varphi(s)$, $s=\beta/\alpha$, where $\varphi=\varphi(s)$ is a C^∞ function on $(-b_0,b_0)$ with certain regularity, $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on M (see [9, 14, 16, 18–20, 22]). A polynomial (α,β)-metric of degree k is given by $\varphi:=(1+s)^k$, $s=\beta/\alpha$, where $k\in\mathbb{N}$. This class of metrics contains Randers metrics ($k=1$) and square metrics ($k=2$) as special cases. In this paper, we consider polynomial (α,β)-metrics with almost vanishing H-curvature and prove the following.

Theorem 1.1 *Let $F=\alpha\varphi(s)$, $s=\beta/\alpha$, be a polynomial (α,β)-metric of degree m $(m\geq 3)$ on an n-dimensional manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a closed 1-form on M. Then F has almost vanishing H-curvature if and only if $\mathbf{H}=0$. In this case, F is a Berwald metric.*

Example 1.2 *Theorem 1.1 does not hold for the polynomial (α,β)-metrics of degree 1, i.e. the Randers-type Finsler metrics. For example, the standard Funk metric on the Euclidean unit ball is defined by*

$$F(x,y):=\frac{\sqrt{|y|^2-(|x|^2|y|^2-<x,y>^2)}}{1-|x|^2}+\frac{<x,y>}{1-|x|^2},\quad y\in T_xB^n(1)\simeq\mathbb{R}^n,$$

where $<,>$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. F is a Randers metric and it is easy to see that β is a closed 1-form. By a simple calculation, it follows that $\mathbf{H}=0$ while F is not Berwald metric.

For an (α,β)-metric $F:=\alpha\varphi(s)$, $s=\beta/\alpha$, let us define $b_{i|j}$ by

$$b_{i|j}\theta^j:=db_i-b_j\theta_i^j,$$

where $\theta^i:=dx^i$ and $\theta_i^j:=\Gamma^j_{ik}dx^k$ denote the Levi-Civita connection forms of α. Let

$$r_{ij}:=\tfrac{1}{2}\big(b_{i|j}+b_{j|i}\big),\quad s_{ij}:=\tfrac{1}{2}\big(b_{i|j}-b_{j|i}\big),$$

$$r_j:=b^ir_{ij},\quad r:=b^ib^jr_{ij},\quad s_j:=b^is_{ij},\quad r_0:=r_jy^j,\quad s_0:=s_jy^j,$$

$$r_{i0}:=r_{ij}y^j,\quad r_{00}:=r_{ij}y^iy^j,\quad s_{i0}:=s_{ij}y^j,\quad s^i{}_j:=a^{im}s_{mj},\quad r^i{}_j:=a^{im}r_{mj},$$

$$q_{ij}:=r_{im}s^m_{\ j},\quad t_{ij}:=s_{im}s^m_{\ j},\quad q_i:=b^iq_{ij},\quad t_i:=b^it_{ij}.$$

Now, we can give another example.

Example 1.3 *Theorem 1.1 does not hold for the polynomial (α,β)-metrics of degree 2, generally. For example, let $F=(\alpha+\beta)^2/\alpha$ be the square metric defined by following*

$$\alpha:=\frac{\sqrt{|y|^2(1-|x|^2)+<x,y>^2}}{(1-|x|^2)^2},\quad \beta:=\frac{<x,y>}{(1-|x|^2)^2}.\tag{1.2}$$

F is an (α,β)-metric on the unit ball $\mathbb{B}^n(1)\subset\mathbb{R}^n$. We have

$$b_{i|j}=2\tau\{(1+2b^2)a_{ij}-3b_ib_j\},$$

where $\tau=(1-|x|^2)/2$. Thus, β is closed with respect to α. F has constant flag curvature then it satisfies $\mathbf{H}=0$. Since β is not parallel with respect to α, then F is not a Berwald metric.

Example 1.4 *For polynomial (α,β)-metric $F=(\alpha+\beta)^3/\alpha^2$, we have*

$$\Theta=\frac{3(4\,s-1)}{2(8\,s^2-6B+s-1)},\quad Q=\frac{3}{1-2s},\quad \Psi=\frac{3}{-8\,s^2+6\,B-s+1}.$$

Suppose that F has almost vanishing H-curvature (1.1). Since β is a closed 1-form, then

$$\begin{aligned}
2H_{jk}= &\Big[\frac{h_3}{A^6\alpha^4}r_{00}^2+\frac{h_4}{A^4\alpha^3}r_{00|0}+\frac{h_5}{A^5\alpha^3}r_{00}r_0+\frac{h_6}{A^3\alpha^2}r_{0|0}+\frac{h_{13}}{A^4\alpha^2}rr_{00}+\frac{h_{14}}{A^4\alpha^2}r_0^2\Big]b_jb_k\\
&+\Big[\frac{h_{17}}{A^6\alpha^4}r_{00}^2+\frac{h_{18}}{A^4\alpha^3}r_{00|0}+\frac{h_{19}}{A^5\alpha^3}r_{00}r_0+\frac{h_{20}}{A^3\alpha^2}r_{0|0}+\frac{h_{27}}{A^4\alpha^2}rr_{00}+\frac{h_{28}}{A^4\alpha^2}r_0^2\Big]l_jl_k\\
&+\Big[\frac{h_{31}}{A^5\alpha^4}r_{00}^2+\frac{h_{32}}{A^3\alpha^3}r_{00|0}+\frac{h_{33}}{A^4\alpha^3}r_{00}r_0+\frac{h_{34}}{A^2\alpha^2}r_{0|0}+\frac{h_{41}}{A^3\alpha^2}rr_{00}+\frac{h_{42}}{A^3\alpha^2}r_0^2\Big]a_{jk}\\
&+\frac{h_{43}}{A^2\alpha}r_{jk|0}+\Big[\frac{h_{45}}{A^4\alpha^2}r_{00}+\frac{h_{46}}{A^3\alpha}r_0\Big]r_{jk}+\frac{h_{48}}{A^4\alpha^2}r_{0j}r_{0k}+\Big[\frac{h_{51}}{A^6\alpha^4}r_{00}^2+\frac{h_{52}}{A^4\alpha^3}r_{00|0}\\
&+\frac{h_{53}}{A^5\alpha^3}r_{00}r_0+\frac{h_{54}}{A^3\alpha^2}r_{0|0}+\frac{h_{61}}{A^4\alpha^2}rr_{00}+\frac{h_{62}}{A^4\alpha^2}r_0^2\Big]\Big[l_kb_j+l_jb_k\Big]\\
&+\Big[\frac{h_{71}}{A^4\alpha^2}r_{00}+\frac{h_{72}}{A^3\alpha}r_0\Big]\Big[l_kr_j+l_jr_k\Big]+\Big[\frac{h_{74}}{A^5\alpha^3}r_{00}+\frac{h_{75}}{A^4\alpha^2}r_0+\frac{h_{76}}{A^3\alpha}r\Big](l_kr_{0j}+l_jr_{0k})\\
&+\Big[\frac{h_{89}}{A^4\alpha^2}r_{00}+\frac{h_{90}}{A^3\alpha}r_0\Big]\Big[b_kr_j+b_jr_k\Big]+\Big[\frac{h_{92}}{A^5\alpha^3}r_{00}+\frac{h_{93}}{A^4\alpha^2}r_0+\frac{h_{94}}{A^3\alpha}r\Big]\Big[b_kr_{0j}+b_jr_{0k}\Big]\\
&+\frac{h_{104}}{A^3\alpha}\Big[r_kr_{j0}+r_jr_{k0}\Big]+\frac{h_{106}}{A^3\alpha^2}\Big[l_kr_{0j|0}+l_jr_{0k|0}\Big]+\frac{h_{107}}{A^3\alpha^2}\Big[r_{0j|0}b_k+r_{0k|0}b_j\Big]\\
&+\frac{h_{108}}{A^2\alpha}\Big[r_{k|0}b_j+r_{j|0}b_k\Big]+\frac{h_{109}}{A^2\alpha}\Big[l_kr_{j|0}+l_jr_{k|0}\Big],
\end{aligned}$$

where $A:=1+6B+6Bs-9s^2-8s^3$, $B:=\|\beta\|_\alpha=\sqrt{b^ib_i}$ and $h_i(i=1,2,\cdots,109)$ are the polynomials of variations s and B. By using Lemma 3.1, it follows that β satisfies $r_{ij}=0$ and then it is parallel with respect to α. In this case, F reduces to a Berwald metric.

A Finsler metric $F=F(x,y)$ on an n-dimensional manifold M is called an Einstein metric if its Ricci curvature satisfies $\mathbf{Ric}=(n-1)\lambda F^2$, where $\lambda=\lambda(x)$ is a scalar function on M. In [2], it is proved that every Einstein polynomial (α,β)-metric is Ricci-flat. In this paper, we prove the following.

Theorem 1.5 *Let $F=\alpha\varphi(s)$, $s=\beta/\alpha$, be a polynomial (α,β)-metric of degree m $(m\geq 3)$ on an n-dimensional manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a closed 1-form on M. Suppose that F is an Einstein metric. Then $\mathbf{H}=0$. In this case, F is a Berwald metric.*

Example 1.6 *The Funk metric is an Einstein metric with a closed 1-form. It satisfies $\mathbf{H}=0$ while it is not a Berwald metric. Then Theorem 1.5 does not hold for the polynomial (α,β)-metrics of degree 1.*

Example 1.7 *The square metric in Example 1.3 is a Ricci-flat Finsler metric. Moreover, F is an Einstein metric. However, F is not a Berwald metric. Thus, Theorem 1.5 does not hold for the polynomial (α, β)-metrics of degree 2, generally.*

Example 1.8 *Let $\varphi(s) = (1+s)^3$ be an Einstein metric. By the Theorem 1.1 in [2], F is Ricci-flat. Suppose that β is a closed 1-form. Then $R^m_m =^\alpha R^m_m + T^m_m = 0$, where ${}^\alpha R^m_m$ denotes the Riemannian curvature of α and*

$$T^m_m = \Big[(n-1)\frac{c_1}{A^3} + \frac{c_2}{A^4}\Big]\frac{r_{00}^2}{\alpha^2} + \frac{1}{\alpha}\Bigg[r_0\Big[(n-1)\frac{c_5}{A^2} + \frac{c_6}{A^3}\Big]r_{00} + \Big[(n-1)\frac{c_7}{A} + \frac{c_8}{A^2}\Big]r_{00|0}\Bigg]$$
$$+ \frac{c_{11}}{A^2}(rr_{00} - r_0^2) + \frac{c_{14}}{A}(r_{00}r^m_m - r_{0m}r^m_0 + r_{00|m}b^m - r_{0m|0}b^m),$$

$A := 1 + 6B + 6Bs - 9s^2 - 8s^3$ and c_i $(i = 1, \cdots, 14)$ are polynomials of variations s and B (see [2] for the corrected version of [25]). It follows that $r_{ij} = 0$. Since β is a closed 1-form, then it is parallel with respect to α. In this case, F reduces to a Berwald metric.

The exponential metric is another important (α, β)-metric which is given by $\varphi(s) = e^s$, $s = \beta/\alpha$, (see [10, 15, 24]). Here, we consider exponential (α, β)-metrics with almost vanishing $\mathbf{H}$-curvature and prove the following.

Theorem 1.9 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be an exponential metric on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemann ian metric and $\beta = b_i(x)y^i$ is a closed 1-form on M. Then F has almost vanishing $\mathbf{H}$-curvature if and only if $\mathbf{H} = 0$. In this case, F is a Berwald metric.*

Example 1.10 *Let $F = \alpha e^{\beta/\alpha}$ be an exponential metric. At a point $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$ and in the direction $y = (y^1, \cdots, y^n) \in T_x\mathbb{R}^n$, consider the following Riemannian metric α and 1-form β as follows*

$$\alpha(x,y) = \sqrt{(y^1)^2 + e^{2x^1}\big[(y^2)^2 + \cdots + (y^n)^2\big]}, \qquad \beta(x,y) := y^1. \tag{1.3}$$

Then $s_{ij} = 0$. In this case, F has constant S-curvature [5]. Thus, F satisfies $\mathbf{H} = 0$ (see [3] and [5]).

Finally, we consider the Einstein exponential metric and prove the following.

Theorem 1.11 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be an exponential metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a closed 1-form on M. Suppose that F is an Einstein metric. Then $\mathbf{H} = 0$. In this case, F is a Berwald metric.*

Example 1.12 Let $F = \alpha e^{\beta/\alpha}$ be an exponential metric, where α and β are defined by (**??**). Suppose that F is an Einstein metric. Thus, $R^m_m =^\alpha R^m_m + T^m_m$, where

$$T^m_m := \Big\{(n-1)\frac{c_1}{A^3} + \frac{c_2}{A^4}\Big\}\alpha^{-2}r_{00}^2 + \alpha^{-1}\Big\{\big[(n-1)\frac{c_5}{A^2} + \frac{c_6}{A^3}\big]r_{00}r_0 + \big[(n-1)\frac{c_7}{A} + \frac{c_8}{A^2}\big]r_{00|0}\Big\}$$
$$+ \frac{c_{11}}{A^2}(rr_{00} - r_0^2) + \frac{c_{14}}{A}(r_{00}r^m_m - r_{0m}r^m_0 + r_{00|m}b^m - r_{0m|0}b^m), \tag{1.4}$$

$A = 1 + B - s - s^2$ and c_i, $(i = 1, \cdots, 14)$, are polynomials of variations s and B (see [2]). Thus, we get $g_2 r_{00}^2 \equiv 0$, mod(A), where $I_2 \equiv g_2$, mod(A), and g_2 is a polynomial of s and B. By Lemma 3.1, it follows that β is a Killing 1-form. Then β is parallel with respect to α. In this case, F reduces to a Berwald metric.

2. Preliminary

Let (M, F) be a Finsler manifold. A global vector field $\mathbf{G}$ is induced by F is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where $G^i = G^i(x, y)$ are given by

$$G^i = \frac{1}{4} g^{il} \left[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right].$$

$\mathbf{G}$ is called the spray of (M, F). The projection of an integral curve of the spray $\mathbf{G}$ is called a geodesic in M [12, 22].

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called Berwald curvature . F is called a Berwald metric if $\mathbf{B} = 0$.

For $y \in T_xM_0$, define $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{E}_y(u, v) := E_{ij}(y) u^i v^j$, where

$$E_{ij} := \frac{1}{2} B^m_{\ ijm}.$$

$\mathbf{E}$ is called mean Berwald curvature. F is called a weakly Berwald metric if $\mathbf{E} = 0$. By definition, every Berwald metric is a weakly Berwald metric.

For $y \in T_xM_0$, define the linear transformations $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$, where $\mathbf{R}_y(u) := R^i{}_k(y) u^k \frac{\partial}{\partial x^i}$ and

$$R^i{}_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature (see [11, 17, 21]).

The Ricci curvature $\mathbf{Ric}(x, y)$ is the trace of the Riemann curvature defined by

$$\mathbf{Ric}(x, y) := R^m_{\ m}(x, y).$$

A Finsler metric F on an n-dimensional manifold M is called an Einstein metric if the Ricci curvature satisfies

$$\mathbf{Ric} = (n - 1)\sigma F^2, \tag{2.2}$$

where $\sigma = \sigma(x)$ is a scalar function on M.

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First, we remark that the spray coefficients G^i of an (α, β)-metric $F = \alpha\varphi(s)$, $s = \beta/\alpha$, and the spray coefficients of the Riemannian metric α are related by following

$$G^i = G^i_\alpha + Q\alpha s^i{}_0 + (r_{00} - 2Q\alpha s_0)(\Psi b^i + \Theta l^i),$$

where $l^i := \alpha^{-1} y_i$ and

$$Q := \frac{\varphi'}{\varphi - s\varphi'} \quad \Theta := \frac{\varphi\varphi' - s(\varphi\varphi'' + \varphi'\varphi')}{2\varphi\big[(\varphi - s\varphi') + (B^2 - s^2)\varphi''\big]}, \quad \Psi := \frac{\varphi''}{2\big[(\varphi - s\varphi') + (B^2 - s^2)\varphi''\big]}.$$

Moreover, $B := \|\beta\|_\alpha = \sqrt{b^i b_i}$, where $b^i := a^{ij} b_j$.

Lemma 3.1 *Suppose r_{00} of an (α,β)-metric $F = \alpha\varphi(s)$, $s = \beta/\alpha$, on a manifold M satisfies*

$$Ir^2_{00} \equiv 0, \quad mod(as^2 + bs + c), \qquad and \qquad I \not\equiv 0, \quad mod(as^2 + bs + c),$$

where I is a polynomial of B, and s, a, b, and c are polynomials of B and $b \neq 0$. Suppose that r_1 and r_2 are the roots of the equation $as^2 + bs + c = 0$ such that $r_1^2 \neq r_2^2$. Then $r_{ij} = 0$.

Proof The following hold

$$Ir^2_{00} \equiv 0, \quad \mathrm{mod}(s - r_1) \quad \text{and} \quad Ir^2_{00} \equiv 0, \quad \mathrm{mod}(s - r_2). \tag{3.1}$$

Let us put

$$I \equiv f_1 \quad \mathrm{mod}(s - r_1) \quad \text{and} \quad I \equiv f_2 \quad \mathrm{mod}(s - r_2),$$

where f_1 and f_2 are polynomials of B. Then we have

$$f_1 r^2_{00} \equiv 0, \quad \mathrm{mod}(s - r_1) \qquad \text{and} \qquad f_2 r^2_{00} \equiv 0, \quad \mathrm{mod}(s - r_2)$$

which imply that

$$r^2_{00} \equiv 0, \quad \mathrm{mod}(s - r_1) \qquad \text{and} \qquad r^2_{00} \equiv 0, \quad \mathrm{mod}(s - r_2).$$

It follows that

$$r_{00} \equiv 0 \quad \mathrm{mod}(s - r_1) \qquad \text{and} \qquad r_{00} \equiv 0 \quad \mathrm{mod}(s - r_2)$$

Suppose that $r_{00} \neq 0$. Then by the Lemma 4.1 in [25], we get

$$r_{00} = \sigma_1 \alpha^2 (s^2 - r_1^2), \quad \text{and} \quad r_{00} = \sigma_2 \alpha^2 (s^2 - r_2^2), \tag{3.2}$$

where $\sigma_1 = \sigma_1(x)$ and $\sigma_2 = \sigma_2(x)$ are scalar functions on M. By (3.2), we have

$$(\sigma_1 - \sigma_2)\beta^2 + (\sigma_1 r_1^2 - \sigma_2 r_2^2)\alpha^2 = 0.$$

Then $\sigma_1 = \sigma_2$ and $r_1^2 = r_2^2$ which contradict with the assumption. Thus, $r_{00} = 0$. Taking vertical derivations of it twice yields $r_{ij} = 0$. □

Lemma 3.2 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be a polynomial (α,β)-metric of degree m on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Suppose that F has almost vanishing **H**-curvature. Then the following holds*

$$f_1 r_{00} s_0 \alpha + f_2 r^2_{00} + f_3 s_0^2 \alpha^2 \equiv 0, \qquad mod\Big((1 - m^2)s^2 + (2 - m)s + m(m-1)B + 1\Big), \tag{3.3}$$

where f_j, $(j = 1, 2, 3)$ are polynomials of variations s and B and they are homogeneous of degree one with respect to s

Proof For the polynomial metric $\varphi(s) = (1+s)^m$, we have

$$Q = \frac{m}{1+s-sm}, \qquad \Theta = \frac{m\,(1+2s-2sm)}{2(-m^2s^2+s^2-sm+2\,s+1-Bm+m^2B)},$$
$$\Psi = \frac{m(m-1)}{2(-m^2s^2+s^2-sm+2\,s+1-Bm+m^2B)}.$$

By assumption, $F = \alpha\varphi(s)$ has almost vanishing **H**-curvature, i.e. there exists a 1-form θ on M such that

$$H_{jk} = \frac{n+1}{2}\theta F_{y^jy^k}, \tag{3.4}$$

where

$$F_{y^jy^k} = \frac{(1+s)^{m-2}}{\alpha}\Big[\big[1-(m-2)s-(m-1)s^2\big]a_{jk} + (m^2-m)b_jb_k - (m^2-m)(b_jl_k+b_kl_j)s + \big[(m^2-1)s^2+(m-2)s-1\big]l_jl_k\Big] \tag{3.5}$$

$l_i := \alpha_{y^i}$ and

$$\begin{aligned}2H_{jk} = \;& \Big[\frac{h_1}{A^6D^3\alpha^3}r_{00}s_0 + \frac{h_2}{A^4D^3\alpha^2}s_{0|0} + \frac{h_3}{A^6\alpha^4}r_{00}^2 + \frac{h_4}{A^4\alpha^3}r_{00|0} + \frac{h_5}{A^5\alpha^3}r_{00}r_0 + \frac{h_6}{A^3\alpha^2}r_{0|0}\\
&+ \frac{h_7}{A^4D^4\alpha}t_0 + \frac{h_8}{A^6D^4\alpha^2}s_0^2 + \frac{h_9}{A^4D^2\alpha^2}q_{00} + \frac{h_{10}}{A^3D^2\alpha}q_0 + \frac{h_{11}}{A^5D^3\alpha^2}r_0s_0 + \frac{h_{12}}{A^4D^2\alpha}rs_0\\
&+ \frac{h_{13}}{A^4\alpha^2}rr_{00} + \frac{h_{14}}{A^4\alpha^2}r_0^2\Big]b_jb_k + \Big[\frac{h_{15}}{A^6D^3\alpha^3}r_{00}s_0 + \frac{h_{16}}{A^4D^3\alpha^2}s_{0|0} + \frac{h_{17}}{A^6\alpha^4}r_{00}^2 + \frac{h_{18}}{A^4\alpha^3}r_{00|0}\\
&+ \frac{h_{19}}{A^5\alpha^3}r_{00}r_0 + \frac{h_{20}}{A^3\alpha^2}r_{0|0} + \frac{h_{21}}{A^4D^4\alpha}t_0 + \frac{h_{22}}{A^6D^4\alpha^2}s_0^2 + \frac{h_{23}}{A^4D^2\alpha^2}q_{00} + \frac{h_{24}}{A^3D^2\alpha}q_0\end{aligned}$$

$$
\begin{aligned}
&+ \frac{h_{25}}{A^5D^3\alpha^2}r_0s_0 + \frac{h_{26}}{A^4D^2\alpha}rs_0 + \frac{h_{27}}{A^4\alpha^2}rr_{00} + \frac{h_{28}}{A^4\alpha^2}r_0^2\Big]l_jl_k \\
&+ \Big[\frac{h_{29}}{A^5D^2\alpha^3}r_{00}s_0 + \frac{h_{30}}{A^3D^2\alpha^2}s_{0|0} + \frac{h_{31}}{A^5\alpha^4}r_{00}^2 + \frac{h_{32}}{A^3\alpha^3}r_{00|0} + \frac{h_{33}}{A^4\alpha^3}r_{00}r_0 + \frac{h_{34}}{A^2\alpha^2}r_{0|0} \\
&+ \frac{h_{35}}{A^4D^2\alpha^2}r_0s_0 + \frac{h_{36}}{A^5D^3\alpha^2}s_0^2 + \frac{h_{37}}{A^3D^3\alpha}t_0 + \frac{h_{38}}{A^3D\alpha^2}q_{00} + \frac{h_{39}}{A^2D\alpha^2}q_0 + \frac{h_{40}}{A^3D\alpha}rs_0 \\
&+ \frac{h_{41}}{A^3\alpha^2}rr_{00} + \frac{h_{42}}{A^3\alpha^2}r_0^2\Big]a_{jk} + \frac{h_{43}}{A^2\alpha}r_{jk|0} + \Big[\frac{h_{44}}{A^4D\alpha}s_0 + \frac{h_{45}}{A^4\alpha^2}r_{00} + \frac{h_{46}}{A^3\alpha}r_0\Big]r_{jk} \\
&+ \frac{h_{47}}{A^4D^2}s_ks_j + \frac{h_{48}}{A^4\alpha^2}r_{0j}r_{0k} + \Big[\frac{h_{49}}{A^6D^3\alpha^3}r_{00}s_0 + \frac{h_{50}}{A^4D^3\alpha^2}s_{0|0} + \frac{h_{51}}{A^6\alpha^4}r_{00}^2 + \frac{h_{52}}{A^4\alpha^3}r_{00|0} \\
&+ \frac{h_{53}}{A^5\alpha^3}r_{00}r_0 + \frac{h_{54}}{A^3\alpha^2}r_{0|0} + \frac{h_{55}}{A^4D^4\alpha}t_0 + \frac{h_{56}}{A^6D^4\alpha^2}s_0^2 + \frac{h_{57}}{A^4D^2\alpha^2}q_{00} + \frac{h_{58}}{A^3D^2\alpha}q_0 \\
&+ \frac{h_{59}}{A^5D^3\alpha^2}r_0s_0 + \frac{h_{60}}{A^4D^2\alpha}rs_0 + \frac{h_{61}}{A^4\alpha^2}rr_{00} + \frac{h_{62}}{A^4\alpha^2}r_0^2\Big](l_kb_j + l_jb_k) + \Big[\frac{h_{63}}{A^5D^3\alpha}s_0 \\
&+ \frac{h_{64}}{A^5D^2\alpha^2}r_{00} + \frac{h_{65}}{A^4D^2\alpha}r_0 + \frac{h_{66}}{A^3D}r\Big](l_ks_j + l_js_k) + \Big[\frac{h_{67}}{A^4D^4\alpha^2}s_0 + \frac{h_{68}}{A^4D^2\alpha^3}r_{00} \\
&+ \frac{h_{69}}{A^3D^2\alpha^2}r_0\Big](l_ks_{j0} + l_js_{k0}) + \Big[\frac{h_{70}}{A^4D^2\alpha}s_0 + \frac{h_{71}}{A^4\alpha^2}r_{00} + \frac{h_{72}}{A^3\alpha}r_0\Big](l_kr_j + l_jr_k) \\
&+ \Big[\frac{h_{73}}{A^5D^2\alpha^2}s_0 + \frac{h_{74}}{A^5\alpha^3}r_{00} + \frac{h_{75}}{A^4\alpha^2}r_0 + \frac{h_{76}}{A^3\alpha}r\Big](l_kr_{0j} + l_jr_{0k}) + \frac{h_{77}}{A^2D}(l_kq_j + l_jq_k) \\
&+ \frac{h_{78}}{A^3D\alpha}(l_kq_{0j} + l_jq_{0k}) + \frac{h_{79}}{A^3D^2\alpha}(l_kq_{j0} + l_jq_{k0}) + \frac{h_{80}}{A^3D^3}(l_kt_j + l_jt_k) \\
&+ \Big[\frac{h_{81}}{A^5D^2\alpha^2}r_{00} + \frac{h_{82}}{A^5D^3\alpha}s_0 + \frac{h_{83}}{A^4D^2\alpha}r_0 + \frac{h_{84}}{A^3D}r\Big](b_ks_j + b_js_k) + \Big[\frac{h_{85}}{A^4D^4\alpha^2}s_0 \\
&+ \frac{h_{86}}{A^4D^2\alpha^3}r_{00} + \frac{h_{87}}{A^3D^2\alpha^2}r_0\Big](b_ks_{j0} + b_js_{k0}) + \frac{h_{95}}{A^3D^2\alpha}(b_kq_{j0} + b_jq_{k0}) \\
&+ \Big[\frac{h_{88}}{A^4D^2\alpha}s_0 + \frac{h_{89}}{A^4\alpha^2}r_{00} + \frac{h_{90}}{A^3\alpha}r_0\Big](b_kr_j + b_jr_k) + \Big[\frac{h_{91}}{A^5D^2\alpha^2}s_0 + \frac{h_{92}}{A^5\alpha^3}r_{00} \\
&+ \frac{h_{93}}{A^4\alpha^2}r_0 + \frac{h_{94}}{A^3\alpha}r\Big](b_kr_{0j} + b_jr_{0k}) + \frac{h_{96}}{A^3D\alpha}(b_kq_{0j} + b_jq_{0k}) + \frac{h_{97}}{A^3D^3}(b_kt_j + b_jt_k) \\
&+ \frac{h_{98}}{A^2D}(b_kq_j + b_jq_k) + \frac{h_{99}}{A^3D^3\alpha}(s_ks_{j0} + s_js_{k0}) + \frac{h_{100}}{A^3D}(s_kr_j + s_jr_k) \\
&+ \frac{h_{101}}{A^4D\alpha}(s_kr_{j0} + s_jr_{k0}) + \frac{h_{102}}{A^3D\alpha^2}(s_{k0}r_{j0} + s_{j0}r_{k0}) + \frac{h_{103}}{A^2D\alpha}(s_{k0}r_j + s_{j0}r_k) \\
&+ \frac{h_{104}}{A^3\alpha}(r_kr_{j0} + r_jr_{k0}) + \frac{h_{105}}{A^2D}(q_{kj} + q_{jk}) + \frac{h_{106}}{A^3\alpha^2}(l_kr_{0j|0} + l_jr_{0k|0}) \\
&+ \frac{h_{107}}{A^3\alpha^2}(r_{0j|0}b_k + r_{0k|0}b_j) + \frac{h_{108}}{A^2\alpha}(r_{k|0}b_j + r_{j|0}b_k) + \frac{h_{109}}{A^2\alpha}(l_kr_{j|0} + l_jr_{k|0}) \\
&+ \frac{h_{110}}{A^3D^2\alpha}(l_js_{k|0} + l_ks_{j|0}) + \frac{h_{111}}{A^3D^2\alpha}(s_{j|0}b_k + s_{k|0}b_j),
\end{aligned} \tag{3.6}
$$

where

$$A := 1 + m(m-1)B - (m-2)s - (m^2-1)s^2, \quad D := (m-1)s - 1,$$

and h_i $(i = 1, 2, \cdots, 111)$ are the polynomials of variations s and B. Substituting (3.6) in (3.4) and multiplying the result with $A^6 D^4 \alpha^4$ implies that

$$H_{jk} A^6 D^4 \alpha^4 - \frac{n+1}{2} \theta F_{y^j y^k} A^6 D^4 \alpha^4 = 0. \tag{3.7}$$

The following holds

$$\theta F_{y^j y^k} A^6 D^4 \alpha^4 \equiv 0, \quad \mathrm{mod(A)}.$$

Then (3.7) is equivalent to the following

$$\Big[h_{49} r_{00} s_0 \alpha + h_{51} D^4 r_{00}^2 + h_{56} s_0^2 \alpha^2\Big](l_j b_k + l_k b_j) + \Big[h_1 r_{00} s_0 \alpha + h_3 D^4 r_{00}^2 + h_8 s_0^2 \alpha^2\Big] b_j b_k$$
$$+ \Big[h_{15} r_{00} s_0 \alpha + h_{17} D^4 r_{00}^2 + h_{22} s_0^2 \alpha^2\Big] l_j l_k \equiv 0, \quad \mathrm{mod(A)}. \tag{3.8}$$

Multiplying (3.8) with $b^j b^k$ yields

$$I_1 r_{00} s_0 \alpha + I_2 r_{00}^2 + I_3 s_0^2 \alpha^2 \equiv 0, \qquad \mathrm{mod(A)}, \tag{3.9}$$

where I_i $(i = 1, 2, 3)$, are polynomials of s and B. Let us put

$$I_1 \equiv f_1 \quad \text{and} \quad I_2 \equiv f_2 \quad \text{and} \quad I_3 \equiv f_3, \quad \mathrm{mod(A)}.$$

Then by (3.9), we get (3.3). □

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1: Let β be a closed 1-form. Then (3.6) reduces to the following:

$$\begin{aligned}
2H_{jk} = \quad & \Big[\frac{h_3}{A^6\alpha^4} r_{00}^2 + \frac{h_4}{A^4\alpha^3} r_{00|0} + \frac{h_5}{A^5\alpha^3} r_{00} r_0 + \frac{h_6}{A^3\alpha^2} r_{0|0} + \frac{h_{13}}{A^4\alpha^2} r r_{00} + \frac{h_{14}}{A^4\alpha^2} r_0^2\Big] b_j b_k \\
+ & \Big[\frac{h_{17}}{A^6\alpha^4} r_{00}^2 + \frac{h_{18}}{A^4\alpha^3} r_{00|0} + \frac{h_{19}}{A^5\alpha^3} r_{00} r_0 + \frac{h_{20}}{A^3\alpha^2} r_{0|0} + \frac{h_{27}}{A^4\alpha^2} r r_{00} + \frac{h_{28}}{A^4\alpha^2} r_0^2\Big] l_j l_k \\
+ & \Big[\frac{h_{31}}{A^5\alpha^4} r_{00}^2 + \frac{h_{32}}{A^3\alpha^3} r_{00|0} + \frac{h_{33}}{A^4\alpha^3} r_{00} r_0 + \frac{h_{34}}{A^2\alpha^2} r_{0|0} + \frac{h_{41}}{A^3\alpha^2} r r_{00} + \frac{h_{42}}{A^3\alpha^2} r_0^2\Big] a_{jk} \\
+ & \frac{h_{43}}{A^2\alpha} r_{jk|0} + \Big[\frac{h_{45}}{A^4\alpha^2} r_{00} + \frac{h_{46}}{A^3\alpha} r_0\Big] r_{jk} + \frac{h_{48}}{A^4\alpha^2} r_{0j} r_{0k} + \Big[\frac{h_{51}}{A^6\alpha^4} r_{00}^2 + \frac{h_{52}}{A^4\alpha^3} r_{00|0} \\
+ & \frac{h_{53}}{A^5\alpha^3} r_{00} r_0 + \frac{h_{54}}{A^3\alpha^2} r_{0|0} + \frac{h_{61}}{A^4\alpha^2} r r_{00} + \frac{h_{62}}{A^4\alpha^2} r_0^2\Big] \Big[l_k b_j + l_j b_k\Big] \\
+ & \Big[\frac{h_{71}}{A^4\alpha^2} r_{00} + \frac{h_{72}}{A^3\alpha} r_0\Big] \Big[l_k r_j + l_j r_k\Big] + \Big[\frac{h_{74}}{A^5\alpha^3} r_{00} + \frac{h_{75}}{A^4\alpha^2} r_0 + \frac{h_{76}}{A^3\alpha} r\Big] (l_k r_{0j} + l_j r_{0k}) \\
+ & \Big[\frac{h_{89}}{A^4\alpha^2} r_{00} + \frac{h_{90}}{A^3\alpha} r_0\Big] \Big[b_k r_j + b_j r_k\Big] + \Big[\frac{h_{92}}{A^5\alpha^3} r_{00} + \frac{h_{93}}{A^4\alpha^2} r_0 + \frac{h_{94}}{A^3\alpha} r\Big] \Big[b_k r_{0j} + b_j r_{0k}\Big] \\
+ & \frac{h_{104}}{A^3\alpha} \Big[r_k r_{j0} + r_j r_{k0}\Big] + \frac{h_{106}}{A^3\alpha^2} \Big[l_k r_{0j|0} + l_j r_{0k|0}\Big] + \frac{h_{107}}{A^3\alpha^2} \Big[r_{0j|0} b_k + r_{0k|0} b_j\Big] \\
+ & \frac{h_{108}}{A^2\alpha} \Big[r_{k|0} b_j + r_{j|0} b_k\Big] + \frac{h_{109}}{A^2\alpha} \Big[l_k r_{j|0} + l_j r_{k|0}\Big],
\end{aligned} \tag{3.10}$$

By substituting (3.10) in (3.4) and multiplying the result with $A^6\alpha^4$, we get

$$H_{jk}A^6\alpha^4 - \frac{n+1}{2}\theta F_{y^jy^k}A^6\alpha^4 = 0. \tag{3.11}$$

Since

$$\frac{n+1}{2}\theta F_{y^jy^k}A^6\alpha^4 \equiv 0, \quad \mathrm{mod(A)}$$

then (3.11) is equal to the following

$$\Big[h_{51}(l_jb_k + l_kb_j) + h_3b_jb_k + h_{17}l_jl_k\Big]r_{00}^2 \equiv 0, \quad \mathrm{mod(A)}. \tag{3.12}$$

Multiplying (3.12) with b^jb^k yields

$$I_2r_{00}^2 \equiv 0, \quad \mathrm{mod(A)},$$

where I_2 is a polynomial of s and B. Then we get

$$f_2r_{00}^2 \equiv 0 \quad \mathrm{mod(A)},$$

where $I_2 \equiv f_2 \quad \mathrm{mod(A)}$, and f_2 is a polynomial of s and B and of degree 1 in s. By Lemma 3.1, it follows that β is parallel with respect to α. Plugging this in (3.10) yields $\mathbf{H} = 0$. The converse is trivial. On the other hand, every regular (α,β)-metric is a Berwald metric if and only if β is parallel with respect to α. This completes the proof.

4. Proof of Theorem 1.5

In this section, we are going to prove Theorem 1.5. First, we prove the following.

Lemma 4.1 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be a polynomial (α,β)-metric of degree m on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Suppose that F is an Einstein metric. Then the following holds*

$$g_1r_{00}s_0\alpha + g_2r_{00}^2 + g_3s_0^2\alpha^2 \equiv 0, \qquad mod\Big((1-m^2)s^2 + (2-m)s + m(m-1)B + 1\Big), \tag{4.1}$$

where g_j $(j = 1,2,3)$, are polynomials of variations B and s.

Proof Let $\varphi(s) = (1+s)^m$ $(m \geq 3)$ be an Einstein metric. By the Theorem 1.1 in [2], F is Ricci-flat. Then

$$R^m_m =^{\alpha} R^m_m + T^m_m = 0, \tag{4.2}$$

where ${}^{\alpha}R^m_m$ denotes the Riemannian curvature of α and

$$\begin{aligned} T^m_m =& \Big[(n-1)\frac{c_1}{A^3} + \frac{c_2}{A^4}\Big]\frac{r_{00}^2}{\alpha^2} + \frac{1}{\alpha}\Bigg[\Big[(n-1)\frac{c_3}{A^3D} + \frac{c_4}{A^4D}\Big]r_{00}s_0 + r_0\Big[(n-1)\frac{c_5}{A^2} + \frac{c_6}{A^3}\Big]r_{00} \\ &+ \Big[(n-1)\frac{c_7}{A} + \frac{c_8}{A^2}\Big]r_{00|0}\Bigg] + \Bigg[\Big[(n-1)\frac{c_9}{A^3D^3} + \frac{c_{10}}{A^4D^3}\Big]s_0^2 + \frac{c_{11}}{A^2}(rr_{00} - r_0^2) \end{aligned}$$

$$+\Big[(n-1)\frac{c_{12}}{A^2D}+\frac{c_{13}}{A^3D}\Big]r_0s_0+\frac{c_{14}}{A}(r_{00}r^m_m-r_{0m}r^m_0+r_{00|m}b^m-r_{0m|0}b^m)$$
$$+\Big[(n-1)\frac{c_{15}}{AD}+\frac{c_{16}}{A^2D}\Big]r_{0m}s^m_0+\Big[(n-1)\frac{c_{17}}{AD}+\frac{c_{18}}{A^2D}\Big]s_{0|0}+\frac{c_{19}}{D^3}s_{0m}s^m_0\Bigg]$$
$$+\Bigg[\frac{c_{20}}{A^2D}rs_0+\Big[(n-1)\frac{c_{21}}{AD^2}+\frac{c_{22}}{A^2D^2}\Big]s_ms^m_0+\frac{c_{23}}{AD}(3s_mr^m_0-2s_0r^m_m+2r_ms^m_0$$
$$-2s_{0|m}b^m+s_{m|0}b^m)+\frac{c_{24}}{D}s^m_{0|m}\Bigg]\alpha+\Big[\frac{c_{25}}{AD^2}s_ms^m+\frac{c_{26}}{D^2}s^i_ms^m_i\Big]\alpha^2, \tag{4.3}$$

$A=1+m(m-1)B-(m-2)s-(m^2-1)s^2$, $D:=(m-1)s-1$ and c_i $(i=1,\cdots,26)$, are polynomials of variations s and B (see [2] for the corrected version of [25]). Putting (4.3) in (4.2) and multiplying the result with $A^4D^3\alpha^2$ implies that

$${}^{\alpha}R^m_mA^4D^3\alpha^2+T^m_mA^4D^3\alpha^2=0.$$

${}^{\alpha}R^m_m$ is a polynomial with respect to s and B. Since ${}^{\alpha}R^m_mA^4D^3\alpha^2\equiv 0,\ \mathrm{mod(A)}$, then we get $T^m_mA^4D^3\alpha^2\equiv 0,\ \mathrm{mod(A)}$. By (4.3), we obtain

$$r_{00}s_0\alpha c_4D^2+r^2_{00}c_2D^3+s^2_0\alpha^2c_{10}\equiv 0,\quad \mathrm{mod(A)}.$$

Put

$$c_4D^2\equiv g_1\quad\text{and}\quad c_2D^3\equiv g_2\quad\text{and}\quad c_{10}\equiv g_3\quad \mathrm{mod(A)}.$$

Then we get (4.1). □

Proof of the Theorem 1.5: Let β be a closed 1-form on M. By Lemma 4.1, we get

$$g_2r^2_{00}\equiv 0,\ \mathrm{mod(A)},$$

where $I_2\equiv g_2,\ \mathrm{mod(A)}$, and g_2 is polynomials of s and B and of degree 1 in s. By Lemma 3.1, it follows that β is Killing. Then β is parallel with respect to α. In this case, F reduces to a Berwald metric.

5. Proof of Theorem 1.9

In this section, we are going to prove Theorem 1.9. First, we prove the following.

Lemma 5.1 *Let $F=\alpha\varphi(s)$, $s=\beta/\alpha$, be an exponential metric on an n-dimensional manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on M. Suppose that F has almost vanishing* **H***-curvature. Then the following holds*

$$h_1r_{00}s_0\alpha+h_2r^2_{00}+h_3s^2_0\alpha^2\equiv 0,\qquad mod(-s^2-s+B+1), \tag{5.1}$$

where h_j $(j=1,2,3)$ are polynomials of variations s and B and of degree one in s.

Proof For the exponential metric $\varphi(s)=e^s$, we have

$$Q=\frac{1}{1-s},\qquad \Theta=\frac{2s-1}{2(s^2+s-B-1)},\qquad \Psi=\frac{1}{2(1+B-s-s^2)}.$$

By assumption, $F=\alpha\varphi(s),\ s=\beta/\alpha$, has almost vanishing **H**-curvature, i.e. there exists a 1-form θ on M such that

$$H_{jk}=\frac{n+1}{2}\theta F_{y^jy^k}, \tag{5.2}$$

where

$$F_{y^jy^k}=\frac{e^s}{\alpha}\Big[(1-s)a_{jk}+b_jb_k-s(b_jl_k+b_kl_j)+(s^2+s-1)l_jl_k\Big],$$

and

$$\begin{aligned}
2H_{jk}= &\Big[\frac{h_1}{\alpha^3A^6(s-1)^3}r_{00}s_0+\frac{h_2}{\alpha^2A^4(s-1)^3}s_{0|0}+\frac{h_3}{\alpha^4A^6}r_{00}^2+\frac{h_4}{\alpha^3A^4}r_{00|0}+\frac{h_5}{\alpha^3A^5}r_{00}r_0\\
&+\frac{h_6}{\alpha^2A^3}r_{0|0}+\frac{h_7}{\alpha A^4(s-1)^4}t_0+\frac{h_8}{\alpha^2A^6(s-1)^4}s_0^2+\frac{h_9}{\alpha^2A^4(s-1)^2}q_{00}+\frac{h_{10}}{\alpha A^3(s-1)^2}q_0\\
&+\frac{h_{11}}{\alpha^2A^5(s-1)^3}r_0s_0+\frac{h_{12}}{\alpha A^4(s-1)^2}rs_0+\frac{h_{13}}{\alpha^2A^4}rr_{00}+\frac{h_{14}}{\alpha^2A^4}r_0^2\Big]b_jb_k\\
&+\Big[\frac{h_{15}}{\alpha^3A^6(s-1)^3}r_{00}s_0+\frac{h_{16}}{\alpha^2A^4(s-1)^3}s_{0|0}+\frac{h_{17}}{\alpha^4A^6}r_{00}^2+\frac{h_{18}}{\alpha^3A^4}r_{00|0}+\frac{h_{19}}{\alpha^3A^5}r_{00}r_0\\
&+\frac{h_{20}}{\alpha^2A^3}r_{0|0}+\frac{h_{21}}{\alpha A^4(s-1)^4}t_0+\frac{h_{22}}{\alpha^2A^6(s-1)^4}s_0^2+\frac{h_{23}}{\alpha^2A^4(s-1)^2}q_{00}+\frac{h_{24}}{\alpha A^3(s-1)^2}q_0\\
&+\frac{h_{25}}{\alpha^2A^5(s-1)^3}r_0s_0+\frac{h_{26}}{\alpha A^4(s-1)^2}rs_0+\frac{h_{27}}{\alpha^2A^4}rr_{00}+\frac{h_{28}}{\alpha^2A^4}r_0^2\Big]l_jl_k\\
&+a_{jk}\Big[\frac{h_{29}}{\alpha^3A^5(s-1)^2}r_{00}s_0+\frac{h_{30}}{\alpha^2A^3(s-1)^2}s_{0|0}+\frac{h_{31}}{\alpha^4A^5}r_{00}^2+\frac{h_{32}}{\alpha^3A^3}r_{00|0}+\frac{h_{33}}{\alpha^3A^4}r_{00}r_0\\
&+\frac{h_{34}}{\alpha^2A^2}r_{0|0}+\frac{h_{35}}{\alpha^2A^4(s-1)^2}r_0s_0+\frac{h_{36}}{\alpha^2A^5(s-1)^3}s_0^2+\frac{h_{37}}{\alpha A^3(s-1)^3}t_0+\frac{h_{38}}{\alpha^2A^3(s-1)}q_{00}\\
&+\frac{h_{39}}{\alpha^2A^2(s-1)}q_0+\frac{h_{40}}{\alpha A^3(s-1)}rs_0+\frac{h_{41}}{\alpha^2A^3}rr_{00}+\frac{h_{42}}{\alpha^2A^3}r_0^2\Big]+\frac{h_{43}}{\alpha A^2}r_{jk|0}+r_{jk}\Big[\frac{h_{44}s_0}{\alpha A^4(s-1)}\\
&+\frac{h_{45}}{\alpha^2A^4}r_{00}+\frac{h_{46}}{\alpha A^3}r_0\Big]+\frac{h_{47}}{A^4(s-1)^2}s_ks_j+\frac{h_{48}}{\alpha^2A^4}r_{0j}r_{0k}+\Big[\frac{h_{49}}{\alpha^3A^6(s-1)^3}r_{00}s_0\\
&+\frac{h_{50}}{\alpha^2A^4(s-1)^3}s_{0|0}+\frac{h_{51}}{\alpha^4A^6}r_{00}^2+\frac{h_{52}}{\alpha^3A^4}r_{00|0}+\frac{h_{53}}{\alpha^3A^5}r_{00}r_0+\frac{h_{54}}{\alpha^2A^3}r_{0|0}+\frac{h_{55}}{\alpha A^4(s-1)^4}t_0\\
&+\frac{h_{56}}{\alpha^2A^6(s-1)^4}s_0^2+\frac{h_{57}}{\alpha^2A^4(s-1)^2}q_{00}+\frac{h_{58}}{\alpha A^3(s-1)^2}q_0+\frac{h_{59}r_0}{\alpha^2A^5(s-1)^3}s_0+\frac{h_{60}r}{\alpha A^4(s-1)^2}s_0\\
&+\frac{h_{61}}{\alpha^2A^4}rr_{00}+\frac{h_{62}}{\alpha^2A^4}r_0^2\Big](l_kb_j+l_jb_k)+\Big[\frac{h_{63}}{\alpha A^5(s-1)^3}s_0+\frac{h_{64}}{\alpha^2A^5(s-1)^2}r_{00}+\frac{h_{65}}{\alpha A^4(s-1)^2}r_0\\
&+\frac{h_{66}}{A^3(s-1)}r\Big](l_ks_j+l_js_k)+(l_ks_{j0}+l_js_{k0})\Big[\frac{h_{67}}{\alpha^2A^4(s-1)^4}s_0+\frac{h_{68}}{\alpha^3A^4(s-1)^2}r_{00}
\end{aligned}$$

$$+ \frac{h_{69}}{\alpha^2 A^3(s-1)^2}r_0\Big] + (l_k r_j + l_j r_k)\Big[\frac{h_{70}}{\alpha A^4(s-1)^2}s_0 + \frac{h_{71}}{\alpha^2 A^4}r_{00} + \frac{h_{72}}{\alpha A^3}r_0\Big]$$

$$+ (l_k r_{0j} + l_j r_{0k})\Big[\frac{h_{73}}{\alpha^2 A^5(s-1)^2}s_0 + \frac{h_{74}}{\alpha^3 A^5}r_{00} + \frac{h_{75}}{\alpha^2 A^4}r_0 + \frac{h_{76}}{\alpha A^3}r\Big] + \frac{h_{77}}{A^2(s-1)}(l_k q_j + l_j q_k)$$

$$+ \frac{h_{78}}{\alpha A^3(s-1)}(l_k q_{0j} + l_j q_{0k}) + \frac{h_{79}}{\alpha A^3(s-1)^2}(l_k q_{j0} + l_j q_{k0}) + \frac{h_{80}}{A^3(s-1)^3}(l_k t_j + l_j t_k)$$

$$+ (b_k s_j + b_j s_k)\Big[\frac{h_{81}}{\alpha^2 A^5(s-1)^2}r_{00} + \frac{h_{82}}{\alpha A^5(s-1)^3}s_0 + \frac{h_{83}}{\alpha A^4(s-1)^2}r_0 + \frac{h_{84}}{A^3(s-1)}r\Big]$$

$$+ (b_k s_{j0} + b_j s_{k0})\Big[\frac{h_{85}}{\alpha^2 A^4(s-1)^4}s_0 + \frac{h_{86}}{\alpha^3 A^4(s-1)^2}r_{00} + \frac{h_{87}}{\alpha^2 A^3(s-1)^2}r_0\Big]$$

$$+ (b_k r_j + b_j r_k)\Big[\frac{h_{88}}{\alpha A^4(s-1)^2}s_0 + \frac{h_{89}}{\alpha^2 A^4}r_{00} + \frac{h_{90}}{\alpha A^3}r_0\Big] + (b_k r_{0j} + b_j r_{0k})\Big[\frac{h_{91}}{\alpha^2 A^5(s-1)^2}s_0$$

$$+ \frac{h_{92}}{\alpha^3 A^5}r_{00} + \frac{h_{93}}{\alpha^2 A^4}r_0 + \frac{h_{94}}{\alpha A^3}r\Big] + \frac{h_{95}}{\alpha A^3(s-1)^2}(b_k q_{j0} + b_j q_{k0}) + \frac{h_{96}}{\alpha A^3(s-1)}(b_k q_{0j} + b_j q_{0k})$$

$$+ \frac{h_{97}}{A^3(s-1)^3}(b_k t_j + b_j t_k) + \frac{h_{98}}{A^2(s-1)}(b_k q_j + b_j q_k) + \frac{h_{99}}{\alpha A^3(s-1)^3}(s_k s_{j0} + s_j s_{k0})$$

$$+ \frac{h_{100}}{A^3(s-1)}(s_k r_j + s_j r_k) + \frac{h_{101}}{\alpha A^4(s-1)}(s_k r_{j0} + s_j r_{k0}) + \frac{h_{102}}{\alpha^2 A^3(s-1)}(s_{k0} r_{j0} + s_{j0} r_{k0})$$

$$+ \frac{h_{103}}{\alpha A^2(s-1)}(s_{k0} r_j + s_{j0} r_k) + \frac{h_{104}}{\alpha A^3}(r_k r_{j0} + r_j r_{k0}) + \frac{h_{105}}{A^2(s-1)}(q_{kj} + q_{jk})$$

$$+ \frac{h_{106}}{\alpha^2 A^3}(l_k r_{0j|0} + l_j r_{0k|0}) + \frac{h_{107}}{\alpha^2 A^3}(r_{0j|0} b_k + r_{0k|0} b_j) + \frac{h_{108}}{\alpha A^2}(r_{k|0} b_j + r_{j|0} b_k)$$

$$+ \frac{h_{109}}{\alpha A^2}(l_k r_{j|0} + l_j r_{k|0}) + \frac{h_{110}}{\alpha A^3(s-1)^2}(l_j s_{k|0} + l_k s_{j|0}) + \frac{h_{111}}{A^3(s-1)^2}(s_{j|0} b_k + s_{k|0} b_j),$$

$A = 1 + B - s - s^2$ and h_i $(i = 1, 2, \cdots, 111)$ are the polynomials of s and B. Putting (5.3) in (3.4) and multiplying the result with $A^6(s-1)^4\alpha^4$ implies that

$$H_{jk}A^6\alpha^4(s-1)^4 - \frac{n+1}{2}\theta F_{y^j y^k}A^6\alpha^4(s-1)^4 = 0. \tag{5.3}$$

Since $\theta F_{y^j y^k}A^6\alpha^4(s-1)^4 \equiv 0, \quad \mathrm{mod(A)}$, then (5.3) is equal to

$$\Big[h_{49}(s-1)\alpha s_0 r_{00} + h_{51}(s-1)^4 r_{00}^2 + h_{56}\alpha^2 s_0^2\Big](l_j b_k + l_k b_j) + \Big[h_1(s-1)\alpha s_0 r_{00} + h_8\alpha^2 s_0^2$$

$$+ h_3(s-1)^4 r_{00}^2\Big]b_j b_k + \Big[h_{15}(s-1)\alpha s_0 r_{00} + h_{17}(s-1)^4 r_{00}^2 + h_{22}\alpha^2 s_0^2\Big]l_j l_k \equiv 0, \ \mathrm{mod(A)}. \tag{5.4}$$

Multiplying (5.4) with $b^j b^k$ yields $I_1 r_{00} s_0 \alpha + I_2 r_{00}^2 + I_3 s_0^2\alpha^2 \equiv 0, \ \mathrm{mod(A)}$, where I_i, $(i = 1, 2, 3)$ are polynomials of variations s and B. Put $I_1 \equiv h_1$, $I_2 \equiv h_2$ and $I_3 \equiv h_3 \quad \mathrm{mod(A)}$. Then, we get (5.1). □

Proof of Theorem 1.9: Let β be a closed 1-form on M. By Lemma 5.1, we get $h_2 r_{00}^2 \equiv 0, \ \mathrm{mod(A)}$, where $I_2 \equiv h_2$, $mod(A)$, and h_2 is a polynomial of s and B and of degree 1 in s. By Lemma 3.1, β is Killing. Putting it in (5.3) yields $\mathbf{H} = 0$. The converse is trivial. In this case, it follows that β is parallel with respect to α. Then, F reduces to a Berwald metric.

6. Proof of Theorem 1.11

In this section, we are going to prove Theorem 1.11. For this aim, we need the following.

Lemma 6.1 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be an exponential metric on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Suppose that F is an Einstein metric. Then the following holds*

$$k_1 r_{00} s_0 \alpha + k_2 r_{00}^2 + k_3 s_0^2 \alpha^2 \equiv 0, \qquad mod(-s^2 - s + B + 1), \tag{6.1}$$

where k_j, $(j = 1, 2, 3)$, are polynomials of variations B and s.

Proof For the exponential metric $\varphi(s) = e^s$, we have

$$R^m_m =^{\alpha} R^m_m + T^m_m = \mathbf{Ric}(x)F^2, \tag{6.2}$$

where

$$\begin{aligned}
T^m_m := & \Big[(n-1)\frac{c_1}{A^3} + \frac{c_2}{A^4}\Big]\frac{r_{00}^2}{\alpha^2} + \frac{1}{\alpha}\Bigg[\Big[(n-1)\frac{c_3}{A^3 D} + \frac{c_4}{A^4 D}\Big] r_{00} s_0 + \Big[(n-1)\frac{c_5}{A^2} + \frac{c_6}{A^3}\Big] r_{00} r_0 \\
& + \Big[(n-1)\frac{c_7}{A} + \frac{c_8}{A^2}\Big] r_{00|0}\Bigg] + \Bigg[\Big[(n-1)\frac{c_9}{A^3 D^3} + \frac{c_{10}}{A^4 D^3}\Big] s_0^2 + \frac{c_{11}}{A^2}(r r_{00} - r_0^2) \\
& + \Big[(n-1)\frac{c_{12}}{A^2 D} + \frac{c_{13}}{A^3 D}\Big] r_0 s_0 + \frac{c_{14}}{A}(r_{00} r^m_m - r_{0m} r^m_0 + r_{00|m} b^m - r_{0m|0} b^m) \\
& + \Big[(n-1)\frac{c_{15}}{AD} + \frac{c_{16}}{A^2 D}\Big] r_{0m} s^m_0 + \Big[(n-1)\frac{c_{17}}{AD} + \frac{c_{18}}{A^2 D}\Big] s_{0|0} + \frac{c_{19}}{D^3} s_{0m} s^m_0\Bigg] \\
& + \Bigg[\frac{c_{20}}{A^2 D} r s_0 + \Big[(n-1)\frac{c_{21}}{AD^2} + \frac{c_{22}}{A^2 D^2}\Big] s_m s^m_0 + \frac{c_{23}}{AD}(3 s_m r^m_0 - 2 s_0 r^m_m + 2 r_m s^m_0 \\
& - 2 s_{0|m} b^m + s_{m|0} b^m) + \frac{c_{24}}{D} s^m_{0|m}\Bigg]\alpha + \Bigg[\frac{c_{25}}{AD^2} s_m s^m + \frac{c_{26}}{D^2} s^i_m s^m_i\Bigg]\alpha^2,
\end{aligned} \tag{6.3}$$

$A = 1 + B - s - s^2$, $D = s - 1$ and c_i, $(i = 1, \cdots, 26)$, are polynomials of variations s and B (see [2]). Putting T^m_m into (6.2) and multiplying the result with $A^4 D^3 \alpha^2$ implies that

$${}^{\alpha}R^m_m A^4 D^3 \alpha^2 + T^m_m A^4 D^3 \alpha^2 - \mathbf{Ric}(x) F^2 A^4 D^3 \alpha^2 = 0.$$

${}^{\alpha}R^m_m - \mathbf{Ric}(x)F^2$ is a polynomial of s and B. Thus,

$${}^{\alpha}R^m_m A^4 D^3 \alpha^2 - \mathbf{Ric}(x) F^2 A^4 D^3 \alpha^2 \equiv 0, \qquad \text{mod(A)}.$$

Then $T^m_m A^4 D^3 \alpha^2 \equiv 0$, mod(A). By (6.3), we get $r_{00} s_0 \alpha c_4 D^2 + r_{00}^2 c_2 D^3 + s_0^2 \alpha^2 c_{10} \equiv 0$, mod(A). Put

$$c_4 D^2 \equiv h_1 \quad \text{and} \quad c_2 D^3 \equiv h_2 \quad \text{and} \quad c_{10} \equiv h_3, \quad \text{mod(A)}.$$

Then, we get (6.1). □

Proof of Theorem 1.11: By Lemma 6.1, we have (6.1). Let $\beta = b_i(x)y^i$ be a closed 1-form. Then $k_2 r_{00}^2 \equiv 0$, mod(A), where $I_2 \equiv k_2$ mod(A), and k_2 is a polynomial of variations s and B. By Lemma 3.1, β is a Killing 1-form. It follows that β is parallel with respect to α. In this case, F reduces to a Berwald metric.

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