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A reduced computational matrix approach with convergence estimation for solving model differential equations involving specific nonlinearities of quartic type

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Abstract: This study aims to efficiently solve model differential equations involving specific nonlinearities of quartic type by proposing a reduced computational matrix approach based on the generalized Mott polynomial. This method presents a reduced matrix expansion of the generalized Mott polynomial with the parameter-α, matrix equations, and Chebyshev–Lobatto collocation points. The simplicity of the method provides fast computation while eliminating an algebraic system of nonlinear equations, which arises from the matrix equation. The method also scrutinizes the consistency of the solutions due to the parameter-α. The oscillatory behavior of the obtained solutions on long time intervals is simulated via a coupled methodology involving the proposed method and Laplace–Padé technique. The convergence estimation is established via residual function. Numerical and graphical results are indicated to discuss the validity and efficiency of the method.

Key words: Matrix method, error estimation, Mott polynomial, oscillation, nonlinearity

1. Introduction

In recent years, nonlinear differential equations (NDEs) have become one of the fundamental mathematical tools, with the development of natural physical phenomena occurring in mathematics, heat conduction and transfer, physiology, physics, engineering, mechanics, acoustics, and astronomy [2, 3, 6, 7, 9, 13, 18, 19, 22, 23, 25, 30, 32]. As a specific example, the deflection of a cantilever beam exposed to a concentrated load can be governed with a nonlinear differential equation involving functional nonlinearity [14, 23]. The reason why NDEs are widely modeled all over the world is that they present a simple and efficient mathematical structure to deal with complex phenomena whose physical behaviors vary with respect to time, location, and ambiguous environmental conditions. Hence, NDEs give a consistent response of these phenomena subjected to initial, boundary, and initial-boundary conditions. It is known that it is difficult to solve NDEs analytically. The analytical solutions of NDEs can hardly be found since the nonlinearity degree of a model is modified. Thus, the analytical procedures remain insufficient for nonlinear differential equations involving specific nonlinearities, such as strongly, fully, functional, and singular forms. More efforts than ever before have been directed towards numerically treating these types of problems. So far, with this aim, Kürkçü et al. [18] employed the Dickson matrix-collocation method to solve some model problems arising in science. Bülbül and Sezer [4, 5] established the Taylor polynomial method to find the approximate solutions of nonlinear differential equations of Abel and Duffing

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types. Rajaraman and Hariharan [26] proposed the shifted second-kind Chebyshev wavelet method to solve singular boundary value problems. The Green function-based Adomian decomposition method and variational iteration method have been used for singular nonlinear boundary value problems [15, 28]. Odibat and Bataineh [24] employed the homotopy analysis method to deal with strongly nonlinear problems. He [14] applied several asymptotic methods to strongly nonlinear equations.

Differently from the mentioned studies, this study deals with model differential equations involving specific nonlinearities of quartic type under a unique formulation by employing a reduced computational matrix approach based on the generalized Mott polynomial with the parameter-α. The reduced matrix structure provides a fast computation for eliminating a highly stiff algebraic system of nonlinear equations, which arises from a matrix equation. The oscillatory behaviors of the solutions are handled with the aid of a coupled methodology based on the proposed method and Laplace–Padé technique [20, 31]. The framework of this paper is as follows: Section 1.1 states a governing equation under a unique formulation and its solution form to be found. Section 2 describes some basic properties of the Mott polynomial. Section 3 presents the reduced matrix relations and method of solution. Section 4 expresses the residual convergence estimation. Section 5 establishes the oscillatory behavior of the solutions via the coupled methodology. Section 6 contains six illustrative model problems, which are treated by the proposed method. Section 7 discusses the validity and efficiency of the proposed method by evaluating the results found in Section 6.

1.1. Statement of governing equation

In this study, some model differential equations involving specific nonlinearities of quartic type are governed by

$$\sum_{k=0}^{3} P_k(t)y^{(k)}(t) + Q_{ijkl}^{pq}(t)\left[\left(y^{(i)}\right)\left(y^{(j)}\right)\left(y^{(k)}\right)^p\left(y^{(l)}\right)^q\right](t) = f(t), \ a \le t \le b, \tag{1.1}$$

subject to the initial and boundary conditions

$$y(a) = \gamma_1, \ y'(a) = \gamma_2 \text{ and } y(b) = \gamma_3, \tag{1.2}$$

where p and q determine the nonlinear force of Eq. (1.1) and can be chosen in $\{0,1\}$ in accordance with the nonlinearity degree of a considered problem; i,j,k,l represent integer order derivatives and take their values as $0 \le i,j,k,l \le 3$ independently; and $P_k(t)$, $Q_{pq}(t)$, and an external force $f(t)$ are defined on $[a,b]$.

Since Eq. (1.1) contains both a linear form and a quartic nonlinearity form with derivatives, it can be readily adopted to strongly, fully, functional, and singular nonlinear model differential equations by determining the proper values of p, q, $P_k(t)$, and $Q_{ijkl}^{pq}(t)$. For example, the fully nonlinear and strongly nonlinear differential equations can be obtained via $P_k(t) = 0$ and $p = q = 1$, respectively.

The aim of this study is to numerically solve model nonlinear differential equations derived from Eq. (1.1) by proposing a reduced computational matrix approach based on the generalized Mott polynomial. In doing so, we seek an approximate solution of Eq. (1.1) in the following form [17]:

$$y(t) \cong y_N(t;\alpha) = \sum_{n=0}^{N} y_n S_n(t,\alpha), \tag{1.3}$$

where the y_ns are unknown Mott coefficients to be determined by the method and $S_n(t,\alpha)$ represents the generalized Mott polynomials (see [10, 16, 21, 27]). It is promisingly stated here that the parameter α can be used to control the optimal approximation of the solution (1.3) with respect to the exact solution.

On the other hand, the Chebyshev–Lobatto collocation points, which are integrated into the matrix relations, are defined to as follows (see [12]):

$$t_i = \frac{a+b}{2} + \frac{a-b}{2}\cos\left(\frac{\pi i}{N}\right),\ i = 0, 1, \ldots, N. \tag{1.4}$$

2. Some basic properties of the Mott polynomial

The Mott polynomial $S_n(t)$, which forms a basis of the matrix-collocation method in this study, was introduced by Mott [21] while investigating the roaming behaviors of electrons for a problem in the theory of electrons. Then Erdèlyi et al. [10] proposed its explicit formula as follows:

$$S_n(t) = \left(-\frac{t}{2}\right)^n (n-1)! \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{t^{-2l}}{l!\,(n-l)!\,(n-2l-1)!} = (n!)^{-1}\left(-\frac{t}{2}\right)^n {}_3F_0\left(-n, \frac{1}{2} - \frac{n}{2}, 1 - \frac{n}{2}; -4t^{-2}\right),$$

where ${}_3F_0$ is a generalized hypergeometric function.

In 1984, Roman [27] established its associated Sheffer sequence and generating function as follows:

$$f(t) = \frac{-2t}{1-t^2} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{S_k(t)}{k!} s^k = \exp\left(\frac{t\sqrt{1-s^2} - t}{s}\right),$$

where the first five polynomials are

$$\{S_0(t), S_1(t), S_2(t), S_3(t), S_4(t)\} = \left\{1, -\frac{t}{2}, \frac{t^2}{4}, -\frac{3t}{4} - \frac{t^3}{8}, \frac{t^2}{2} + \frac{t^4}{16}\right\}.$$

On the other hand, a triangle coefficient matrix of the Mott polynomial is available in A137378 of OEIS [29]. Recently, Kruchinin [16] gave a generalized form of the Mott polynomial with a parameter-α as follows:

$$S_n(t,\alpha) = \sum_{p=1}^{n} \sum_{q=0}^{p} (-1)^{p-q+(n+p)/2} \frac{n!\left(1 + (-1)^{n+p}\right)}{2p!} \binom{p}{q} \binom{\alpha q}{(n+p)/2} t^p,\ n > 0,$$

where the Mott polynomial is obtained for $\alpha = 0.5$. One can refer to [10, 16, 21, 27] for more properties about the Mott polynomial.

3. Description of matrix relations and method of solution

A reduced computational matrix relation based on the Mott polynomial and method of solution are constructed in this section. Due to this matrix relation, the matrix relations of nonlinear terms in Eq. (1.1) are also properly established. The matrix relation of the Mott polynomial solution (1.3) is of the form [17]

$$y(t;\alpha) = \boldsymbol{S}(t,\alpha)\ \boldsymbol{Y}, \tag{3.1}$$

and its differentiated matrix relation

$$y^{(k)}(t;\alpha)=\boldsymbol{S}^{(k)}(t,\alpha)\,\boldsymbol{Y}, \tag{3.2}$$

where

$$\boldsymbol{S}(t,\alpha)=\begin{bmatrix} S_0(t,\alpha) & S_1(t,\alpha) & \cdots & S_N(t,\alpha)\end{bmatrix},$$

$$\boldsymbol{S}^{(k)}(t,\alpha)=\begin{bmatrix} S_0^{(k)}(t,\alpha) & S_1^{(k)}(t,\alpha) & \cdots & S_N^{(k)}(t,\alpha)\end{bmatrix} \text{ and } \boldsymbol{Y}=\begin{bmatrix} y_0 & y_1 & \cdots & y_N\end{bmatrix}^T.$$

By substituting the matrix relation (3.2) and the collocation points (1.4) into Eq. (1.1), the matrix relation of the linear part of Eq. (1.1) turns out to be

$$[L[y(t_i)]]=\sum_{k=0}^{3}\boldsymbol{P}_k\boldsymbol{S}^{(k)}(\alpha)\,\boldsymbol{Y}, \tag{3.3}$$

where

$$L[y(t)]=\sum_{k=0}^{3}P_k(t)y^{(k)}(t),\ \ \boldsymbol{P}_k=\begin{bmatrix} P_k(t_0) & 0 & \cdots & 0\\ 0 & P_k(t_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & P_k(t_N)\end{bmatrix}_{(N+1)\times(N+1)},$$

and

$$\boldsymbol{S}^{(k)}(\alpha)=\begin{bmatrix}\boldsymbol{S}^{(k)}(t_0,\alpha)\\ \boldsymbol{S}^{(k)}(t_1,\alpha)\\ \vdots\\ \boldsymbol{S}^{(k)}(t_N,\alpha)\end{bmatrix}=\begin{bmatrix} S_0^{(k)}(t_0,\alpha) & S_1^{(k)}(t_0,\alpha) & \cdots & S_N^{(k)}(t_0,\alpha)\\ S_0^{(k)}(t_1,\alpha) & S_1^{(k)}(t_1,\alpha) & \cdots & S_N^{(k)}(t_1,\alpha)\\ \vdots & \vdots & \ddots & \vdots\\ S_0^{(k)}(t_N,\alpha) & S_1^{(k)}(t_N,\alpha) & \cdots & S_N^{(k)}(t_N,\alpha)\end{bmatrix}_{(N+1)\times(N+1)}.$$

Let us now introduce the matrix relation of the nonlinear part of Eq. (1.1). Using the fundamental matrix relation (3.2) and the collocation points (1.4), the matrix relation of the nonlinear part, which consists of a quartic nonlinear term, can be constructed as

$$[N[y(t_i)]]=\boldsymbol{Q}_{ijkl}^{pq}\boldsymbol{S}^{(i)}(\alpha)\,\boldsymbol{S}_{1*}^{(j)}(\alpha)\left(\boldsymbol{S}_{2*}^{(k)}(\alpha)\right)^p\left(\boldsymbol{S}_{3*}^{(l)}(\alpha)\right)^q\boldsymbol{Y}_{(p+q+1)*}, \tag{3.4}$$

where $N[y(t)]=\left[Q_{ijkl}^{pq}y^{(i)}y^{(j)}\left(y^{(k)}\right)^p\left(y^{(l)}\right)^q\right](t),\ p,q=\{0,1\}$, and $i,j,k,l=\{0,1,2,3\}$,

$$\boldsymbol{Q}_{ijkl}^{pq}=\begin{bmatrix} Q_{ijkl}^{pq}(t_0) & 0 & \cdots & 0\\ 0 & Q_{ijkl}^{pq}(t_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & Q_{ijkl}^{pq}(t_N)\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\boldsymbol{S}_{1*}^{(j)}(\alpha)=\begin{bmatrix} \boldsymbol{S}^{(j)}(t_0,\alpha) & 0 & \cdots & 0 \\ 0 & \boldsymbol{S}^{(j)}(t_1,\alpha) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \boldsymbol{S}^{(j)}(t_N,\alpha) \end{bmatrix}_{(N+1)\times(N+1)^2},$$

$$\left(\boldsymbol{S}_{2*}^{(k)}(\alpha)\right)^p=\operatorname{diag}\left[\boldsymbol{S}_{1*}^{(k)}(\alpha)\right]_{(N+1)^{p+1}\times(N+1)^{p+2}},\ \left(\boldsymbol{S}_{3*}^{(l)}(\alpha)\right)^q=\operatorname{diag}\left[\boldsymbol{S}_{2*}^{(l)}(\alpha)\right]_{(N+1)^{p+2}\times(N+1)^{p+q+2}},$$

$$\boldsymbol{Y}_{1*}=\begin{bmatrix} y_0\boldsymbol{Y} & y_1\boldsymbol{Y} & \cdots & y_N\boldsymbol{Y}\end{bmatrix}^T_{1\times(N+1)^2},\ \boldsymbol{Y}_{2*}=\begin{bmatrix} y_0\boldsymbol{Y}_{1*} & y_1\boldsymbol{Y}_{1*} & \cdots & y_N\boldsymbol{Y}_{1*}\end{bmatrix}^T_{1\times(N+1)^3},$$

and

$$\boldsymbol{Y}_{(p+q+1)*}=\begin{bmatrix} y_0\boldsymbol{Y}_{(p+q)*} & y_1\boldsymbol{Y}_{(p+q)*} & \cdots & y_N\boldsymbol{Y}_{(p+q)*}\end{bmatrix}^T_{1\times(N+1)^{p+q+2}}.$$

In view of the matrix relations (3.3) and (3.4), the fundamental matrix equation of Eq. (1.1) is now expressed by

$$\sum_{k=0}^{3}\boldsymbol{P}_k\boldsymbol{S}^{(k)}(\alpha)\,\boldsymbol{Y}+\boldsymbol{Q}_{ijkl}^{pq}\,\boldsymbol{S}^{(i)}(\alpha)\,\boldsymbol{S}_{1*}^{(j)}(\alpha)\left(\boldsymbol{S}_{2*}^{(k)}(\alpha)\right)^p\left(\boldsymbol{S}_{3*}^{(l)}(\alpha)\right)^q\boldsymbol{Y}_{(p+q+1)*}=\boldsymbol{F},\tag{3.5}$$

such that p and q are equal to 0 or 1 and

$$\boldsymbol{F}=\begin{bmatrix} f(t_0) & f(t_1) & \cdots & f(t_N)\end{bmatrix}^T_{1\times(N+1)}.$$

Alternatively, Eq. (3.5) can be written briefly as

$$\boldsymbol{W}\boldsymbol{Y}+\boldsymbol{Z}_{(p+q+1)*}\boldsymbol{Y}_{(p+q+1)*}=\boldsymbol{F}\ \text{ or }\ \left[\boldsymbol{W};\,\boldsymbol{Z}_{(p+q+1)*}:\boldsymbol{F}\right],$$

where

$$\boldsymbol{W}=\sum_{k=0}^{3}\boldsymbol{P}_k\boldsymbol{S}^{(k)}(\alpha)\ \text{and}\ \boldsymbol{Z}_{(p+q+1)*}=\boldsymbol{Q}_{ijkl}^{pq}\,\boldsymbol{S}^{(i)}(\alpha)\,\boldsymbol{S}_{1*}^{(j)}(\alpha)\left(\boldsymbol{S}_{2*}^{(k)}(\alpha)\right)^p\left(\boldsymbol{S}_{3*}^{(l)}(\alpha)\right)^q$$

represent the linear and nonlinear matrix equations, respectively.

On the other hand, using the matrix relations (3.1) and (3.2), the matrix forms of the initial and boundary conditions (1.2) are written respectively as

$$\left.\begin{aligned} \boldsymbol{U}_1&=\begin{bmatrix} S_0(a,\alpha) & S_1(a,\alpha) & \cdots & S_N(a,\alpha)\end{bmatrix},\\ \boldsymbol{U}_2&=\begin{bmatrix} S_0^{(1)}(a,\alpha) & S_1^{(1)}(a,\alpha) & \cdots & S_N^{(1)}(a,\alpha)\end{bmatrix},\\ \boldsymbol{U}_3&=\begin{bmatrix} S_0(b,\alpha) & S_1(b,\alpha) & \cdots & S_N(b,\alpha)\end{bmatrix}.\end{aligned}\right\}\tag{3.6}$$

In accordance with the number of the conditions of Eq. (1.1) subjected to Eq. (1.2), replacing the row matrices (3.6) by any row matrices of $\boldsymbol{W}$, the augmented matrix is then stated as

$$\left[\widetilde{\boldsymbol{W}}\,;\,\widetilde{\boldsymbol{Z}}_{(p+q+1)*}\,:\,\widetilde{\boldsymbol{F}}\right]=\left[\begin{array}{ccccc}\boldsymbol{W}^{*} & ; & \boldsymbol{Z}^{*}_{(p+q+1)*} & : & \boldsymbol{F}^{*}\\ \boldsymbol{U}_1 & ; & \boldsymbol{0}_{1\times(N+1)^{p+q+2}} & : & \gamma_1\\ \boldsymbol{U}_2 & ; & \boldsymbol{0}_{1\times(N+1)^{p+q+2}} & : & \gamma_2\\ \boldsymbol{U}_3 & ; & \boldsymbol{0}_{1\times(N+1)^{p+q+2}} & : & \gamma_3\end{array}\right], \tag{3.7}$$

where $\boldsymbol{W}^{*}$, $\boldsymbol{Z}^{*}_{(p+q+1)*}$ and $\boldsymbol{F}^{*}$ are in $(N-2)\times(N+1)$, $(N-2)\times(N+1)^{p+q+2}$ and $(N-2)\times 1$ dimensions, respectively. Also, $\boldsymbol{0}_{1\times(N+1)^{p+q+2}}$ is a zero matrix in $1\times(N+1)^{p+q+2}$ dimension.

We here draw attention to the fact that the matrix forms (3.6) are replaced by the last zero matrices of $\widetilde{\boldsymbol{Z}}_{(p+q+1)*}$, in order to treat the fully nonlinear differential equations derived from Eq. (1.1). It is also evident that $\boldsymbol{W}$ is a null matrix when $P_k(t)=0$. We are then ready to solve the augmented matrix (3.7) by eliminating an algebraic system of nonlinear equations. After substituting the Mott coefficients obtained from the augmented matrix (3.7) into Eq. (1.3), Eq. (1.3) yields the Mott polynomial solution with the parameter-α.

4. Residual convergence estimation

In order to establish the convergence estimation of the method, the residual function in terms of the computation limit N is given first. Letting the residual function $R_N(t)$ be a function defined in $L_1[a,b]$, then it is obtained by substituting the Mott polynomial solution $y_N(t;\alpha)$ into Eq. (1.1), so

$$R_N(t)=L\left[y_N\left(t;\alpha\right)\right]+N\left[y_N\left(t;\alpha\right)\right]-f(t),$$

where $L[\bullet]$ and $N[\bullet]$ represent linear and nonlinear parts of Eq. (1.1), respectively.

Then the residual convergence estimation (RCE_N) can be prescribed as

$$RCE_N=\|R_N(t)-R_{N-1}(t)\|=\int_a^b\left|R_N(t)-R_{N-1}(t)\right|dt<\epsilon,$$

where ϵ is a sufficiently small value and estimates a convergence accuracy between the computation limits N and $N-1$ as N is increased.

RCE_N thereby provides a clear observation about the convergence estimation related to N. It is worth mentioning that this estimation can be used when an exact solution of the model problem is unknown.

5. Oscillatory behavior of solutions via Mott–Laplace–Padé methodology

In this section, the oscillatory behavior of the solutions is constructed by means of a coupled methodology based on the proposed method and Laplace–Padé technique [20, 31]. Previously, Momani and Ertürk [20] investigated the oscillations of the solutions of the nonlinear oscillator equations by proposing the differential transform method based on the Laplace–Padé technique. Then Sweilam and Khader [31] made use of the homotopy perturbation method along with the Laplace–Padé technique to obtain the exact solutions of the nonlinear partial differential equations. Now, using the Mott polynomial solution (1.3), the coupled methodology is stated as follows:

$$H(s) = \mathbb{L}\{y_N(t;\alpha)\} = \int_0^\infty y_N(t;\alpha)\, e^{-st} dt,$$

where $\mathbb{L}\{\bullet\}$ is the Laplace transform.

Inserting $s \to 1/t$ into $H(s)$, it then follows that

$$\mathbb{P}\left[H\left(\frac{1}{t}\right)\right] = G\left(\frac{1}{t}\right),$$

where $\mathbb{P}[\bullet]$ is the Padé approximant [1].

Now, inserting $t \to 1/s$ into $G\left(\frac{1}{t}\right)$ and then taking its inverse Laplace transform, the Mott–Laplace–Padé solution $y_{P,N}(t;\alpha)$, which highly determines the oscillatory behavior of the Mott polynomial solution on long time intervals, is finally reached. Note here that the Padé approximant is a well-known mathematical operation and its details can be found in [1]. Also, it can be easily processed with symbolic software, such as Mathematica, MATLAB, and Maple.

6. Illustrative models

In this section, the various model differential equations involving specific nonlinearities of quartic type are handled with the aid of the proposed method. To do this, a unique computer program module of the method is developed in Mathematica 11, which is run by a PC equipped with 8 GB RAM and 3.30 GHz CPU. It is thus aimed that very clear results be obtained for illustrative model problems. It is also worth specifying that we apply the NDSolve module built in Mathematica to some problems whose analytical solutions are unknown so that the validity of the present method can be observed. After obtaining the outcomes of the programs, the solutions are simulated and indicated in figures and tables. Note that the absolute error computation is denoted as $|e_N(t;\,\alpha)|$, where α is the parameter of the Mott polynomial solution (1.3).

Model 6.1. [3, 22] Consider the second-order differential equation of functional nonlinear type determining the deflection of a cantilever beam exposed to a concentrated load:

$$y''(t) + \lambda t \cos(y(t)) = 0, \;\; 0 \leq t \leq 1, \tag{6.1}$$

subject to the boundary conditions $y'(0) = 0$ and $y(1) = 0$. Here, λ represents a ratio between the concentrated vertical load at the free end and the flexural rigidity [3, 22], and the analytical solution of this problem is also unknown. In order to handle this problem using the present method, Eq. (6.1) can be written alternatively as a differential equation of quartic nonlinear type:

$$y''(t) + \lambda t - \frac{\lambda t y^2(t)}{2} + \frac{\lambda t y^4(t)}{24} = 0.$$

Now, by employing the present method, we can solve the above equation constrained by boundary conditions for various λ, α, and N. As previously stated by Na [22], we immediately reach $y'(1) = -3.194$ thanks to $N = 5$ and $\alpha = 6.04$ for $\lambda = 8$. Table 1 indicate the consistency between the Mott polynomial solutions with α and the solution obtained by Mathematica (MS). The deflections of a cantilever exposed to different forces λ are also clearly observed via the Mott polynomial solution $y_5(t;\,2)$ in Figure 1. That is, as λ

increases, the visibility of the deflection becomes more clear. It can be inferred from Table 2 that the residual convergence estimation yields consistent values for $\alpha = 2$ and $\lambda = \{0.05, 0.5, 1, 2\}$.

Table 1. Comparison of the absolute errors and CPU time (s) for Model 6.1 with $\lambda = 1$.

t_i	$\|e_3(t_i; 0.5)\|$	$\|e_3(t_i; 2)\|$	$\|e_4(t_i; 0.5)\|$	$\|e_4(t_i; 2)\|$
0.0	3.6341e–03	2.9045e–04	4.1988e–03	7.7609e–05
0.1	3.6311e–03	2.9086e–04	4.1985e–03	7.7615e–05
0.2	3.6107e–03	2.9377e–04	4.1932e–03	7.8289e–05
0.3	3.5546e–03	3.0124e–04	4.1700e–03	8.1329e–05
0.4	3.4436e–03	3.1398e–04	4.1060e–03	8.8717e–05
0.5	3.2555e–03	3.2983e–04	3.9674e–03	1.0121e–04
0.6	2.9635e–03	3.4206e–04	3.7073e–03	1.1660e–04
0.7	2.5345e–03	3.3768e–04	3.2641e–03	1.2807e–04
0.8	1.9284e–03	2.9670e–04	2.5608e–03	1.2339e–04
0.9	1.0993e–03	1.9321e–04	1.5064e–03	8.6093e–05
Time	0.047	0.062	0.375	0.281

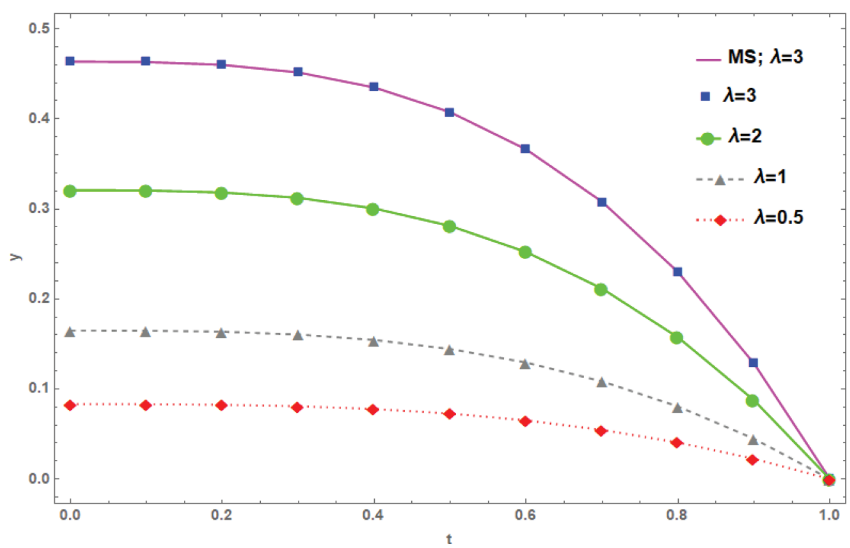


Figure 1. Deflections of a cantilever beam exposed to different λ for Model 6.1.

Table 2. Residual convergence estimation with respect to λ and N for Model 6.1.

$\lambda \,\|\, N$	4	5	6
0.05	1.82e–07	6.69e–08	4.61e–08
0.50	1.81e–04	6.72e–05	4.73e–05
1.00	1.43e–03	5.45e–04	4.10e–05
2.00	1.10e–02	4.68e–03	4.82e–03

Model 6.2. [14, 23] Consider the second-order nonlinear differential equation of the motion of a mass that follows a parabolic path:

$$y''(t) + y(t) + 4\lambda^2 \left(y^2(t)y''(t) + y(t)\left(y''(t)\right)^2 \right) = 0,\ 0 \le t \le 1,$$

subject to the initial conditions $y(0) = 1.8$ and $y'(0) = 0$. Here, λ determines a nonlinear force and the exact solution of this problem is unknown. The problem is solved by implementing the present method for different α and λ. In order to observe the validity of the present method, the problem is also treated with the aid of the NDSolve module, which is of the form

$$\text{NDSolve}[\{y''[t] + y[t] + 4\lambda^2 * \left(y''[t] * (y[t])^2 + y[t] * (y''[t])^2\right) == 0,$$
$$y[0] == 1.8, y'[0] == 0\}, y[t], \{t, 0, 1\}] [[1, 1, 2]].$$

The consistency between the Mott polynomial and Mathematica solutions is observed in Figure 2. Table 3 shows that highly precise approximations and considerable CPU time are obtained in terms of N and α. These approximations take six decimal place error on average. In addition, as N is increased from 4 to 7, the residual convergence estimation RCE_N is determined for $\lambda = 1/2$:

$$RCE_N = \{3.8e-02,\ 9.7e-03,\ 1.2e-02,\ 4.5e-03\},$$

and for $\lambda = \sqrt{2}/2$,

$$RCE_N = \{2.4e-02,\ 8.3e-03,\ 9.2e-03, 2.7e-03\}.$$

Model 6.3. [8] Consider the third-order differential equation of the radical nonlinear type

$$y'''(t) + \sqrt{1 - y^2(t)} = 0,\ 0 \le t \le \frac{\pi}{2}, \tag{6.2}$$

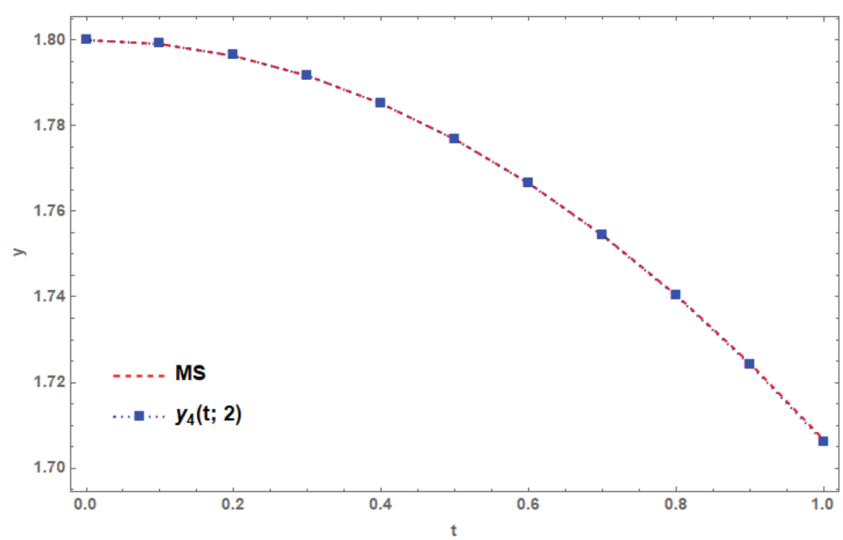


Figure 2. Displacement of the Mott polynomial and Mathematica (MS) solutions for Model 6.2 with $\lambda = \sqrt{3}/2$.

Table 3. Comparison of the absolute errors and CPU time (s) for Model 6.2 with $\lambda = \sqrt{3}/2$.

t_i	$\lvert e_5(t_i;\,2)\rvert$	$\lvert e_6(t_i;\,2)\rvert$	$\lvert e_6(t_i;\,0.5)\rvert$
0.1	2.1662e–06	2.2433e–08	2.2808e–08
0.2	1.4807e–05	8.8424e–08	2.5536e–06
0.3	4.2277e–05	1.2117e–07	1.1806e–05
0.4	8.2496e–05	6.2039e–08	3.5975e–05
0.5	1.2694e–04	7.0930e–07	8.6264e–05
0.6	1.6072e–04	2.1743e–06	1.7708e–04
0.7	1.6250e–04	4.8544e–06	3.2610e–04
0.8	1.0451e–04	9.1691e–06	5.5437e–04
0.9	4.7487e–05	1.5517e–05	8.8637e–04
1.0	3.3431e–04	2.4225e–05	1.3501e–03
Time	0.234	0.641	0.813

subject to the initial and boundary conditions $y\,(0) = 0$, $y'\,(0) = 1$, and $y\left(\frac{\pi}{2}\right) = 1$. Here, the exact solution of this problem is $y(t) = \sin(t)$. In order to bring Eq. (6.2) into compliance with the proposed method, we can write Eq. (6.2) as a fully nonlinear form:

$$\left(y'''\,(t)\right)^2 - y^2(t) = 1.$$

The equation above can now be solved under the linear initial and boundary conditions. The displacement of the approximate solution is illustrated in Figure 3. It can be seen from Figure 3 that a good approximation to the exact solution is obtained. We obtain three decimal place error for $N = 6$, while Duan and Rach [8] recently obtained two decimal place maximum error for a sixth-order polynomial approach corresponding to their computation limit $n = 2$. As N is increased from 4 to 10, the residual convergence estimation RCE_N is prescribed respectively as

$$RCE_N = \{3.8e-01, 3.5e-01, 9.7e-01, 2.5e-01, 9.8e-02, 8.8e-02, 2.5e-02\}.$$

Model 6.4. [11, 30] Consider the Rayleigh–Duffing equation (or oscillator) appearing in acoustics:

$$y''\,(t) + 2p_1 y'(t) + p_0 y(t) + \lambda_1 y^3(t) + \lambda_2\left(y'(t)\right)^3 = 0,\ 0 \le t \le L,$$

subject to the initial conditions $y\,(0) = 0$ and $y'\,(0) = 0.5$. Here, the exact solution of this problem is unknown, p_1 is a damping force, p_0 is the stiffness, and $\{\lambda_1,\,\lambda_2\}$ are real nonlinear forces [11]. This problem is solved using the present method along with the Laplace–Padé technique for $L = \{1,\,20\}$. In Figure 4, according to the damping force, we investigate the oscillatory behavior of the Mott polynomial solution and the Mathematica solution, which is returned by

$$\text{NDSolve}[\{y''[t] + 2p_1 * y'[t] + p_0 * y[t] + \lambda_1 * (y[t])^3 + \lambda_2 * (y'[t])^3 == 0,$$
$$y[0] == 0, y'[0] == 0.5\}, y[t], \{t, 0, 20\}][[1, 1, 2]].$$

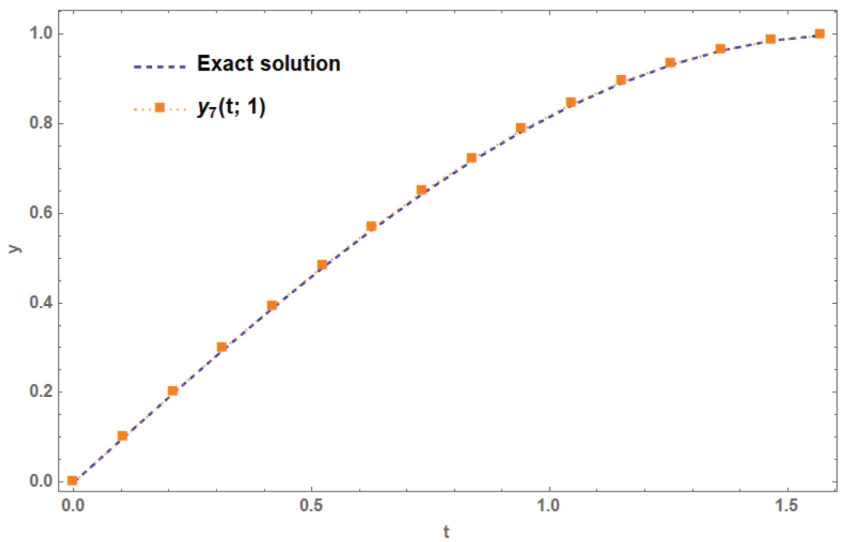


Figure 3. Displacement of the Mott polynomial and exact solutions for Model 6.3.

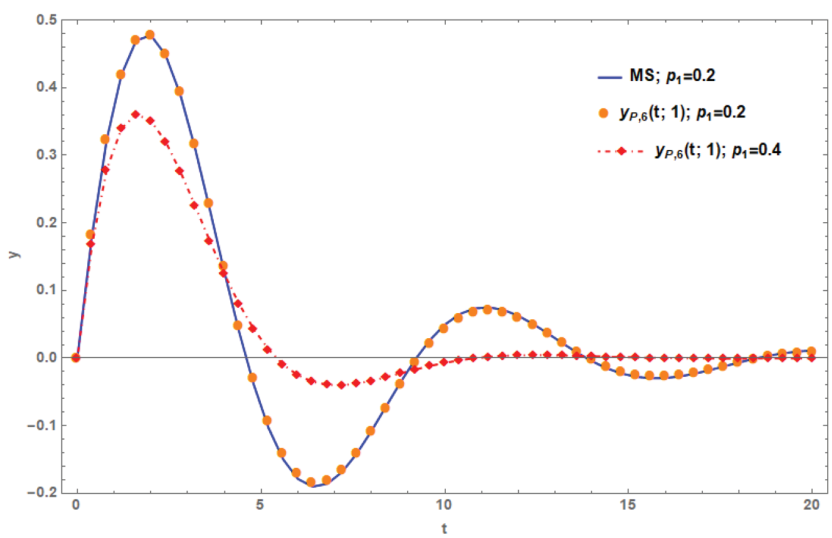


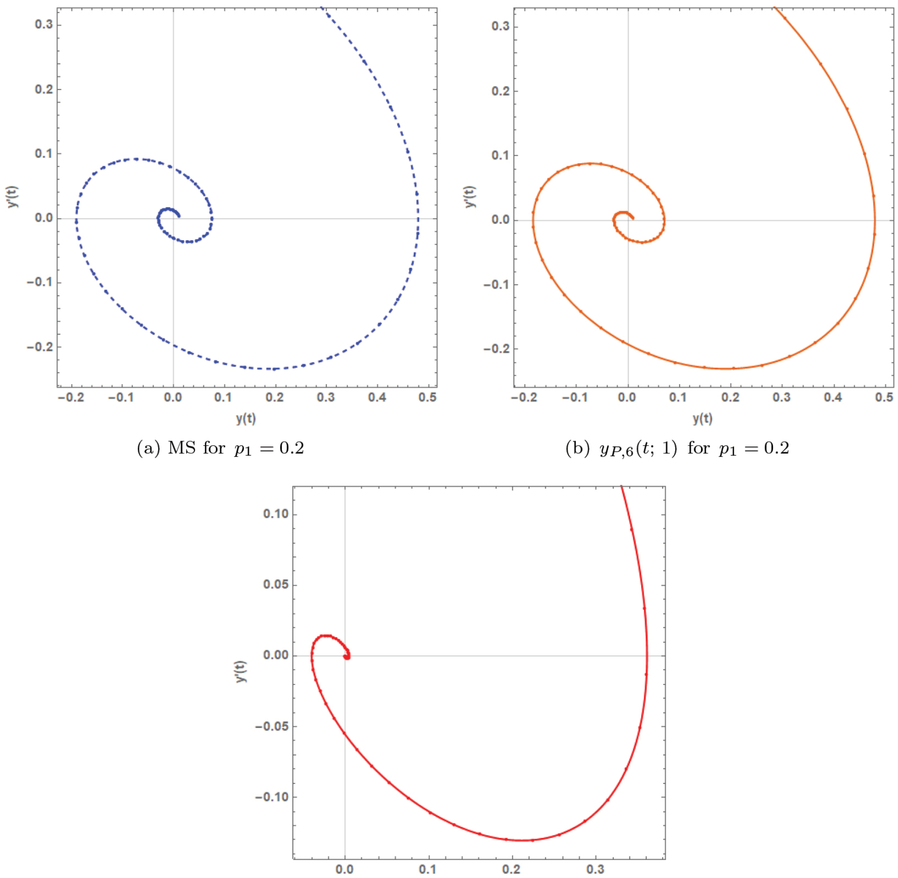
Figure 4. Damping oscillations of the solutions on $[0, 20]$ for Model 6.4 with $p_0 = 0.5$ and $\lambda_1 = \lambda_2 = 0.05$.

Futhermore, Figure 5 shows the phase planes of these solutions in Figure 4 for $L = 20$. It can be noticed that the Mott polynomial solution coincides well with the Mathematica solution and also their damping oscillation is clearly varied in accordance with the damping force p_1. On the other hand, as N increases, the residual convergence estimation is illustrated by means of a logarithmic scale in Figure 6.

Model 6.5. [19, 23] Consider the nonlinear oscillator equation

$$y''(t) + y(t) - \lambda y(t)\, y'(t)\, y''(t) = 0,\ 0 \leq t \leq L,$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$. Here, the exact solution of this problem is unknown and λ stands for a nonlinear force. After applying the present method along with the Laplace-Padé



(a) MS for $p_1 = 0.2$ (b) $y_{P,6}(t;\, 1)$ for $p_1 = 0.2$

(c) $y_{P,6}(t;\, 1)$ for $p_1 = 0.4$

Figure 5. Phase planes of the solutions with respect to the damping force on $[0, 20]$ for Model 6.4 with $p_0 = 0.5$ and $\lambda_1 = \lambda_2 = 0.05$.

technique for $L = \{1,\, 20\}$ and $\lambda = 0.01$, the oscillatory behaviors of the Mott polynomial and Mathematica (MS) solutions are simulated in Figure 7 and their phase plane diagrams are established in Figure 8. From $N = 3$ to $N = 8$, the residual convergence estimation is determined as logarithmic behavior in Figure 9.

Model 6.6. [2, 7, 15, 26, 28] Consider the singular nonlinear differential equation modeling the radial stress on a rotationally symmetric shallow membrane cap:

$$y''\,(t) + \frac{3}{t} y'(t) + \frac{1}{8y^2(t)} = \frac{1}{2},\;\; 0 < t < 1, \tag{6.3}$$

subject to the boundary conditions $y'\,(0) = 0$ and $y\,(1) = 1$. Here, the exact solution to this problem is unknown. Before this problem is solved, some operations on Eq. (6.3) should be performed to employ the proposed method. Therefore, Eq. (6.3) can be written as a fully nonlinear form:

$$ty^2(t)y''\,(t) + 3y^2(t)y'(t) - \frac{ty^2(t)}{2} = -\frac{t}{8}.$$

The above equation constrained by the boundary conditions can be solved via the proposed method.

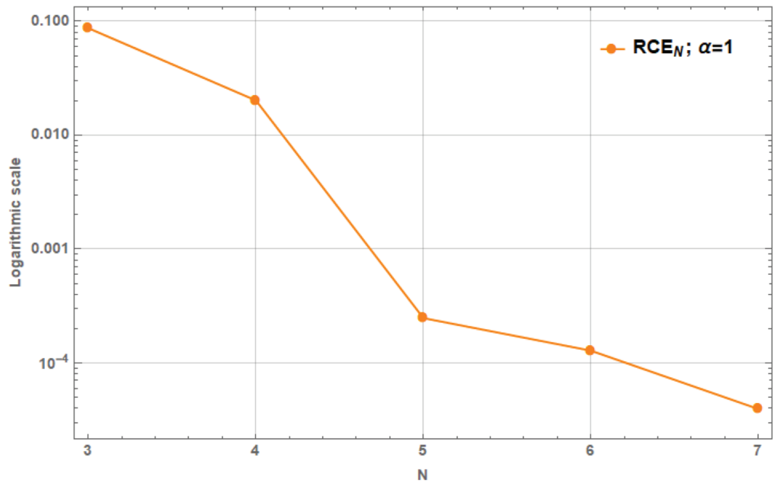


Figure 6. Logarithmic scale of the residual convergence estimation with respect to N for Model 6.4 with $L = 1$, $p_1 = 0.2$, $p_0 = 0.5$, and $\lambda_1 = \lambda_2 = 0.05$.

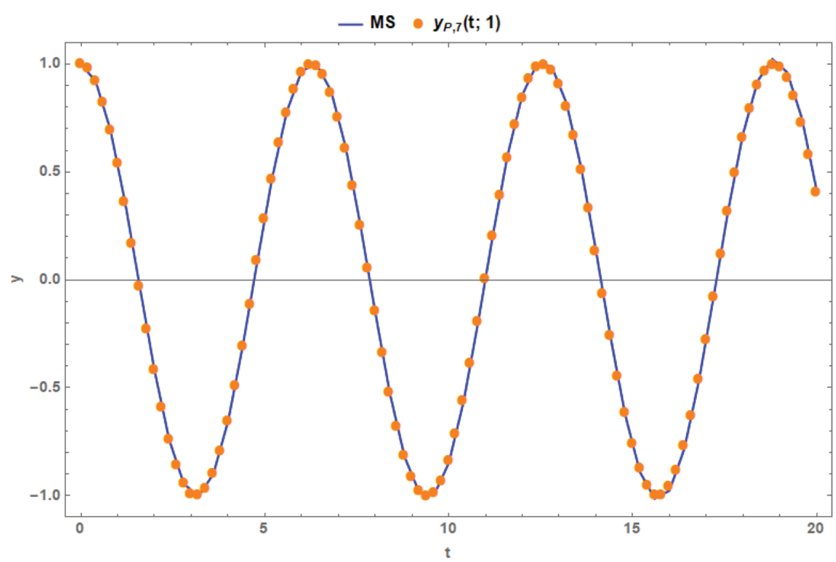


Figure 7. Oscillations of the solutions on $[0, 20]$ for Model 6.5 with $\lambda = 0.01$.

In Table 4, we compare the numerical results of the Mott polynomial solutions with the existing ones, which were previously obtained by the shifted second-kind Chebyshev wavelet method (CWM) [26], the Adomian decomposition method with Green functions [28], and the variational iteration method (VIM) [15]. It is easy to see that the present results are in good agreement with the others. The residual convergence estimation is demonstrated as logarithmic behavior in Figure 10, where the decaying diagram can be seen.

7. Conclusions

A reduced computational matrix approach based on the Mott polynomial and Chebyshev–Lobatto collocation points has been proposed to solve model differential equations involving specific nonlinearities of quartic type. These type equations have been considered in this study for the first time. The reduced matrix system has provided a fast and efficient approximation to the considered problems as seen from Tables 1 and 3. The

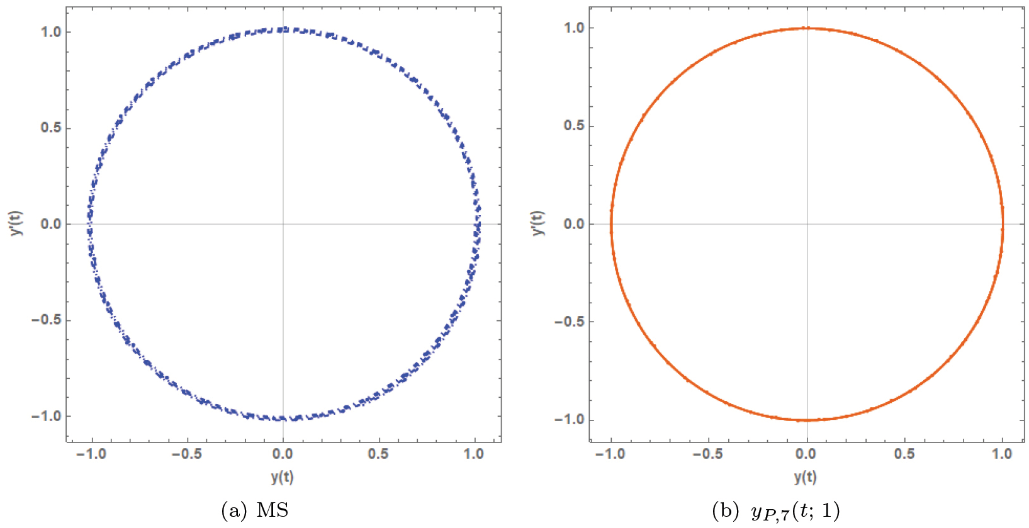


Figure 8. Phase planes of the solutions on $[0,20]$ for Model 6.5 with $\lambda = 0.01$.

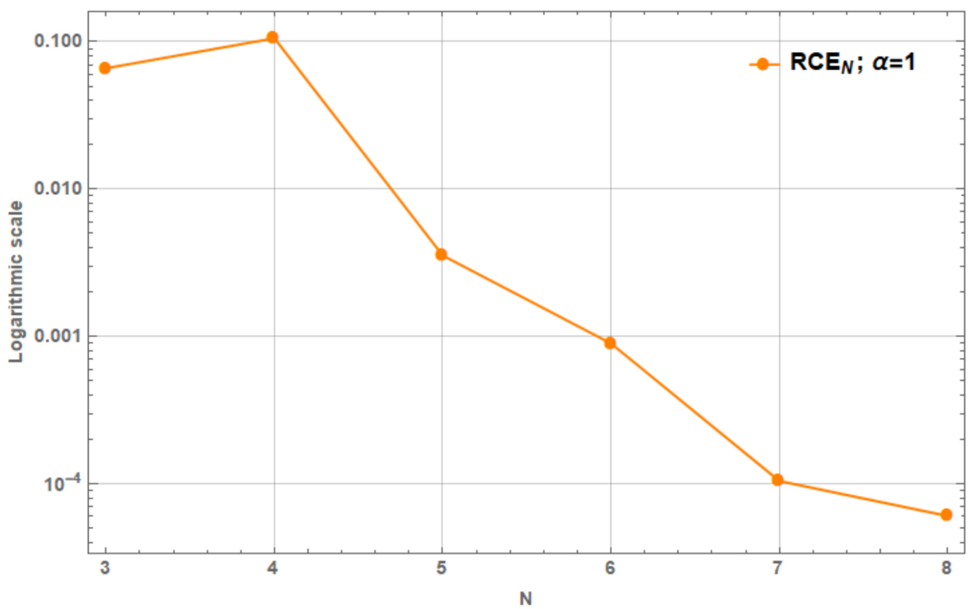


Figure 9. Logarithmic scale of the residual convergence estimation with respect to N for Model 6.5 with $L = 1$ and $\lambda = 0.01$.

consistency of the solutions has been varied via the parameter-α. Since α is a binomial coefficient in the generalized Mott polynomial, it easily changes the behavior of the solutions. Thus, the optimal consistency of the parameter α has been investigated in $(0,10]$. Note that the augmented matrix (3.7) has not returned any Mott coefficient for $\alpha = 0$. The residual convergence estimation has presented clear convergence accuracy with respect to the computational limit of the method as observed in Table 2 and Figures 6, 9, and 10. Although some problems have no exact solution, it can be noticed from Figures 1, 2, 4, 5, 7, and 8 that the displacements of the approximate solutions have been accurately compared with the Mathematica solution. Besides, the

Table 4. Comparison of the numerical results of the solutions for Model 6.6.

t_i	$y_5(t_i; 0.6)$	CWM [26]	GFADM [28]	VIM [15]
0.0	0.954215	0.9546	0.954135	0.952148
0.1	0.954663	0.9551	0.954589	0.952632
0.2	0.956009	0.9564	0.955950	0.954081
0.3	0.958257	0.9587	0.958220	0.956495
0.4	0.961411	0.9619	0.961403	0.959870
0.5	0.965481	0.9660	0.965503	0.964202
0.6	0.970476	0.9710	0.970526	0.969487
0.7	0.976409	0.9769	0.976479	0.975717
0.8	0.983295	0.9837	0.983369	0.982885
0.9	0.991152	0.9914	0.991206	0.990983

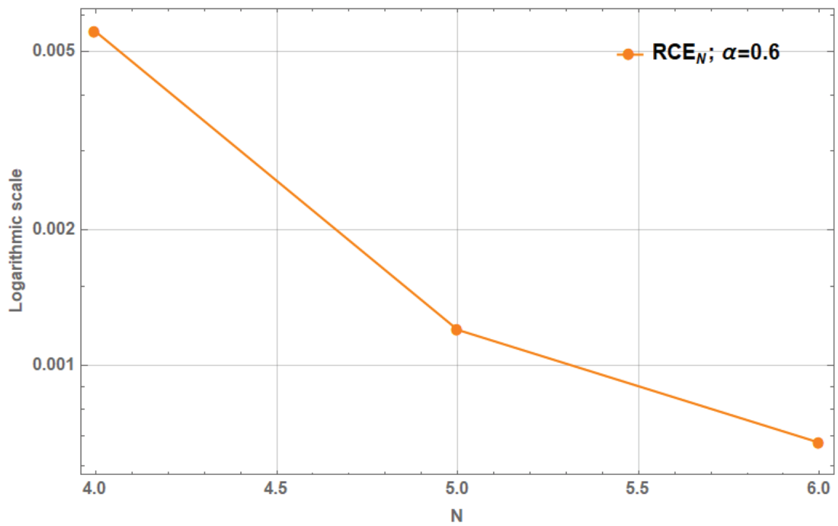


Figure 10. Logarithmic scale of the residual convergence estimation with respect to N for Model 6.6.

oscillatory behaviors and phase planes of the solutions have been provided in Figures 4, 5, 7, and 8. Eventually, it can be inferred from all results that the proposed method yields a fast, reliable, and simple scheme to treat model problems derived from Eq. (1.1) type. It is also evident that the proposed method can be developed for fractional differential and partial differential equations.

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