Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 240 – 251
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doi:10.3906/mat-1910-100

General coefficient estimates for bi-univalent functions: a new approach

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Received: 27.10.2019 • **Accepted/Published Online:** 03.12.2019 • **Final Version:** 20.01.2020

Abstract: We prove for univalent functions $f(z) = z + \sum_{k=n}^{\infty} a_k z^k; (n \geq 2)$ in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ with $f^{-1}(w) = w + \sum_{k=n}^{\infty} b_k w^k$; $(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4})$ that

$$b_{2n-1} = na_n^2 - a_{2n-1} \text{ and } b_k = -a_k \text{ for } (n \leq k \leq 2n-2).$$

As applications, we find estimates for $|a_n|$ whenever f is bi-univalent, bi-close-to-convex, bi-starlike, bi-convex, or for bi-univalent functions having positive real part derivatives in $\mathbb{U}$. Moreover, we estimate $|na_n^2 - a_{2n-1}|$ whenever f is univalent in $\mathbb{U}$ or belongs to certain subclasses of univalent functions. The estimation method can be applied for various subclasses of bi-univalent functions in $\mathbb{U}$ and it helps to improve well-known estimates and to generalize some known results as shown in the last section.

Key words: Univalent functions, bi-univalent functions, starlike functions, convex functions, close-to-convex functions, Faber polynomials, coefficient estimates

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Furthermore, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$. The class $\mathcal{P}$ consists of analytic functions p satisfying $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$, $(z \in \mathbb{U})$. The Carathéodory lemma states that the coefficients of $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ satisfy $|c_n| \leq 2$ for $n \geq 1$. Denote by $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{CV}$ respectively the subclasses of $\mathcal{S}$ consisting of close-to-convex, starlike, and convex functions in $\mathbb{U}$. Analytically, $f(z) \in \mathcal{S}^*$ if and only if $zf'(z)/f(z) \in \mathcal{P}$, $(z \in \mathbb{U})$, while $f(z) \in \mathcal{CV}$ if and only if $1 + zf''(z)/f'(z) \in \mathcal{P}$, $(z \in \mathbb{U})$. In addition, $f(z) \in \mathcal{C}$ if and only if $f'(z)/g'(z) \in \mathcal{P}$, $(z \in \mathbb{U})$ for some $g \in \mathcal{CV}$. Alexander's relation states that $f(z) \in \mathcal{CV}$ if and only if $zf'(z) \in \mathcal{S}^*$. Indeed, $\mathcal{CV} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}$.

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2010 *AMS Mathematics Subject Classification:* 30C45

We know, for every $f \in \mathcal{S}$ defined by (1.1), that the inverse function f^{-1} exists and has the form

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots, \quad (|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}). \tag{1.2}$$

That is, $f^{-1}(f(z)) = z,\ (z \in \mathbb{U})$ and $f(f^{-1}(w)) = w,\ (|w| < 1/4)$ according to the Koebe one-quarter theorem (see [15]). A function $f \in \mathcal{A}$ is said to be bi-property if both f and f^{-1} satisfy that property. The class of bi-univalent functions in $\mathbb{U}$ is denoted by Σ. Some examples of functions in Σ (see [7, 31]) are:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

However, the familiar Koebe function $z/(1-z)^2$ and the functions

$$z - \frac{1}{2}z^2, \quad \frac{z}{1-z^2}$$

are in $\mathcal{S}$ but not members of Σ. Finding bounds for the coefficients of classes of bi-univalent functions dates back to 1967, since Lewin showed in [26] that $|a_2| < 1.51$. However, Brannan and Clunie [7] conjectured that $|a_2| \leq \sqrt{2}$. Later, Netanyahu [29] found that $\max_{f\in\Sigma} |a_2| = 4/3$. The interest in the bounds of $|a_n|$ for classes of Σ increased with the publications [17, 31], where the nonsharp estimates for the first two coefficients were provided (see, for example, [8, 32]). In recent years, these works revived the investigation of the coefficient estimates for various subclasses of analytic and meromorphic bi-univalent functions (see [6, 9–11, 14, 19–22, 24, 28, 30, 34, 36]). Not much is known about the higher coefficients of bi-univalent functions as Ali et al. [4] also declared finding the bounds for $|a_n|$; $n \geq 4$ an open problem, because the condition of bi-univalency makes the behavior of the higher coefficients unpredictable. In this work, however, we find particular solutions.

It is well known for $f \in \mathcal{S}$, defined by (1.1), and $f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n$, see [18, pp. 56–57], that

$$b_n = \frac{(-1)^{n+1}}{n!}\begin{vmatrix} na_2 & 1 & 0 & \dots & 0 \\ 2na_3 & (n+1)a_2 & 2 & \dots & 0 \\ 3na_4 & (2n+1)a_3 & (n+2)a_2 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & n-2 \\ (n-1)na_n & [(n-2)n+1]a_{n-1} & [(n-3)n+2]a_{n-2} & \dots & (2n-2)a_2 \end{vmatrix}. \tag{1.3}$$

The elements in the above determinants $|A_{ij}|$ are given by

$$A_{ij} = \begin{cases} [(i-j+1)n + j - 1]a_{i-j+2}, & \text{if } i+1 \geq j \\ 0, & \text{if } i+1 < j. \end{cases} \tag{1.4}$$

In particular, according to (1.3), we have $b_2 = -a_2$,

$$b_3 = \frac{(-1)^4}{3!}\begin{vmatrix} 3a_2 & 1 \\ 6a^3 & 4a_2 \end{vmatrix} = 2a_2^2 - a_3,$$

$$b_4 = \frac{(-1)^5}{4!}\begin{vmatrix} 4a_2 & 1 & 0 \\ 8a_3 & 5a_2 & 2 \\ 12a_4 & 9a_3 & 6a_2 \end{vmatrix} = 5a_2a_3 - 5a_2^3 - a_4,$$

and

$$b_5 = \frac{(-1)^6}{5!}\begin{vmatrix} 5a_2 & 1 & 0 & 0 \\ 10a_3 & 6a_2 & 2 & 0 \\ 15a_4 & 11a_3 & 7a_2 & 3 \\ 20a_5 & 16a_4 & 12a_3 & 8a_2 \end{vmatrix} = 6a_2a_4 - 21a_2^2a_3 + 14a_2^4 + 3a_3^2 - a_5.$$

Loewner, using his parametric method (see [27] and [23, p. 222]), proved that if f, defined by (1.1), belongs to $\mathcal{S}$ or $\mathcal{S}^*$, then

$$|b_n| \leq \frac{\Gamma(2n+1)}{\Gamma(n+2)\Gamma(n+1)}, \quad n \in \{2, 3, \cdots\}, \tag{1.5}$$

where the extremal function that satisfies the equality in (1.5) is the inverse of the Koebe function.

Many authors have used the Faber polynomials, introduced by Faber [16], to estimate $|a_n|$ for various subclasses of Σ (see, for example, [5, 12, 13]). In fact, the coefficients b_n can be expressed, using the Faber polynomials, in the form

$$b_n = \frac{1}{n}K_{n-1}^{-n}(a_2, a_3, ..., a_n),$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}[a_5 + (-n+2)a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6 + (-2n+5)a_3a_4] + \sum_{j\geq 7} a_2^{n-j}V_j, \end{aligned}$$

where such expressions as (for example) $(-n)!$ are to be interpreted symbolically by

$$(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2)\cdots, \qquad (n \in \{0, 1, 2, ...\})$$

and V_j is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ (see [3]). In particular,

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3(2a_2^2 - a_3) \quad \text{and} \quad K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, an expansion of K_{n-1}^p is given by (see for details [2])

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}D_{n-1}^2 + \frac{p!}{(p-3)!3!}D_{n-1}^3 + \cdots + \frac{p!}{(p-n+1)!(n-1)!}D_{n-1}^{n-1}, \tag{1.6}$$

where p is an integer number and $D_{n-1}^p = D_{n-1}^p(a_2, a_3, ...)$, and alternatively (see [33]),

$$D_{n-1}^m(a_2, a_3, ..., a_n) = \sum \frac{m!}{\mu_1!\mu_2!\ldots\mu_{n-1}!}a_2^{\mu_1}a_3^{\mu_2}\ldots a_n^{\mu_{n-1}},$$

where the sum is taken over all nonnegative integers $\mu_1, ..., \mu_{n-1}$ satisfying the conditions

$$\begin{cases} \mu_1 + \mu_2 + ... + \mu_{n-1} = m, \\ \mu_1 + 2\mu_2 + ... + (n-1)\mu_{n-1} = n-1. \end{cases}$$

Evidently, $D_{n-1}^{n-1}(a_2, a_3, ..., a_n) = a_2^{n-1}$.

In this paper, for a univalent function $f(z) = z + \sum_{k=n}^{\infty} a_k z^k;\ (n \geq 2)$, we give the coefficients $b_k;\ (n \leq k \leq 2n-1)$ of the inverse function $f^{-1}(w) = w + \sum_{k=n}^{\infty} b_k w^k$. This leads to estimate $|a_n|$ for $f \in \Sigma$ or f belongs to certain subclasses of Σ, whereby some of them are obtained here. Moreover, for $f \in \mathcal{S}$ or f belongs to certain subclasses of $\mathcal{S}$, we estimate $|na_n^2 - a_{2n-1}|$.

2. Coefficients for inverses of univalent functions and estimates

Our first main result is given in the following theorem.

Theorem 2.1 *Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k;\ (n \geq 2)$ be a univalent function in $\mathbb{U}$ and $f^{-1}(w) = w + \sum_{k=n}^{\infty} b_k w^k$; $(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4})$. Then,*

$$b_{2n-1} = na_n^2 - a_{2n-1} \ \ \text{and} \ \ b_k = -a_k \ \ \text{for} \ \ (n \leq k \leq 2n-2).$$

Proof According to (1.3), the conclusion is trivial for $n = 2$. Since $a_k = 0;\ (2 \leq k \leq n-1)$, we have

$$\begin{aligned} b_n &= \frac{(-1)^{n+1}}{n!} \begin{vmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & n-2 \\ (n-1)na_n & 0 & 0 & \dots & 0 \end{vmatrix} \\ &= \frac{(-1)^{n+1}}{n!} \times (n-3)!(-1)^{n-3} \begin{vmatrix} 0 & n-2 \\ (n-1)na_n & 0 \end{vmatrix} \\ &= -a_n. \end{aligned}$$

Next, for $n+1 \leq k \leq 2n-1$ in (1.3), b_k can be expressed as $b_k = \frac{(-1)^{k+1}}{k!} \times$

$$\begin{vmatrix} ka_2 & 1 & 0 & \dots & . & \dots & 0 \\ 2ka_3 & (k+1)a_2 & 2 & \dots & . & \dots & 0 \\ 3ka_4 & (2k+1)a_3 & (k+2)a_2 & \dots & . & \dots & . \\ . & . & . & \dots & . & \dots & . \\ . & . & . & \dots & . & \dots & . \\ (n-1)ka_n & . & . & \dots & n-1 & \dots & . \\ nka_{n+1} & [(n-1)k+1]a_n & . & \dots & . & \dots & . \\ . & . & . & \dots & . & \dots & 0 \\ . & . & . & \dots & . & \dots & k-2 \\ (k-1)ka_k & [(k-2)k+1]a_{k-1} & [(k-3)k+2]a_{k-2} & \dots & (k+1-n)(k-1)a_{k+1-n} & \dots & (2k-2)a_2. \end{vmatrix}$$

Therefore, since $a_k = 0;\ (2 \le k \le n-1)$, we get $b_k = \frac{(-1)^{k+1}}{k!} \times$

$$\begin{vmatrix} 0 & 1 & 0 & \dots & . & \dots & 0 \\ 0 & 0 & 2 & \dots & . & \dots & 0 \\ . & . & 0 & \dots & . & \dots & . \\ . & . & . & \dots & . & \dots & . \\ 0 & 0 & . & \dots & . & \dots & . \\ (n-1)ka_n & 0 & 0 & \dots & n-1 & \dots & . \\ nka_{n+1} & [(n-1)k+1]a_n & 0 & \dots & . & \dots & . \\ . & . & . & \dots & . & \dots & 0 \\ . & . & . & \dots & . & \dots & k-2 \\ (k-1)ka_k & [(k-2)k+1]a_{k-1} & [(k-3)k+2]a_{k-2} & \dots & (k+1-n)(k-1)a_{k+1-n} & \dots & 0 \end{vmatrix}$$

$$= \frac{1}{k!} \begin{vmatrix} 0 & 2 & 0 & \dots & . & \dots & 0 \\ 0 & 0 & 3 & \dots & . & \dots & 0 \\ . & . & . & \dots & . & \dots & . \\ . & . & . & \dots & . & \dots & . \\ 0 & 0 & . & \dots & . & \dots & . \\ (n-1)ka_n & 0 & 0 & \dots & n-1 & \dots & . \\ nka_{n+1} & 0 & 0 & \dots & . & \dots & . \\ . & . & . & \dots & . & \dots & 0 \\ . & . & . & \dots & . & \dots & k-2 \\ (k-1)ka_k & [(k-3)k+2]a_{k-2} & . & \dots & (k+1-n)(k-1)a_{k+1-n} & \dots & 0 \end{vmatrix}.$$

Continue simplifying in this way by multiplying the entry A_{12} by the determinant of the resulting matrix formed by removing the first row and the second column to reach

$$\begin{aligned} b_k &= \frac{-(n-2)!}{k!} \begin{vmatrix} (n-1)ka_n & n-1 & \dots & . \\ nka_{n+1} & . & \dots & . \\ . & . & \dots & 0 \\ . & . & \dots & k-2 \\ (k-1)ka_k & (k+1-n)(k-1)a_{k+1-n} & \dots & 0 \end{vmatrix} \\ &= \frac{-(n-2)!}{k!} \begin{vmatrix} 0 & k-2 & 0 & \dots & . & 0 \\ 0 & 0 & k-3 & \dots & . & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & n & . \\ (k+1-n)(k-1)a_{k+1-n} & . & . & \dots & . & n-1 \\ (k-1)ka_k & . & . & \dots & . & (n-1)ka_n \end{vmatrix} \\ &= \frac{(n-2)!(k-2)!}{k!(n-1)!} \begin{vmatrix} (k+1-n)(k-1)a_{k+1-n} & n-1 \\ (k-1)ka_k & (n-1)ka_n \end{vmatrix} \\ &= (k+1-n)a_{k+1-n}a_n - a_k \\ &= \begin{cases} na_n^2 - a_{2n-1}, & \text{if } k = 2n-1, \\ -a_k, & \text{if } n+1 \le k \le 2n-2. \end{cases} \end{aligned}$$

This completes the proof of Theorem 2.1. □

Corollary 2.2 *Let f and f^{-1} be defined as in Theorem 2.1. Then*

$$|a_n| \le \sqrt{\frac{|a_{2n-1}| + |b_{2n-1}|}{n}}.$$

Corollary 2.3 *If $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$; $(n \ge 2)$ is bi-univalent or bi-close-to-convex or bi-starlike function in $\mathbb{U}$, then*

$$|a_n| \le \sqrt{4 - \frac{2}{n}}.$$

Proof Let f and f^{-1} defined as in Theorem 2.1 be univalent or close-to-convex or starlike functions in $\mathbb{U}$. It is well known that $|a_k| \le k$ and $|b_k| \le k$, so $|a_{2n-1}| \le 2n-1$ and $|b_{2n-1}| \le 2n-1$. Hence, in view of Theorem 2.1, we obtain

$$|a_n| \le \sqrt{\frac{|a_{2n-1}| + |b_{2n-1}|}{n}} \le \sqrt{\frac{2(2n-1)}{n}} = \sqrt{4 - \frac{2}{n}}.$$

□

Corollary 2.4 *If $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \mathcal{S}$ $(n \ge 2)$ and f^{-1} belongs to $\mathcal{S}^*$ or $\mathcal{C}$ or $\mathcal{S}$, then*

$$|na_n^2 - a_{2n-1}| \le 2n - 1.$$

Using Theorem 2.1 and (1.5), we obtain the following:

Corollary 2.5 *If $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$, $(n \ge 2)$ belongs to $\mathcal{S}$ or $\mathcal{S}^*$, and then*

$$|na_n^2 - a_{2n-1}| \le \frac{\Gamma(4n-1)}{\Gamma(2n+1)\Gamma(2n)}. \tag{2.1}$$

Note that if $n = 2$, then the equality in (2.1) is attained for the Koebe function. It would be of interest to know the maximal function that satisfies the equality in (2.1) whenever $n > 2$.

Corollary 2.6 *If $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$; $(n \ge 2)$ is bi-convex function in $\mathbb{U}$, then*

$$|a_n| \le \sqrt{\frac{2}{n}}.$$

Proof Let f and f^{-1} defined as in Theorem 2.1 be convex functions in $\mathbb{U}$. It is well known that $|a_k| \le 1$ and $|b_k| \le 1$, so $|a_{2n-1}| \le 1$ and $|b_{2n-1}| \le 1$. Therefore, by Theorem 2.1, we get

$$|a_n| \le \sqrt{\frac{|a_{2n-1}| + |b_{2n-1}|}{n}} \le \sqrt{\frac{2}{n}}.$$

□

Corollary 2.7 *If $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \mathcal{S}$; $(n \ge 2)$ and f^{-1} belongs to $\mathcal{CV}$, then*

$$|na_n^2 - a_{2n-1}| \le 1.$$

According to Theorem 2.1, if $f(z) = z + \sum_{k=n}^{\infty} a_k z^k;\ (n \geq 2)$ is univalent in $\mathbb{U}$, and then its inverse function f^{-1} has the form

$$f^{-1}(w) = w - \sum_{k=n}^{2n-2} a_k w^k + (na_n^2 - a_{2n-1})w^{2n-1} + \cdots,\quad (|w| < r_0(f),\ r_0(f) \geq 1/4).$$

Example 2.8 *The inverse of the univalent function* $f(z) = z + a_n z^n;\ (|a_n| \leq 1/n,\ n \geq 2)$ *is given in the form*

$$f^{-1}(w) = w - a_n w^n + na_n^2 w^{2n-1} + \cdots,\quad (|w| < r_0(f),\ r_0(f) \geq 1/4).$$

Note that f *is a starlike function and it is convex whenever* $|a_n| \leq 1/n^2$.

3. Coefficient estimates for bi-univalent functions having positive real part derivatives

Using Theorem 2.1 and Faber polynomial expansion, we obtain coefficient estimates for the following subclass of Σ.

Definition 3.1 *For* $n \geq 2$, $p \in \mathbb{N}$, *and* $0 \leq \alpha < 1$, *a function* $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \Sigma$ *is said to belong to the class* $R(n,p;\alpha)$ *if*

$$\mathrm{Re}\{(f'(z))^p\} > \alpha,\quad (z \in \mathbb{U}) \tag{3.1}$$

and

$$\mathrm{Re}\{(g'(w))^p\} > \alpha,\quad (w \in \mathbb{U}), \tag{3.2}$$

where $g = f^{-1}$.

Note that the functions of $R(n,1;0)$ are bi-close-to-convex in $\mathbb{U}$.

Theorem 3.2 *If* $f(z) \in R(n,p;\alpha)$, *then*

(i) for $p = 1$, *we have*

$$|a_n| \leq \begin{cases} \sqrt{\frac{4(1-\alpha)}{n(2n-1)}}, & \text{if } 0 \leq \alpha \leq \frac{n-1}{2n-1} \\ \frac{2(1-\alpha)}{n}, & \text{if } \frac{n-1}{2n-1} \leq \alpha < 1, \end{cases}$$

(ii) for $p \geq 2$, *we have*

$$|a_n| \leq \frac{2(1-\alpha)}{np},$$

(iii)

$$|a_k| \leq \frac{2(1-\alpha)}{kp},\quad (k > n \geq 2,\ \ p \in \mathbb{N}),$$

(iv)

$$|na_n^2 - a_{2n-1}| \leq \frac{2(1-\alpha)}{(2n-1)p},\quad (p \in \mathbb{N}).$$

Proof According to [1, Equation (4), p. 449], if $\psi(z) = 1 + \sum_{k=1}^{\infty} \psi_k z^k$ is analytic in $\mathbb{U}$ and $p \in \mathbb{N}$, then

$$(\psi(z))^p = 1 + \sum_{k=1}^{\infty} K_k^p(\psi_1, \psi_2, ..., \psi_k) z^k.$$

Thus,

$$\begin{aligned} (f'(z))^p &= 1 + \sum_{k=1}^{\infty} K_k^p(2a_2, 3a_3, ..., (k+1)a_{k+1}) z^k \\ &= 1 + \sum_{k=2}^{\infty} K_{k-1}^p(2a_2, 3a_3, ..., ka_k) z^{k-1}. \end{aligned} \tag{3.3}$$

Similarly, for $g = f^{-1}$, we have

$$\begin{aligned} g'(w) &= 1 + \sum_{k=2}^{\infty} kb_k w^k \\ &= 1 + \sum_{k=2}^{\infty} K_{k-1}^{-k}(a_2, a_3, ..., a_k) w^{k-1} \end{aligned}$$

and

$$(g'(w))^p = 1 + \sum_{k=2}^{\infty} K_{k-1}^p(2b_2, 3b_3, ..., kb_k) w^{k-1}. \tag{3.4}$$

By (3.1) and (3.2), there exist two positive real part functions $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}$ and $q(w) = 1 + \sum_{k=1}^{\infty} q_k w^k \in \mathcal{P}$ such that

$$\begin{aligned} (f'(z))^p &= \alpha + (1-\alpha)p(z) \\ &= 1 + (1-\alpha)p_1 z + (1-\alpha)p_2 z^2 + \cdots \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} (g'(w))^p &= \alpha + (1-\alpha)q(w) \\ &= 1 + (1-\alpha)q_1 w + (1-\alpha)q_2 w^2 + \cdots . \end{aligned} \tag{3.6}$$

Comparing the corresponding coefficients of (3.3) and (3.5) gives

$$K_{k-1}^p(2a_2, 3a_3, ..., ka_k) = (1-\alpha)p_{k-1}. \tag{3.7}$$

Similarly, from (3.4) and (3.6), we obtain

$$K_{k-1}^p(2b_2, 3b_3, ..., kb_k) = (1-\alpha)q_{k-1}. \tag{3.8}$$

Therefore, equations (3.7) and (3.8) in conjunction with (1.6) yield

$$kpa_k = (1-\alpha)p_{k-1}, \quad (k \geq n \geq 2)$$

and

$$kpb_k = (1-\alpha)q_{k-1}, \quad (k \geq n \geq 2).$$

Hence, using the Carathéodory lemma, we get

$$|a_k| \leq \frac{(1-\alpha)|p_{k-1}|}{kp} \leq \frac{2(1-\alpha)}{kp}, \quad (k \geq n \geq 2)$$

and

$$|b_k| \leq \frac{(1-\alpha)|q_{k-1}|}{kp} \leq \frac{2(1-\alpha)}{kp}, \quad (k \geq n \geq 2).$$

In particular, we have

$$|a_n| \leq \frac{2(1-\alpha)}{np}, \tag{3.9}$$

$$|a_{2n-1}| \leq \frac{2(1-\alpha)}{(2n-1)p}, \quad \text{and} \quad |b_{2n-1}| \leq \frac{2(1-\alpha)}{(2n-1)p}. \tag{3.10}$$

Thus, in view of Theorem 2.1 and (3.10), we obtain

$$|a_n| \leq \sqrt{\frac{|a_{2n-1}| + |b_{2n-1}|}{n}} \leq \sqrt{\frac{4(1-\alpha)}{(2n-1)np}} \tag{3.11}$$

and

$$|na_n^2 - a_{2n-1}| = |b_{2n-1}| \leq \frac{2(1-\alpha)}{(2n-1)p}.$$

Considering the estimates (3.9) and (3.11) implies, for $p = 1$ and $0 \leq \alpha \leq (n-1)/(2n-1)$, that

$$\sqrt{\frac{4(1-\alpha)}{(2n-1)np}} \leq \frac{2(1-\alpha)}{np}.$$

On the other hand, for ($p = 1$ and $(n-1)/(2n-1) \leq \alpha < 1$) or for ($p \geq 2$ and $0 \leq \alpha < 1$), we have

$$\frac{2(1-\alpha)}{np} \leq \sqrt{\frac{4(1-\alpha)}{(2n-1)np}}.$$

This completes the proof of Theorem 3.2. □

Remark 3.3 *(1) The estimate of $|a_n|$ given in Theorem 3.2 (i) for $p = 1$ is much better than that given by Jahangiri et al. in [25, Theorem 2.1].*
(2) Setting $n = 2$, $p = 1$, and $k = 3$ in Theorem 3.2 gives [13, Corollary 7]. The estimates of $|a_2|$ and $|a_3|$ are much better than those given by Srivastava et al. [31] and the estimate of $|a_2|$ is much better than that given by Xu et al. [35].
(3) In [25, Example 2.1], it is stated wrongly that the inverse of $f(z) = z + \frac{1-\alpha}{np} z^n$ is given by $g(w) = w - \frac{1-\alpha}{np} w^n$. It can be easily checked that $f(g(w)) \neq w$. Indeed, $g(w)$ must be in the following form (see Example 2.8):

$$g(w) = w - \frac{1-\alpha}{np} w^n + n\left(\frac{1-\alpha}{np}\right)^2 w^{2n-1} + \cdots .$$

The following is an example of a function in $R(2,1;0)$ that satisfies the conclusions of Theorem 3.2.

Example 3.4 *Consider the function* $f(z) = -\log(1-z)$. *Then*

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1}{k} z^k$$

and

$$f^{-1}(w) = 1 - e^{-w} = w + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k!} w^k.$$

Now $\operatorname{Re}\{f'(z)\} = \operatorname{Re}\{1/(1-z)\} > 0$ *and* $\operatorname{Re}\{(f^{-1})'(w)\} = \operatorname{Re}\{e^{-w}\} > 0$ *implies that* $f \in R(2,1;0)$. *In view of Theorem 3.2 (i) and (iv), we have*

$$|a_2| = \frac{1}{2} \leq \sqrt{\frac{2}{3}}$$

and

$$|b_3| = |2a_2^2 - a_3| = \frac{1}{6} \leq \frac{2}{3}.$$

Acknowledgment

The authors declare that they have no conflicts of interest.

References

[1] Airault H. Remarks on Faber polynomials. International Mathematical Forum 2008; 3: 449–456.

[2] Airault H, Bouali A. Differential calculus on the Faber polynomials. Bulletin des Sciences Mathématiques 2006; 130 (3): 179–222.

[3] Airault H, Ren J. An algebra of differential operators and generating functions on the set of univalent functions. Bulletin des Sciences Mathématiques 2002; 126 (5): 343–367.

[4] Ali RM, Lee SK, Ravichandran V, Supramaniam S. Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. Applied Mathematics Letters 2012; 25: 344–351.

[5] Altınkaya Ş, Yalçın S, Çakmak S. A subclass of bi-univalent functions based on the Faber polynomial expansions and the Fibonacci numbers. Mathematics 2019; 7: 160.

[6] Aouf MK, El-Ashwah RM, Abd-Eltawab AM. New subclasses of biunivalent functions involving Dziok-Srivastava operator. ISRN Mathematical Analysis 2013; 2013: 387178.

[7] Brannan DA, Clunie JG (editors). Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute Held at the University of Durham; July 20, 1979). New York, NY, USA: Academic Press, 1980.

[8] Brannan DA, Taha TS. On some classes of bi-univalent functions. In: Mazhar SM, Hamoui A, Faour NS (editors). Mathematical Analysis and Its Applications; Kuwait; 1985. KFAS Proceedings Series, Vol. 3. Oxford, UK: Pergamon Press, 1988, pp. 53–60.

[9] Bulut S. Coefficient estimates for initial Taylor-Maclaurin coefficients for a subclass of analytic and bi-univalent functions defined by Al-Oboudi differential operator. Scientific World Journal 2013; 2013: 171039.

[10] Bulut S. Coefficient estimates for new subclasses of analytic and bi-univalent functions defined by Al-Oboudi differential operator. Journal of Function Spaces and Applications 2013; 2013: 181932.

[11] Bulut S. Coefficient estimates for a class of analytic and bi-univalent functions. Novi Sad Journal of Mathematics 2013; 43 (2): 59–65.

[12] Bulut S. Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions. Comptes rendus de l'Académie des Sciences Paris Series I 2014; 352 (6): 479–484.

[13] Bulut S. Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions. Filomat 2016; 30 (6): 1567–1575.

[14] Caglar M, Orhan H, Ya N. Coefficient bounds for new subclasses of bi-univalent functions. Filomat 2013; 27 (7): 1165–1171.

[15] Duren PL. Univalent Functions. Grundlehren der Mathematischen Wissenschaften, Vol. 259. New York, NY, USA: Springer, 1983.

[16] Faber G. Uber polynomische Entwickelungen. Mathematische Annalen 1903; 57 (3): 389–408 (in German).

[17] Frasin BA, Aouf MK. New subclasses of bi-univalent functions. Applied Mathematics Letter 2011; 24: 1569–1573.

[18] Goodman AW. Univalent Functions, Vol. I. Tampa, FL, USA: Mariner Publishing Company, 1983.

[19] Goyal SP, Goswami P. Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives. Journal of Egyptian Mathematical Society 2012; 20: 179–182.

[20] Hamidi SG, Halim SA, Jahangiri JM. Coefficient estimates for a class of meromorphic bi-univalent functions. Comptes rendus de l'Académie des Sciences Paris Series I 2013; 351 (9-10): 349–352.

[21] Hamidi SG, Janani T, Murugusundaramoorthy G, Jahangiri JM. Coefficient estimates for certain classes of meromorphic bi-univalent functions. Comptes rendus de l'Académie des Sciences Paris Series I 2014; 352 (4): 277–282.

[22] Hayami T, Owa S. Coefficient bounds for bi-univalent functions. Pan-American Mathematical Journal 2012; 22 (4): 15–26.

[23] Hayman WK. Multivalent Functions, Second Edition. Cambridge, UK: Cambridge University Press, 1994.

[24] Jahangiri JM, Hamidi SG. Coefficient estimates for certain classes of bi-univalent functions. International Journal of Mathematics and Mathematical Sciences 2013; 2013: 190560.

[25] Jahangiri JM, Hamidi SG, Halim SA. Coefficients of bi-univalent functions with positive real part derivatives. Bulletin of the Malaysian Mathematical Sciences Society 2014; (2) 37: 633–640.

[26] Lewin M. On a coefficient problem for bi-univalent functions. Proceedings of the American Mathematical Society 1967; 18: 63–68.

[27] Loewner C. Untersuchungen uber schlichte konforme Abbildungen des Einheitskreises. Mathematische Annalen 1923; 89: 103-121 (in German).

[28] Murugusundaramoorthy G, Magesh N, Prameela V. Coefficient bounds for certain subclasses of bi-univalent functions. Abstract and Applied Analysis 2013; 2013: 573017.

[29] Netanyahu E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. Archive for Rational Mechanics and Analysis 1969; 32: 100–112.

[30] Porwal S, Darus M. On a new subclass of bi-univalent functions. Journal of Egyptian Mathematical Society 2013; 21 (3): 190–193.

[31] Srivastava HM, Mishra AK, Gochhayat P. Certain subclasses of analytic and bi-univalent functions. Applied Mathematics Letters 2010; 23: 1188-1192.

[32] Taha TS. Topics in univalent function theory. PhD, University of London, London, UK, 1981.

[33] Todorov PG. On the Faber polynomials of the univalent functions of class Σ. Journal of Mathematical Analysis and Applications 1991; 162: 268-276.

[34] Wang ZG, Bulut S. A note on the coefficient estimates of bi-close-to-convex functions. Comptes Rendus Mathematique 2017; 355 (8): 876-880.

[35] Xu QH, Gui YC, Srivastava HM. Coefficient estimates for a certain subclass of analytic and bi-univalent functions. Applied Mathematics Letters 2012; 25: 990-994.

[36] Xu QH, Xiao HG, Srivastava HM. A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. Applied Mathematics and Computation 2012; 218: 11461–11465.