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Conformally flat Willmore spacelike hypersurfaces in $\mathbb{R}_1^{n+1}$

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Abstract: In this paper, we give the equation satisfied by umbilics-free Willmore spacelike hypersurfaces using the conformal invariants in Lorentzian space forms. At the same time, we give the equation satisfied by hyperelastic spacelike curves in 2-dimensional Lorentzian space forms and classify the closed hyperelastic spacelike curves. Finally conformally flat Willmore spacelike hypersurfaces are classified in terms of the hyperelastic spacelike curves in 2-dimensional Lorentzian space forms.

Key words: Willmore spacelike hypersurface, conformal metric, elastic spacelike curve

1. Introduction

Recently Willmore submanifolds in a sphere $\mathbb{S}^{n+1}$ have been studied extensively. In particular, Willmore surfaces are the most interesting and long-term point of focus. An important result is that Marques solved the famous Willmore conjecture [14]. The high-dimensional generalization of the Willmore surface is attributed to Guo et al. [6], who proposed the high-dimensional version of the Willmore conjecture. Since then, there have been many studies on the rigidity of Willmore hypersurfaces (see [1, 4–7, 9, 10, 12]). In [11], the second author classified the Willmore hypersurfaces with two distinct principal curvatures in $\mathbb{S}^{n+1}$; therefore, the conformally flat Willmore hypersurfaces in $\mathbb{S}^{n+1}$ have been classified.

As its parallel generalization, the Willmore spacelike submanifold in Lorentzian space forms is another important submanifold. However, there are fewer results. In this paper, we investigate the Willmore spacelike hypersurfaces in Lorentzian space form. One of our main goals is to generalize the results in [11] from sphere space to Lorentzian space forms.

There exists a standard conformal mapping between the Lorentzian space forms $\mathbb{R}_1^{n+1}$, $\mathbb{S}_1^{n+1}(1)$ and $\mathbb{H}_1^{n+1}(-1)$ (see [14], or Section 2). Since Willmore spacelike hypersurfaces are conformal invariants, the results are the same between the Lorentzian space forms. In this paper we only consider the Willmore spacelike hypersurfaces in $\mathbb{R}_1^{n+1}$, whose results also hold in other space.

Let $f : M^n \to \mathbb{R}_1^{n+1}, (n \geq 4)$ be a spacelike hypersurface in Lorentzian space $\mathbb{R}_1^{n+1}$. Given the first fundamental form $I = \langle df, df\rangle_1$ and the second fundamental form $II = \sum_{ij} h_{ij}\theta_i \otimes \theta_j$, we denote by $H = \frac{1}{n}\sum_i h_{ii}$ the mean curvature and $|II|$ the norm of the second fundamental form. The spacelike hypersurface

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f is called a Willmore spacelike hypersurface if it is a critical point of the following Willmore functional:

$$W(f) = \int_{M^n} (|II|^2 - nH^2)^{\frac{n}{2}} dv,$$

where dv is the volume element with respect to I. The functional $W(f)$ is invariant under the conformal transformation of $\mathbb{R}^{n+1}_1$ and so the Willmore spacelike hypersurfaces are conformal invariants. In this paper we investigate the Willmore spacelike hypersurfaces using the framework of conformal geometry of $\mathbb{R}^{n+1}_1$. We define the conformal metric of f by

$$g = \rho^2 df \cdot df = \frac{n}{n-1}(\|II\|^2 - nH^2)I.$$

When the spacelike hypersurface is umbilics-free, then g is a Riemannian metric that is invariant under the conformal transformations of $\mathbb{R}^{n+1}_1$. Together with another quadratic form (called the conformal second fundamental form), they form a complete system of invariants for the spacelike hypersurface ($n \geq 3$) in the conformal geometry of the Lorentzian space $\mathbb{R}^{n+1}_1$ (see Section 2). In fact, the functional $W(f)$ is the volume with respect to the conformal metric g up to a constant.

The one-dimensional version of the functional $W(f)$ is the r-elastic energy functional $W^r(\gamma)$ on a spacelike curve γ for some natural number r. Let $\gamma : I \to M^2_1(c)$ be a spacelike curve in 2-dimensional Lorentzian space form $M^2_1(c)$ with Gauss curvature c, and let s denote the arclength parametrization and κ the oriented curvature of γ. The r-elastic energy functional $W^r(\gamma)$ is defined as follows:

$$W^r(\gamma) = \int_\gamma \kappa^r ds.$$

The critical point of the functional $W^r(\gamma)$ is call an r-hyperelastic spacelike curve. In this paper, we compute the Euler–Lagrange equation of the functional $W^r(\gamma)$, i.e. the r-hyperelastic curve equation.

Theorem 1.1 *Let $\gamma : I \to M^2_1(c)$ be a spacelike curve in 2-dimensional Lorentzian space form $M^2_1(c)$ with Gauss curvature c, and let s denote the arclength parametrization and κ the oriented curvature of γ. Then γ is an r-hyperelastic spacelike curve if and only if its oriented curvature satisfies*

$$-r(r-1)\kappa^{r-3}[\kappa\kappa_{ss} + (r-2)\kappa_s^2 - \frac{\kappa^4}{r} - \frac{\kappa^2}{r-1}c] = 0.$$

Because of the causal character, there are no closed spacelike curves in $\mathbb{R}^2_1$, but there are closed spacelike curves in $M^2_1(c)(c \neq 0)$. The following theorem gives the classification of closed r-hyperelastic spacelike curves.

Theorem 1.2 *Let $\gamma : I \to M^2_1(c)$ be an r-hyperelastic spacelike curve in 2-dimensional Lorentzian space form $M^2_1(c)$. If γ is closed, then γ is the totally geodesic closed curve in $\mathbb{S}^2_1(c)(c > 0)$, or the totally umbilical closed curve with the oriented curvature $\sqrt{\frac{-rc}{r-1}}$ in $\mathbb{H}^2_1(c)(c < 0)$.*

One of our purposes is to classify the conformally flat Willmore spacelike hypersurfaces by n-hyperelastic spacelike curves up to a conformal transformation. Two hypersurfaces $f, \tilde{f} : M^n \to \mathbb{R}^{n+1}_1$ are conformally equivalent if there exists a conformal transformation φ such that $f(M^n) = \varphi \circ \tilde{f}(M^n)$.

Theorem 1.3 *Let $f : M^n \to \mathbb{R}^{n+1}_1$, $n \geq 4$, be a conformally flat Willmore spacelike hypersurface without umbilical point. Then f is locally conformally equivalent to one of the following spacelike hypersurfaces in $\mathbb{R}^{n+1}_1$:*

(1) *a cylinder over an n-hyperelastic spacelike curve in $\mathbb{R}^2_1 \subset \mathbb{R}^{n+1}_1$,*

(2) *a cone over an n-hyperelastic spacelike curve in $\mathbb{S}^2_1(-1) \subset \mathbb{R}^3_1 \subset \mathbb{R}^{n+1}_1$,*

(3) *a rotational hypersurface over an n-hyperelastic spacelike curve in $\mathbb{R}^2_{1+} \subset \mathbb{R}^{n+1}_1$.*

The Lorentzian hyperbolic 2-plane $\mathbb{R}^2_{1+}$ is defined by

$$\mathbb{R}^2_{1+} = \{(x, y) \in \mathbb{R}^2 | y > 0\},$$

and it is endowed with the Lorentzian metric $ds^2 = \frac{1}{y^2}(-dx^2 + dy^2)$. The Gauss curvature of $\mathbb{R}^2_{1+}$ is -1 with respect to the Lorentzian metric ds^2. Letting $\mathbb{H}^2_1(-1)$ be a 2-dimensional anti-de Sitter sphere, there exists the following standard isometric embedding:

$$\varphi : \mathbb{R}^2_{1+} \to \mathbb{H}^2_1(-1), \quad \varphi(x, y) = (\frac{y^2 - x^2 + 1}{2y}, \frac{x}{y}, \frac{y^2 - x^2 - 1}{2y}). \tag{1.1}$$

In this paper, all manifolds, maps, etc. will be assumed C^∞. The paper is organized as follows. In Section 2, we give the elementary facts about conformal geometry for spacelike hypersurfaces in $\mathbb{R}^{n+1}_1$. In Section 3, we compute the Euler–Lagrange equation of the Willmore functional $W(f)$ by conformal invariants. In Section 4, we compute the Euler–Lagrange equation of the functional $W^r(\gamma)$ and prove Theorem 1.2. In Section 5, we present some examples of Willmore spacelike hypersurfaces in terms of the n-hyperelastic spacelike curves. In Section 5, we give the proof of Theorem 1.3.

2. Conformal geometry of spacelike hypersurfaces

In this section, following Wang's idea in [16], we define some conformal invariants on a spacelike hypersurface and give a congruent theorem of the spacelike hypersurfaces under the conformal transformation group of $M^{n+1}_1(c)$ (or see [13]).

Let $\mathbb{R}^{n+2}_s$ be the real vector space $\mathbb{R}^{n+2}$ with the Lorentzian product $\langle , \rangle_s$ given by

$$\langle X, Y \rangle_s = -\sum_{i=1}^{s} x_i y_i + \sum_{j=s+1}^{n+2} x_j y_j.$$

For any $a > 0$, the standard sphere $\mathbb{S}^{n+1}(a)$, the hyperbolic space $\mathbb{H}^{n+1}(-a)$, the de Sitter space $\mathbb{S}^{n+1}_1(a)$, and the anti-de Sitter space $\mathbb{H}^{n+1}_1(-a)$ are defined by

$$\mathbb{S}^{n+1}(a) = \{x \in \mathbb{R}^{n+2} | x \cdot x = a^2\}, \quad \mathbb{H}^{n+1}(-a) = \{x \in \mathbb{R}^{n+2}_1 | \langle x, x \rangle_1 = -a^2\},$$
$$\mathbb{S}^{n+1}_1(a) = \{x \in \mathbb{R}^{n+2}_1 | \langle x, x \rangle_1 = a^2\}, \quad \mathbb{H}^{n+1}_1(-a) = \{x \in \mathbb{R}^{n+2}_2 | \langle x, x \rangle_2 = -a^2\},$$

respectively. Let $M^{n+1}_1(c)$ be a Lorentzian space form. When $c = 0$, $M^{n+1}_1(c) = \mathbb{R}^{n+1}_1$; when $c = 1$, $M^{n+1}_1(c) = \mathbb{S}^{n+1}_1(1)$; and when $c = -1$, $M^{n+1}_1(c) = \mathbb{H}^{n+1}_1(-1)$.

Denoting by C^{n+2} the cone in $\mathbb{R}_2^{n+3}$ and by $\mathbb{Q}_1^{n+1}$ the conformal compactification space in $\mathbb{R}P^{n+3}$,

$$C^{n+2} = \{X \in \mathbb{R}_2^{n+3} | \langle X, X\rangle_2 = 0, X \neq 0\}, \quad \mathbb{Q}_1^{n+1} = \{[X] \in \mathbb{R}P^{n+2} | \langle X, X\rangle_2 = 0\}.$$

Let $O(n+3,2)$ be the Lorentzian group of $\mathbb{R}_2^{n+3}$ keeping the Lorentzian product $\langle X, Y\rangle_2$ invariant. Then $O(n+3,2)$ is a transformation group on $\mathbb{Q}_1^{n+1}$ defined by

$$T([X]) = [XT], \quad X \in C^{n+2}, \quad T \in O(n+3,2).$$

Topologically, $\mathbb{Q}_1^{n+1}$ is identified with the compact space $\mathbb{S}^n \times \mathbb{S}^1/\mathbb{S}^0$, which is endowed by a standard Lorentzian metric $h = g_{\mathbb{S}^n} \oplus (-g_{\mathbb{S}^1})$, where $g_{\mathbb{S}^k}$ denotes the standard metric of the k-dimensional sphere $\mathbb{S}^k$. Therefore, $\mathbb{Q}_1^{n+1}$ has conformal metric $[h] = \{e^{\tau}h\}, \tau \in C^\infty(\mathbb{Q}_1^{n+1})$, and $[O(n+3,2)]$ is the conformal transformation group of $\mathbb{Q}_1^{n+1}$ (see [2, 15]).

Setting $P = \{[X] \in \mathbb{Q}_1^{n+1} | x_1 = x_{n+3}\}$, $P_- = \{[X] \in \mathbb{Q}_1^{n+1} | x_{n+3} = 0\}$, $P_+ = \{[X] \in \mathbb{Q}_1^{n+1} | x_1 = 0\}$, we can define the following conformal diffeomorphisms:

$$\begin{array}{ll} \sigma_0: \ \mathbb{R}_1^{n+1} \to \mathbb{Q}_1^{n+1} \backslash P, & u \mapsto [(\frac{\langle u,u\rangle_1+1}{2}, u, \frac{\langle u,u\rangle_1-1}{2})], \\ \sigma_1: \ \mathbb{S}_1^{n+1}(1) \to \mathbb{Q}_1^{n+1} \backslash P_+, & u \mapsto [(1,u)], \\ \sigma_{-1}: \ \mathbb{H}_1^{n+1}(-1) \to \mathbb{Q}_1^{n+1} \backslash P_-, & u \mapsto [(u,1)]. \end{array} \tag{2.1}$$

We may regard $\mathbb{Q}_1^{n+1}$ as the common compactification of $\mathbb{R}_1^{n+1}, \mathbb{S}_1^{n+1}(1), \mathbb{H}_1^{n+1}(-1)$.

Let $f: M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface. Using σ_c, we obtain the hypersurface $\sigma_c \circ f : M^n \to \mathbb{Q}_1^{n+1}$ in $\mathbb{Q}_1^{n+1}$. Thus, from [15], we have the following results:

Theorem 2.1 *Two hypersurfaces $f, \bar{f} : M^n \to M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T \in O(n+3,2)$ such that $\sigma_c \circ f = T(\sigma_c \circ \bar{f}) : M^n \to \mathbb{Q}_1^{n+1}$.*

Since $f: M^n \to M_1^{n+1}(c)$ is a spacelike hypersurface, $(\sigma_c \circ f)_*(TM^n)$ is a positive definite subbundle of $T\mathbb{Q}_1^{n+1}$. For any local lift Z of the standard projection $\pi : C^{n+2} \to \mathbb{Q}_1^{n+1}$, we get a local lift $y = Z \circ \sigma_c \circ f : U \to C^{n+1}$ of $\sigma_c \circ f : M \to \mathbb{Q}_1^{n+1}$ in an open subset U of M^n. Thus, $\langle \mathrm{d}y, \mathrm{d}y\rangle_2 = \rho^2 \langle df, df\rangle_s$ is a local metric, where $\rho \in C^\infty(U)$. We denote by Δ and κ the Laplacian operator and the normalized scalar curvature with respect to the local positive definite metric $\langle \mathrm{d}y, \mathrm{d}y\rangle_2$, respectively. Similar to Wang's proof of Theorem 1.2 in [16], we get the following theorem:

Theorem 2.2 *Let $f: M^n \to M_1^{n+1}(c)$ be a spacelike hypersurface. Then the 2-form*

$$g = -(\langle \Delta y, \Delta y\rangle_2 - n^2\kappa)\langle dy, dy\rangle_2$$

is a globally defined conformal invariant. Moreover, g is positive definite at any nonumbilical point of M^n.

We call g the conformal metric of the spacelike hypersurface f, and there exists a unique lift

$$Y : M \to C^{n+2}$$

such that $g = \langle \mathrm{d}Y, \mathrm{d}Y\rangle_2$. We call Y the conformal position vector of the spacelike hypersurface f. Theorem 2.2 implies the following:

Theorem 2.3 *Two spacelike hypersurfaces $f, \bar{f} : M^n \to M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T \in O(n+3,2)$ such that $\bar{Y} = YT$, where $Y, \bar{Y}$ are the conformal position vectors of $f, \bar{f}$, respectively.*

Let $\{E_1, \cdots, E_n\}$ be a local orthonormal basis of M^n with respect to g with dual basis $\{\omega_1, \cdots, \omega_n\}$. Denote $Y_i = E_i(Y)$ and define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle_2 Y,$$

where Δ is the Laplace operator of g. Then we have

$$\langle N, Y\rangle_2 = 1,\ \langle N, N\rangle_2 = 0,\ \langle N, Y_k\rangle_2 = 0,\ \langle Y_i, Y_j\rangle_2 = \delta_{ij}, \quad 1 \le i, j, k \le n.$$

We may decompose $\mathbb{R}_2^{n+3}$ such that

$$\mathbb{R}_2^{n+3} = \mathrm{span}\{Y, N\} \oplus \mathrm{span}\{Y_1, \cdots, Y_n\} \oplus \mathbb{V},$$

where $\mathbb{V} \perp \mathrm{span}\{Y, N, Y_1, \cdots, Y_n\}$. We call $\mathbb{V}$ the conformal normal bundle of f, which is a linear bundle. Let ξ be a local section of $\mathbb{V}$ and $\langle \xi, \xi\rangle_2 = -1$, and then $\{Y, N, Y_1, \cdots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}_2^{n+3}$ along M^n. We write the structure equations as follows:

$$\begin{aligned}
\mathrm{d}Y &= \sum_i \omega_i Y_i,\\
\mathrm{d}N &= \sum_{ij} A_{ij}\omega_j Y_i + \sum_i C_i \omega_i \xi,\\
\mathrm{d}Y_i &= -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij}\omega_j \xi,\\
\mathrm{d}\xi &= \sum_i C_i \omega_i Y + \sum_{ij} B_{ij}\omega_j Y_i,
\end{aligned} \tag{2.2}$$

where $\omega_{ij} (= -\omega_{ji})$ are the connection 1-forms on M^n with respect to $\{\omega_1, \cdots, \omega_n\}$. It is clear that $A = \sum_{ij} A_{ij}\omega_j \otimes \omega_i$, $B = \sum_{ij} B_{ij}\omega_j \otimes \omega_i$, $C = \sum_i C_i \omega_i$ are globally defined conformal invariants. We call A, B, and C the Blaschke tensor, the conformal second fundamental form, and the conformal 1-form, respectively. The covariant derivatives of these tensors are defined by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_k C_k \omega_{kj},$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki},$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}.$$

By exterior differentiation of the structure equations (2.2), we can get the integrable conditions of the structure equations:

$$A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \tag{2.3}$$

$$A_{ij,k} - A_{ik,j} = B_{ij}C_k - B_{ik}C_j, \tag{2.4}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.5}$$

$$C_{i,j} - C_{j,i} = \textstyle\sum_k (B_{ik}A_{kj} - B_{jk}A_{ki}), \tag{2.6}$$

$$R_{ijkl} = B_{il}B_{jk} - B_{ik}B_{jl} + A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{il}\delta_{jk} - A_{jk}\delta_{il}. \tag{2.7}$$

Furthermore, we have

$$\begin{aligned} &\mathrm{tr}(A) = \frac{1}{2n}(n^2\kappa - 1), \quad R_{ij} = \mathrm{tr}(A)\delta_{ij} + (n-2)A_{ij} + \sum_k B_{ik}B_{kj}, \\ &(1-n)C_i = \sum_j B_{ij,j}, \quad \sum_{ij} B_{ij}^2 = \frac{n-1}{n}, \quad \sum_i B_{ii} = 0, \end{aligned} \tag{2.8}$$

where κ is the normalized scalar curvature of g. From (2.8), we see that when $n \geq 3$, all coefficients in the structure equations are determined by the conformal metric g and the conformal second fundamental form B, and thus we get the congruent theorem:

Theorem 2.4 *Two spacelike hypersurfaces $f, \bar{f} : M^n \to M_1^{n+1}(c)(n \geq 3)$ are conformally equivalent if and only if there exists a diffeomorphism $\varphi : M^n \to M^n$ that preserves the conformal metric g and the conformal second fundamental form B.*

Next we give the relations between the conformal invariants and the isometric invariants of a spacelike hypersurface in $\mathbb{R}_1^{n+1}$.

Assume that $f : M^n \to \mathbb{R}_1^{n+1}$ is an umbilic-free spacelike hypersurface. Let $\{e_1, \cdots, e_n\}$ be an orthonormal local basis with respect to the induced metric $I = \langle df, df\rangle_1$ with dual basis $\{\theta_1, \cdots, \theta_n\}$. Let e_{n+1} be a normal vector field of f , $\langle e_{n+1}, e_{n+1}\rangle_1 = -1$. Let $II = \sum_{ij} h_{ij}\theta_i \otimes \theta_j$ denote the second fundamental form and $H = \frac{1}{n}\sum_i h_{ii}$ the mean curvature. Denote by Δ_M the Laplacian operator and κ_M the normalized scalar curvature for I. By the structure equation of $f : M^n \to \mathbb{R}_1^{n+1}$ we get that

$$\Delta_M f = nHe_{n+1}. \tag{2.9}$$

There is a local lift of f:

$$y : M^n \to C^{n+2}, \quad y = (\frac{\langle f, f\rangle_1 + 1}{2}, f, \frac{\langle f, f\rangle_1 - 1}{2}).$$

It follows from (2.9) that $\langle \Delta y, \Delta y\rangle_2 - n^2\kappa_M = \frac{n}{n-1}(-|II|^2 + n|H|^2) = -e^{2\tau}$. Therefore, the conformal metric g, conformal position vector of f, and ξ have the following expressions:

$$\begin{aligned} &g = \frac{n}{n-1}(|II|^2 - n|H|^2)\langle df, df\rangle_1 = e^{2\tau}I, \quad Y = e^\tau y, \\ &\xi = -Hy + (\langle f, e_{n+1}\rangle_1, e_{n+1}, \langle f, e_{n+1}\rangle_1). \end{aligned} \tag{2.10}$$

By a direct calculation we get the following expressions of the conformal invariants:

$$\begin{aligned} A_{ij} &= e^{-2\tau}[\tau_i\tau_j - h_{ij}H - \tau_{i,j} + \frac{1}{2}(-|\nabla\tau|^2 + |H|^2)\delta_{ij}], \\ B_{ij} &= e^{-\tau}(h_{ij} - H\delta_{ij}), \quad C_i = e^{-2\tau}(H\tau_i - H_i - \sum_j h_{ij}\tau_j), \end{aligned} \tag{2.11}$$

where $\tau_i = e_i(\tau)$ and $|\nabla\tau|^2 = \sum_i \tau_i^2$, and $\tau_{i,j}$ is the Hessian of τ for I and $H_i = e_i(H)$.

The eigenvalue of the conformal second fundamental form is called the conformal principal curvature of the spacelike hypersurface. Clearly from (2.11) we know that the number of distinct conformal principal curvatures is the same as that of its distinct principal curvatures.

3. The first variation of the Willmore functional

Let $f_0 : M^n \to M_1^{n+1}(c)$ be a compact spacelike hypersurface with boundary ∂M^n. The generalized Willmore functional $W(f_0)$ is as the volume functional of the conformal metric g:

$$\mathrm{Vol}_g(M^n) = \int_{\mathbf{M}} \mathrm{d}M_g = (\frac{n}{n-1})^{\frac{n}{2}} \int_{M^n} (|II|^2 - nH^2)^{\frac{n}{2}} dM_I = (\frac{n}{n-1})^{\frac{n}{2}} W(f_0).$$

Let $f : M^n \times (-\epsilon, \epsilon) \to M_1^{n+1}(c)$ be an admissible variation of f_0 such that

$$f(\cdot, t)|_{\partial M^n} = f_0|_{\partial M^n}, \quad df_t(\mathrm{T}_p M^n)|_{\partial M^n} = df_0(\mathrm{T}_p M^n)|_{\partial M^n}$$

for each small t. For each t, f_t is a spacelike hypersurface and g_t denotes its conformal metric. As in Section 2, we have a moving frame $\{Y, N, Y_i, \xi\}$ in $\mathbb{R}_2^{n+3}$ and the conformal volume functional $W(f_t)$. Let ξ be a local basis for the conformal normal bundle $\mathbb{V}_t$ of f_t. Denote by d and d_M the differential operators on $M^n \times (-\epsilon, \epsilon)$ and M^n, respectively. Then we have

$$d = d_M + \mathrm{d}t \wedge \frac{\partial}{\partial t}. \tag{3.1}$$

By (2.10) we can find functions $w, v_i, v : M^n \times (-\epsilon, \epsilon) \to \mathbb{R}$ such that

$$\frac{\partial Y}{\partial t} = wY + \sum_i v_i Y_i + v\xi. \tag{3.2}$$

Since $\{Y, N, Y_i, \xi\}$ is a moving frame along $M^n \times (-\epsilon, \epsilon)$, it follows from (3.1) and (3.2) that

$$\begin{aligned} dY &= w\mathrm{d}tY + \sum_i \Omega_i Y_i + V\xi, \\ dN &= -w\mathrm{d}tN + \sum_i \Psi_i Y_i + \Phi\xi, \\ dY_i &= -\Psi_i Y - \Omega_i N + \sum_j \Omega_{ij} Y_j + F_i\xi, \\ d\xi &= -\Phi Y - VN - \sum_i F_i Y_i, \end{aligned} \tag{3.3}$$

where $\Omega_{ij} = -\Omega_{ji}, \Omega_i = \omega_i + v_i dt, V = vdt$. By exterior differentiation of (3.3) we get

$$\begin{aligned} d\Omega_i &= \sum_j \Omega_{ij} \wedge \Omega_j + w\mathrm{d}t \wedge \Omega_i + v\mathrm{d}t \wedge F_i, \\ dv \wedge \mathrm{d}t &= \sum_i \Omega_i \wedge F_i, \\ dF_i &= \sum_j F_j \wedge \Omega_{ji} - \Omega_i \wedge \Phi - \Psi_i \wedge V, \\ d\Omega_{ij} &= \sum_k \Omega_{ik} \wedge \Omega_{kj} - \Omega_i \wedge \Psi_j - \Psi_i \wedge \Omega_j + F_i \wedge F_j. \end{aligned} \tag{3.4}$$

Since $\mathrm{T}^*(M^n \times (-\epsilon, \epsilon)) = \mathrm{T}^*M^n \oplus \mathrm{T}^*\mathbb{R}$, we have the following decomposition:

$$\Omega_{ij} = \omega_{ij} + L_{ij}\mathrm{d}t, \quad \Psi_i = A_i + u_i\mathrm{d}t, \quad \Phi = C + u\mathrm{d}t; F_i = B_i + b_i dt,$$

where $\{u_i, u, L_{ij}, b_i\}$ are local functions on $\mathbf{M} \times (-\epsilon, \epsilon)$. Using (3.4) and comparing the terms in $\mathrm{T}^*\mathbf{M} \wedge \mathrm{d}t$ we get

$$\frac{\partial \omega_i}{\partial t} = \sum_j (v_{i,j} + L_{ij} + B_{ij}v)\omega_j + w\omega_i, \tag{3.5}$$

where $\{v_{i,j}\}$ is the covariant derivative of $\sum v_i E_i$ with respect to g_t. Here we have used the notations of conformal invariants $\{A_{ij}, B_{ij}, C_i\}$ for x_t. In the same way we get from (3.4) that

$$\begin{aligned} b_i &= e_i(v) + \sum_j B_{ij}v_j, \\ \frac{\partial F_i}{\partial t} &= \sum_j (b_{i,j} + \sum_k L_{ik}B_{kj} + A_{ij}v - v_i C_j)\omega_j + u\omega_i, \end{aligned} \tag{3.6}$$

where $\{b_{i,j}\}$ are covariant derivatives of $\sum_i b_i\omega_i$. Using (3.5) and (3.6) we get

$$\begin{aligned} \frac{\partial B_{ij}}{\partial t} + wB_{ij} = b_{i,j} &+ \sum_k (L_{ik}B_{ik} - B_{ik}L_{kj}) \\ &- v_i C_j + A_{ij}v + u\delta_{ij} - \sum_k B_{ik}(v_{k,j} + B_{kj}v), \end{aligned}$$

and

$$\begin{aligned} b_{i,j} &= v_{,ij} + \sum_k (B_{ik,j}v_k + B_{ik}v_{k,j}), \\ v_{,ij} &= d_M e_i(v) + \sum_j e_j(v)\omega_{ji}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{\partial B_{ij}}{\partial t} + wB_{ij} = v_{,ij} &+ \sum_k (L_{ik}B_{ik} - B_{ik}L_{kj} + B_{ij,k}v_k) \\ &+ (A_{ij} - \sum_k B_{ik}B_{kj}) + (u - \sum_k v_k C_k)\delta_{ij}, \end{aligned}$$

and

$$\frac{n-1}{n}w = \sum_{ij} B_{ij}[v_{,ij} + (A_{ij} - \sum_k B_{ik}B_{kj})v]. \tag{3.7}$$

Now we calculate the first variation of the conformal volume functional

$$W(t) = \mathrm{vol}(g_t) = \int_{M^n} \omega_1 \wedge \cdots \wedge \omega_n = \int_{M^n} \mathrm{d}M_t,$$

where $\mathrm{d}M$ is the volume for g_t. From (3.5) and (3.7) we get

$$\begin{aligned} W'(t) &= \sum_i \int_{M^n} \omega_1 \wedge \cdots \wedge \frac{\partial \omega_i}{\partial t} \wedge \cdots \wedge \omega_n = n\int_{M^n} w\mathrm{d}M_t \\ &= \frac{n^2}{n-1}\int_{M^n} \{\sum_{ij} B_{ij}[v_{,ij} + (A_{ij} - \sum_k B_{ik}B_{kj})v]\}\mathrm{d}M_t. \end{aligned} \tag{3.8}$$

From the fact that the variation is admissible we know $v_i = 0, v = 0$, and $e_i(v) = 0$ on ∂M^n. It follows from (3.8) and Green's formula that

$$W'(t) = \frac{n^2}{n-1}\int_{M^n} \{\sum_{ij} B_{ij,ij} + \sum_{ij} A_{ij}B_{ij} - \sum_{ijk} B_{ik}B_{kj}B_{ij}\}v\mathrm{d}M_t.$$

Thus, we have the following results:

Theorem 3.1 *A spacelike hypersurface $x : M^n \to M_1^{n+1}(c)$ is a Willmore spacelike hypersurface (i.e. a critical hypersurface to the conformal volume functional) if and only if*

$$\sum_{ij} B_{ij,ij} + \sum_{ij} A_{ij}B_{ij} - \sum_{ijk} B_{ik}B_{kj}B_{ij} = 0.$$

Using (2.8) we can write the Euler–Lagrange equations as

$$\sum_i C_{i,i} + \sum_{ij}(\frac{1}{n-1}R_{ij} - A_{ij})B_{ij} = 0. \tag{3.9}$$

4. n-Hyperelastic spacelike curve

The Lorentzian metric of $M_1^2(c)$ will be denoted by $\langle,\rangle_1$ and its Levi-Civita connection by ∇. For vector fields X, Y, Z on $M_1^2(c)$, we write the structure equation $\nabla_X Y - \nabla_Y X = [X, Y]$ and $\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = R(X, Y)Z$, where $[,]$ is the Lie bracket and R the curvature tensor.

Let $\gamma : I \to M_1^2(c)$ be an immersed curve, and $V(t)$ will denote the tangent vector to $\gamma(t)$. If $\langle V(t), V(t)\rangle_1 > 0$, then we call the curve a spacelike curve. Now we always assume that the curve $\gamma(t)$ is a spacelike curve. Let $T(t), N(t)$ be unit tangent and normal vectors, respectively. Thus, $\langle T(t), T(t)\rangle_1 = 1$ and $\langle N(t), N(t)\rangle_1 = -1$. The Frenet equations for γ are given by

$$\nabla_T T = \kappa N, \qquad \nabla_T N = \kappa T,$$

where κ is the oriented curvature of γ.

The letter γ will also denote a variation $\gamma = \gamma_\mu(t) : (-\epsilon, \epsilon) \times I \to M_1^2(c)$ with $\gamma(0,t) = \gamma(t)$. For each μ, the curve $\gamma_\mu : I \to M_1^2(c)$ is a spacelike curve. Associated with such a variation is the variation vector field $\Lambda = \Lambda(t) = \frac{\partial \gamma}{\partial \mu}(0,t)$ along the curve $\gamma(t)$. We will also write $V = V(\mu,t)$, $\Lambda = \Lambda(\mu,t)$, $T = T(\mu,t)$, $v = v(\mu,t) = \langle V(\mu,t), V(\mu,t)\rangle_1^{\frac{1}{2}}$, etc., with the obvious meanings. Let s denote the arclength parametrization and L the length of γ. For a fixed natural number r we consider the functional

$$W^r(\gamma) = \int_\gamma \kappa^r ds = \int_0^1 \kappa^r v dt. \tag{4.1}$$

The following lemma collects some elementary facts that facilitate the derivations of the variational formulas, whose proof is standard.

Lemma 4.1 *Under the above notation, we have the following results:*
(1) $[V, \Lambda] = 0$.
(2) $\frac{\partial v}{\partial \mu} = \langle \nabla_T \Lambda, T\rangle_1 = -\varpi v$, *where* $\varpi = -\langle \nabla_T \Lambda, T\rangle_1$.
(3) $[\Lambda, T] = \varpi T$.
(4) $\frac{\partial \kappa^2}{\partial \mu} = -2\langle \nabla_T\nabla_T \Lambda, \nabla_T T\rangle_1 + 4\varpi\kappa^2 - 2\langle R(\Lambda, T)T, \nabla_T T\rangle_1$.

Using standard arguments that involve some integrations by parts, the Frenet equations of γ, and Lemma 4.1, we can obtain the first variation formula of the functional $W^r(\gamma)$:

$$\begin{aligned} \frac{dW^r(\gamma)}{d\mu} &= -r(r-1)\int_\gamma \kappa^{r-3}[\kappa\kappa_{ss} + (r-2)\kappa_s^2 - \frac{\kappa^4}{r} - \frac{\kappa^2}{r-1}c]\langle \Lambda, N\rangle_1 ds \\ &+ r(r-2)\kappa^{r-3}\kappa_s\langle \Lambda, \nabla_T T\rangle_1|_0^L + r\kappa^{r-2}\langle \Lambda, \nabla_T\nabla_T T\rangle_1|_0^L \\ &- r\kappa^r\langle \Lambda, T\rangle_1|_0^L - r\kappa^{r-2}\langle \nabla_T\Lambda, \nabla_T T\rangle_1|_0^L. \end{aligned} \tag{4.2}$$

Thus, under suitable boundary conditions, γ is a critical point of $W^r(\gamma)$ if and only if the following Euler–Lagrange equation is satisfied:

$$-r(r-1)\kappa^{r-3}[\kappa\kappa_{ss} + (r-2)\kappa_s^2 - \frac{\kappa^4}{r} - \frac{\kappa^2}{r-1}c] = 0. \tag{4.3}$$

Example 4.2 *Let* $\Pi_d = \{(x,y,z) \in \mathbb{R}_1^3 | ax + by + cz = d\}$ *be a 2-dimensional plane. If* $-a^2 + b^2 + c^2 < 0$, *then* Π_d *is a spacelike plane. Moreover, we have the following:*
(1) The totally geodesic curve $\Pi_0 \cap \mathbb{S}_1^2(c)$ *is an* r*-hyperelastic spacelike curve in* $\mathbb{S}_1^2$.
(2) The totally umbilical curve $\Pi_d \cap \mathbb{H}_1^2(c)$ *with the oriented curvature* $\sqrt{\frac{-rc}{r-1}}$ *is an* r*-hyperelastic spacelike curve in* $\mathbb{H}_1^2$ *for some constant* d.

Now we prove Theorem 1.2. If γ is a closed curve, then there are two points p, q such that

$$\kappa(p) = \min_\gamma\{\kappa(s)\}, \quad \kappa(q) = \max_\gamma\{\kappa(s)\}.$$

Thus,

$$\kappa_s(p) = \kappa_s(q) = 0, \quad \kappa_{ss}(p) \geq 0, \quad \kappa_{ss}(q) \leq 0.$$

If $c > 0$, by the Euler–Lagrange equation (4.3) we get $\kappa(p) = \kappa(q) = 0$ and $\kappa = 0$. Thus, the closed r-hyperelastic spacelike curve is a totally geodesic curve in $\mathbb{S}^2_1(c)$.

If $c = 0$, because there exists no closed spacelike curve in $\mathbb{R}^2_1$, there exists no closed r-hyperelastic spacelike curve.

If $c < 0$, by the Euler–Lagrange equation (4.3) we get

$$\kappa(q) = \max_\gamma\{\kappa(s)\} \leq \sqrt{\frac{-rc}{r-1}}, \quad \kappa(p)^2(\frac{\kappa(p)^2}{r} + \frac{c}{r-1}) \geq 0.$$

If $\kappa(p) = \min_\gamma\{\kappa(s)\} > 0$, then $\sqrt{\frac{-rc}{r-1}} \leq \kappa(p) \leq \kappa(q) \leq \sqrt{\frac{-rc}{r-1}}$ and $\kappa = \sqrt{\frac{-rc}{r-1}}$. Thus, the closed r-hyperelastic spacelike curve is a totally umbilical curve in $\mathbb{H}^2_1(c)$.

If $\kappa(p) = \min_\gamma\{\kappa(s)\} = 0$, we assume that p, q are adjacent, and there are no other extreme points between them, and thus $\kappa_s \neq 0$ between p and q. Since $\kappa(p) = \kappa(q) = 0$, then there exists a point p' between p and q such that $\kappa_{ss}(p') = 0$. We note that $\kappa(p') \neq 0$. By combination with equation (4.3) we have

$$\kappa(p')^2(\frac{\kappa(p')^2}{r} + \frac{c}{r-1}) \geq 0;$$

that is, $\kappa(p') \geq \sqrt{\frac{-rc}{r-1}}$, which is a contradiction with

$$\kappa(p') < \kappa(q) = \max_\gamma\{\kappa(s)\} \leq \sqrt{\frac{-rc}{r-1}}.$$

Thus, $\kappa(p) = \min_\gamma\{\kappa(s)\} > 0$ and $\kappa = \sqrt{\frac{-rc}{r-1}}$. Thus, the closed r-hyperelastic spacelike curve is a totally umbilical curve in $\mathbb{H}^2_1(c)$. Thus, we finish the proof of Theorem 1.2.

5. Some special Willmore spacelike hypersurfaces

In this section, we construct some special Willmore spacelike hypersurfaces by n-hyperelastic curves in 2-dimensional Lorentzian space form.

Example 5.1 *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ be a spacelike curve. The cylinder in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{R}^2_1$ is defined by*

$$f : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}_1, \quad f(s, y) = (\gamma(s), y),$$

where $y \in \mathbb{R}^{n-1}$.

Proposition 5.2 *If the cylinder in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{R}^2_1$ as in Example 5.1 is umbilics-free, then the cylinder is a Willmore spacelike hypersurface if and only if $\gamma(s)$ is an n-hyperelastic spacelike curve in $\mathbb{R}^2_1$.*

Proof Let s denote the arclength parametrization of the curve. Then the first and the second fundamental forms of the cylinder f are given by

$$I = ds^2 + I_{\mathbb{R}^{n-1}}, \quad II = \kappa ds^2,$$

where $\kappa(s)$ is the geodesic curvature of $\gamma(s) \subset \mathbb{R}^2_1$ and $I_{\mathbb{R}^{n-1}}$ denotes the standard metric of the $(n-1)$-dimensional Euclidean space $\mathbb{R}^{n-1}$. Thus, the principal curvatures of the cylinder are $\kappa, 0, ..., 0$, and the mean curvature $H = \frac{\kappa}{n}$. Since the cylinder is umbilics-free, then $\kappa \neq 0$. From (2.10), we see that the conformal metric of the cylinder f is $g = \kappa^2(s)(ds^2 + I_{\mathbb{R}^{n-1}})$. Let $\{e_1 = \frac{\partial}{\partial s}, e_2, \cdots, e_n\}$ be an orthonormal basis of $T(\mathbb{R} \times \mathbb{R}^{n-1})$, and then the coefficients of conformal invariants of the cylinder f with respect to the orthonormal basis can be computed from (2.11) as follows:

$$\begin{aligned} &C_1 = -\frac{1}{\kappa^2} e_1(\kappa) = -\frac{1}{\kappa^2}\kappa_s, C_2 = \cdots = C_n = 0; \\ &(B_{ij}) = diag(\frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n}), \\ &(A_{ij}) = diag(a_1, a_2, \cdots, a_2), \end{aligned} \tag{5.1}$$

where

$$a_1 = -\frac{\kappa_{ss}}{\kappa^3} + \frac{3}{2}\frac{(\kappa_s)^2}{\kappa^4} + \frac{1-2n}{2n^2}, \quad a_2 = \frac{1}{2}[\frac{-(\kappa_s)^2}{\kappa^4} + \frac{1}{n^2}].$$

Using (2.11) and (5.1), we get that

$$C_{1,1} = \frac{-\kappa_{ss}}{\kappa^3} + 2\frac{(\kappa_s)^2}{\kappa^4}, \quad C_{i,i} = -C_1^2 = -\frac{(\kappa_s)^2}{\kappa^4}, 2 \leq i \leq n. \tag{5.2}$$

From (5.1) and (5.2), we have

$$\begin{aligned} &-(n-1)\sum_i C_{i,i} - \sum_{ijk} B_{ij}B_{jk}B_{ki} + \sum_{ij} B_{ij}A_{ij} \\ &= \frac{(n-1)^2}{n\kappa^4}[\kappa\kappa_{ss} + (n-2)\kappa_s^2 - \frac{\kappa^4}{n}]. \end{aligned} \tag{5.3}$$

Thus, from (3.9) the cylinder is a Willmore spacelike hypersurface if and only if $\gamma(s)$ is an n-hyperelastic spacelike curve in $\mathbb{R}^2_1$. □

Example 5.3 *Let $\gamma : R \to \mathbb{S}^2_1(1)$ be a spacelike curve in 2-dimensional de Sitter space $\mathbb{S}^2_1(1)$. The cone in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{S}^2_1(1) \subset \mathbb{R}^3_1$ is defined by*

$$f : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{n-2} \to \mathbb{R}^{n+1}_1, \quad f(s,t,y) = (t\gamma(s), y),$$

where $y \in \mathbb{R}^{n-2}$ and $\mathbb{R}^+ = \{t|t > 0\}$.

Proposition 5.4 *If the cone in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{S}^2_1(1)$ as in Example 5.3 is umbilics-free, then the cone is a Willmore spacelike hypersurface if and only if $\gamma(s)$ is an n-hyperelastic spacelike curve in $\mathbb{S}^2_1(1)$.*

Proof The first and the second fundamental forms of the cone f are given by

$$I = t^2 ds^2 + I_{\mathbb{R}^{n-1}}, \quad II = t\kappa ds^2.$$

Let $\{e_1 = \frac{1}{t}\frac{\partial}{\partial s}, e_2 = \frac{\partial}{\partial t}, \cdots, e_n\}$ be an orthonormal basis of $T(I \times R^+ \times R^{n-2})$ with dual basis $\{\omega_1, \cdots, \omega_n\}$, which consists of principal vectors. Let $\{\omega_{ij}\}$ be connection forms with respect to the basis $\{\omega_1, \omega_2, \cdots, \omega_n\}$. Then

$$\omega_{1i} = 0, \quad for \;\; 3 \leq i \leq n \;\; and \;\; \omega_{12} = e_2(\log \frac{1}{t}\kappa)\omega_1.$$

Under the orthonormal basis $\{e_1, e_2, \cdots, e_n\}$, the coefficients of the second fundamental form of the hypersurface f have the diagonal form: $(h_{ij}) = diag(\frac{1}{t}\kappa, 0, \cdots, 0)$. We assume that the hypersurface f is umbilics-free and locally let $\kappa > 0$, so $\rho = \frac{\kappa}{t}$ and the conformal metric g of the cone f is $g = \rho^2 I = \frac{\kappa^2}{t^2}(t^2 ds^2 + I_{R^{n-1}})$. Since $\{\rho^{-1}e_1, \cdots, \rho^{-1}e_n\}$ is an orthonormal basis with respect to g, the coefficients of conformal invariants of f with respect to the orthonormal basis can be obtained as follows using (2.11):

$$\begin{aligned} &C_1 = -\frac{t}{\kappa^2}e_1(\kappa) = -\frac{1}{\kappa^2}\kappa_s, C_2 = \cdots = C_n = 0; \\ &(B_{ij}) = diag(\frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n}), \\ &(A_{ij}) = diag(a_1, a_2, \cdots, a_2), \end{aligned} \tag{5.4}$$

where

$$a_1 = -\frac{\kappa_{ss}}{\kappa^3} + \frac{3}{2}\frac{(\kappa_s)^2}{\kappa^4} - \frac{1}{2\kappa^2} - \frac{2n-1}{2n^2}, \quad a_2 = \frac{1}{2}[\frac{-(\kappa_s)^2}{\kappa^4} + \frac{1}{\kappa^2} + \frac{1}{n^2}].$$

Using (2.11) and (5.7), we get that

$$C_{1,1} = -\frac{\kappa_{ss}}{\kappa^3} + 2\frac{(\kappa_s)^2}{\kappa^4}, \quad C_{i,i} = -C_1^2 = -\frac{(\kappa_s)^2}{\kappa^4}, 2 \leq i \leq n. \tag{5.5}$$

From (5.7) and (5.8), we have

$$\begin{aligned} &-(n-1)\sum_i C_{i,i} - \sum_{ijk} B_{ij}B_{jk}B_{ki} + \sum_{ij} B_{ij}A_{ij} \\ &= \frac{(n-1)^2}{n\kappa^4}[\kappa\kappa_{ss} + (n-2)\kappa_s^2 - \frac{\kappa^4}{n} - \frac{\kappa^2}{n-1}]. \end{aligned} \tag{5.6}$$

Thus, from (3.9) the cone is a Willmore spacelike hypersurface if and only if $\gamma(s)$ is an n-hyperelastic spacelike curve in $\mathbb{S}_1^2(1)$. □

Example 5.5 *Let $\gamma : R \to \mathbb{R}^2_{1+}$ be a spacelike curve. The rotational hypersurface in $\mathbb{R}^{n+1}_1$ over $\gamma(s) \subset \mathbb{R}^2_{1+}$ is defined by*

$$f : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}_1, \qquad f(s, \theta) = (x(s), y(s)\theta),$$

where $\theta \in \mathbb{S}^{n-1}$ is the standard round sphere.

Proposition 5.6 *If the rotational hypersurface in* $\mathbb{R}_1^{n+1}$ *over* $\gamma(s) \subset \mathbb{R}_{1+}^2$ *as in Example 5.5 is umbilics-free, then the rotational hypersurface is a Willmore spacelike hypersurface if and only if* $\gamma(s)$ *is an* n*-hyperelastic spacelike curve in* $\mathbb{R}_{1+}^2$.

Proof Denote the covariant differentiation of the metric ds^2 by D in $\mathbb{R}_{1+}^2$. For $\gamma(s) = (x(s), y(s)) \subset R_{1+}^2$, let $\dot{x}$ denote derivative $\frac{\partial x}{\partial s}$, and so on. Choose the unit tangent vector $\alpha = \frac{1}{y}(\dot{x}, \dot{y})$ and the unit normal vector $\beta = \frac{1}{y}(\dot{y}, \dot{x})$. The geodesic curvature is computed via $\kappa(s) =< D_\alpha \alpha, \beta >= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{y^2} + \frac{\dot{x}}{y}$. The rotational hypersurface f has the unit normal vector $\eta = \frac{1}{y}(\dot{y}, \dot{x}\theta)$. The first and the second fundamental forms of the rotational hypersurface f are given by

$$I = df \cdot df = y^2(ds^2 + I_{\mathbb{S}^{n-1}}), \quad II = -df \cdot d\eta = (y\kappa - \dot{x})ds^2 - \dot{x} I_{\mathbb{S}^{n-1}}.$$

Thus, the principal curvatures of the rotational hypersurface f are $\frac{y\kappa - \dot{x}}{y^2}$, $\frac{-\dot{x}}{y^2}, \cdots, \frac{-\dot{x}}{y^2}$. From (2.10), we know that the conformal metric of the rotational hypersurface f is $g = \kappa^2(x)(ds^2 + I_{\mathbb{S}^{n-1}})$, and $\rho = \frac{\kappa}{y}$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T(R \times S^{n-1})$, which consists of principal vectors. From (2.11) we can obtain the coefficients of conformal invariants of f under the orthonormal basis $\{\rho^{-1}e_1, \cdots, \rho^{-1}e_n\}$ for g as follows:

$$\begin{aligned} &C_1 = -\frac{t}{\kappa^2} e_1(\kappa) = -\frac{1}{\kappa^2}\kappa_s, C_2 = \cdots = C_n = 0; \\ &(B_{ij}) = diag(\frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n}), \\ &(A_{ij}) = diag(a_1, a_2, \cdots, a_2), \end{aligned} \tag{5.7}$$

where

$$a_1 = -\frac{\kappa_{ss}}{\kappa^3} + \frac{3}{2}\frac{(\kappa_s)^2}{\kappa^4} + \frac{1}{2\kappa^2} - \frac{2n-1}{2n^2}, \quad a_2 = \frac{1}{2}[\frac{-(\kappa_s)^2}{\kappa^4} - \frac{1}{\kappa^2} + \frac{1}{n^2}].$$

Using (2.11) and (5.7), we get that

$$C_{1,1} = -\frac{\kappa_{ss}}{\kappa^3} + 2\frac{(\kappa_s)^2}{\kappa^4}, \quad C_{i,i} = -C_1^2 = -\frac{(\kappa_s)^2}{\kappa^4}, 2 \leq i \leq n. \tag{5.8}$$

From (5.7) and (5.8), we have

$$\begin{aligned} &-(n-1)\sum_i C_{i,i} - \sum_{ijk} B_{ij}B_{jk}B_{ki} + \sum_{ij} B_{ij}A_{ij} \\ &= \frac{(n-1)^2}{n\kappa^4}[\kappa\kappa_{ss} + (n-2)\kappa_s^2 - \frac{\kappa^4}{n} + \frac{\kappa^2}{n-1}]. \end{aligned} \tag{5.9}$$

Thus, from (3.9) the rotational hypersurface is a Willmore spacelike hypersurface if and only if $\gamma(s)$ is an n-hyperelastic spacelike curve in $\mathbb{R}_{1+}^2$. □

6. The proof of Theorem 1.3

In this section we prove our main theorem, Theorem 1.3. For this we need the following lemma, and we refer to [8] for the proof of the following lemma.

Lemma 6.1 *[8] Let (M^n, g) be a Riemannian manifold and $\tilde{g}$ another Riemannian metric on M^n such that $\tilde{g} = e^{2\tau}g$, where τ is a smooth function on M^n. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal basis for g with dual basis $\{\omega_1, \cdots, \omega_n\}$, and let $\{\omega_{ij}\}$ be the connection forms with respect to the basis $\{\omega_1, \cdots, \omega_n\}$. Then $\{\tilde{e}_1 = e^{-\tau}e_1, \cdots, \tilde{e}_n = e^{-\tau}e_n\}$ is a local orthonormal basis for $\tilde{g}$, and $\{\tilde{\omega}_1 = e^{\tau}\omega_1, \cdots, \tilde{\omega}_n = e^{\tau}\omega_n\}$ is the dual basis.*

Moreover, if $\{\tilde{\omega}_{ij}\}$ are the connection forms with respect to the basis $\{\tilde{\omega}_1, \cdots, \tilde{\omega}_n\}$, then

$$\tilde{\omega}_{ij} = \omega_{ij} + e_i(\tau)\omega_j - e_j(\tau)\omega_i, 1 \le i, j \le n.$$

It is a classical result that an n-dimensional hypersurface in space forms has a principle curvature of multiplicity of at least $n-1 (n \ge 4)$ everywhere if and only if it is conformally flat. Similarly, there are the same results for spacelike hypersurfaces in Lorentzian space forms. Let $f: M^n \to \mathbb{R}^{n+1}_1 (n \ge 4)$ be a conformally flat Willmore spacelike hypersurface without umbilical points. We denote by b_1, b_2 the conformal principal curvatures. From (2.8), we can choose a local orthonormal basis $\{E_1, \cdots, E_n\}$ with respect to the conformal metric g such that

$$(B_{ij}) = diag(\frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n}).$$

In the following section we make use of the following convention on the ranges of indices:

$$1 \le i, j, k \le n; \quad 2 \le \alpha, \beta, \gamma \le n.$$

Since $B_{\alpha\beta} = \frac{1}{n}\delta_{\alpha\beta}$, we can rechoose a local orthonormal basis $\{E_1, \cdots, E_n\}$ with respect to the conformal metric g such that

$$(B_{ij}) = diag(\frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n}), \quad (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & a_2 & 0 & \cdots & 0 \\ A_{31} & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & 0 & 0 & \cdots & a_n \end{pmatrix}. \tag{6.1}$$

Let $\{\omega_1, \cdots, \omega_n\}$ be the dual basis and $\{\omega_{ij}\}$ the connection forms.

Lemma 6.2 *Let $f: M^n \to \mathbb{R}^{n+1}_1$ $(n \ge 4)$ be a conformally flat Willmore spacelike hypersurface. If f is umbilics-free, then we can choose a local orthonormal basis $\{E_1, \cdots, E_n\}$ with respect to the conformal metric g such that*

$$(B_{ij}) = diag\{\frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n}\}, \quad (A_{ij}) = diag\{a_1, a_2, \cdots, a_2\}.$$

Moreover, the distribution $span\{E_2, \cdots, E_n\}$ is integrable.

Proof Using $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$, equation (2.5), and (6.1), we get

$$\begin{aligned} &B_{1\alpha,\alpha} = -C_1, \quad otherwise, \quad B_{ij,k} = 0; \\ &\omega_{1\alpha} = -C_1\omega_\alpha, \quad C_\alpha = 0. \end{aligned} \tag{6.2}$$

Thus, we have $d\omega_1 = \sum_\alpha \omega_{1\alpha} \wedge \omega_\alpha = 0$ and the distribution $\mathbb{D} = span\{E_2, \cdots, E_n\}$ is integrable.

Using $dC_i + \sum_k C_k\omega_{ki} = \sum_k C_{i,k}\omega_k$ and (6.2), we can obtain

$$C_{\alpha,\alpha} = -C_1^2, \quad C_{\alpha,k} = 0, \quad \alpha \neq k. \tag{6.3}$$

From (6.2),

$$\begin{aligned} d\omega_{1\alpha} &= -dC_1 \wedge \omega_\alpha - C_1 d\omega_\alpha \\ &= -dC_1 \wedge \omega_\alpha + C_1^2\omega_1 \wedge \omega_\alpha - C_1 \sum_\gamma \omega_\gamma \wedge \omega_{\gamma\alpha}, \end{aligned}$$

and $d\omega_{1\alpha} - \sum_j \omega_{1j} \wedge \omega_{j\alpha} = -\frac{1}{2}\sum_{kl} R_{1\alpha kl}\omega_k \wedge \omega_l$, we get that

$$R_{1\alpha 1\alpha} = C_{1,1} - C_1^2, \qquad R_{1\alpha\beta\alpha} - C_{1,\beta} = 0. \tag{6.4}$$

Since $R_{1\alpha 1\alpha} = \frac{n-1}{n^2} + a_1 + a_\alpha = C_{1,1} - C_1^2$ and $R_{1\alpha\beta\alpha} = A_{1\beta}, \alpha \neq \beta$, we have

$$a_2 = a_3 = \cdots = a_n, \qquad A_{1\beta} = C_{1,\beta}. \tag{6.5}$$

Thus, $A|_{\mathbb{D}} = aI, \ \ a = a_2$. Since E_1 is a principal vector field, then vector $E = A_{12}E_2 + \cdots + A_{1n}E_n$ is well defined. If $E = 0$, then $A_{12} = \cdots = A_{1n} = 0$ and $(A_{ij}) = diag\{a_1, a_2, \cdots, a_2\}$. Thus, Lemma 6.2 holds.

If $E \neq 0$, we can rechoose a local orthonormal basis $\{\tilde{E}_2 = \frac{E}{|E|}, \tilde{E}_3, \cdots, \tilde{E}_n\}$ of $\mathbb{D}$ with respect to the conformal metric g such that

$$(B_{ij}) = diag(\frac{n-1}{n}, \frac{-1}{n}, \cdots, \frac{-1}{n}), \quad (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & 0 & \cdots & 0 \\ A_{21} & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_2 \end{pmatrix}. \tag{6.6}$$

Thus, $C_{1,\alpha} = A_{1\alpha} = 0, \ \ \alpha \geq 3$. To prove the lemma, we only to prove that $A_{12} = 0$. Since f is a Willmore spacelike hypersurface, using equation (3.9),

$$-(n-1)\sum_i C_{i,i} - \sum_i b_i^3 + \sum_i b_i a_i = 0,$$

and (6.3) and (6.4), we get that

$$\begin{aligned} a_1 - a_2 &= nC_{1,1} - n(n-1)C_1^2 + \frac{n-2}{n}, \\ a_1 + a_2 &= C_{1,1} - C_1^2 - \frac{n-1}{n^2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} a_1 &= \frac{n+1}{2}C_{1,1} - \frac{n^2-n+1}{2}C_1^2 + \frac{n^2-3n+1}{2n^2}, \\ a_2 &= \frac{1-n}{2}C_{1,1} + \frac{n^2-n-1}{2}C_1^2 - \frac{n^2-n-1}{2n^2}. \end{aligned} \tag{6.7}$$

Using $dA_{\alpha\beta}+\sum_k A_{k\beta}\omega_{k\alpha}+\sum_k A_{\alpha k}\omega_{k\beta}=\sum_k A_{\alpha\beta,k}\omega_k$ and $C_\alpha=0$ we get that

$$\begin{aligned}&E_\beta(a_2)=A_{\alpha\alpha,\beta}=0,\quad \alpha\geq 3,\quad E_2(a_2)=A_{\alpha 2,\alpha}=-A_{12}C_1,\\ &A_{\alpha\beta,1}=0,\quad A_{\alpha\beta,\gamma}=0,\quad \alpha\neq\beta,\beta\neq\gamma,\alpha\neq\gamma,\\ &E_1(a_2)=A_{\alpha\alpha,1}=A_{\beta\beta,1},\quad A_{1\alpha,\alpha}=A_{\alpha\alpha,1}+\frac{1}{n}C_1.\end{aligned}\tag{6.8}$$

Combining (6.7), we can obtain

$$\begin{aligned}&E_\beta(C_{1,1})=0,\quad E_\beta(a_1)=0,\quad \alpha\geq 3,\\ &E_2(C_{1,1})=2nC_1A_{12},\quad E_2(a_1)=(2n-1)C_1A_{12},\\ &E_1(a_2)=(a_2-a_1-\frac{1}{n})C_1.\end{aligned}\tag{6.9}$$

Similarly, using $dA_{ij}+\sum_k A_{kj}\omega_{ki}+\sum_k A_{ik}\omega_{kj}=\sum_k A_{ij,k}\omega_k$ for $i=1,2,\ j=\alpha\geq 3$ and equation (6.6), we have

$$\begin{aligned}&(a_1-a_2)\omega_{1\alpha}+A_{12}\omega_{2\alpha}=\sum_k A_{1\alpha,k}\omega_k,\quad \alpha\geq 3,\\ &A_{12}\omega_{1\alpha}=\sum_k A_{2\alpha,k}\omega_k.\end{aligned}\tag{6.10}$$

Similarly, using $dA_{ij}+\sum_k A_{kj}\omega_{ki}+\sum_k A_{ik}\omega_{kj}=\sum_k A_{ij,k}\omega_k$ for $i=1,2,\ j=1,2$ and equation (6.6), we have

$$\begin{aligned}&dA_{11}+2A_{12}\omega_{21}=\sum_k A_{11,k}\omega_k,\\ &dA_{22}+2A_{12}\omega_{12}=\sum_k A_{22,k}\omega_k,\\ &dA_{12}+(a_1-a_2)\omega_{12}=\sum_k A_{12,k}\omega_k.\end{aligned}\tag{6.11}$$

From (6.11), we know that $E_\alpha(a_1)=A_{11,\alpha}=0,\quad \alpha\geq 3$. On the other hand, from (2.4) and (6.10), we have

$$A_{11,\alpha}=A_{1\alpha,1}=A_{12}\omega_{2\alpha}(E_1),\quad \alpha\geq 3.$$

If we assume that $A_{12}\neq 0$, from (6.10) we obtain that

$$\begin{aligned}&\omega_{2\alpha}(E_1)=0,\quad \omega_{2\alpha}(E_\beta)=0,\quad \beta\neq\alpha,\\ &A_{12}\omega_{2\alpha}(E_\alpha)=(a_1-a_2)C_1+A_{1\alpha,\alpha}=(a_1-a_2+\frac{1}{n})C_1+E_1(a_2).\end{aligned}\tag{6.12}$$

Since $E_1(a_2)=A_{22,1}=A_{12,2}-\frac{1}{n}C_1=E_2(A_{12})-\frac{1}{n}C_1$, we have

$$A_{12}\omega_{2\alpha}(E_\alpha)=E_2(A_{12}).\tag{6.13}$$

Combining (6.12) and (6.14), we have

$$\omega_{2\alpha}=\frac{E_2(A_{12})}{A_{12}}\omega_\alpha=\varphi\omega_\alpha.\tag{6.14}$$

Using $d\omega_{2\alpha} - \sum_k \omega_{2k} \wedge \omega_{k\alpha} = -\frac{1}{2}\sum_{kl} R_{2\alpha kl}\omega_k \wedge \omega_l$, we can obtain

$$E_1(\varphi) - C_1\varphi = -R_{2\alpha 1\alpha} = -A_{12}. \tag{6.15}$$

Since $E_2(a_1) = (2n-1)C_1A_{12}$, from (6.11), we have $A_{11,2} = (2n+1)C_1A_{12}$, and thus $A_{12,1} = (2n+1)C_1A_{12}$ and

$$E_1(A_{12}) = (2n+1)C_1A_{12}. \tag{6.16}$$

Since $\omega_{1\alpha} = -C_1\omega_\alpha$ and $\omega_{2\alpha} = \varphi\omega_\alpha$, then

$$E_1E_2 - E_2E_1 = [E_1, E_2] = C_1E_2.$$

Thus,

$$\begin{aligned}
E_1(\varphi) - C_1\varphi &= E_1(\frac{E_2(A_{12})}{A_{12}}) - C_1\frac{E_2(A_{12})}{A_{12}} \\
&= \frac{E_1(E_2(A_{12}))}{A_{12}} - \frac{E_2(A_{12})E_1(A_{12})}{A_{12}^2} - C_1\frac{E_2(A_{12})}{A_{12}} \\
&= \frac{E_2(E_1(A_{12})) + C_1E_2(A_{12})}{A_{12}} - \frac{E_2(A_{12})E_1(A_{12})}{A_{12}^2} - C_1\frac{E_2(A_{12})}{A_{12}} \\
&= (2n+1)A_{12}.
\end{aligned}$$

From (6.15) we derive

$$2(n+1)A_{12} = 0.$$

This is a contradiction and thus

$$A_{12} = 0.$$

Thus, we finish the proof of Lemma 6.2. □

Now we choose the local orthonormal basis $\{E_1, \cdots, E_n\}$ as in Lemma 4.1, which consists of principal vectors. Then $\{Y, N, Y_1, \cdots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}_2^{n+3}$ along M^n. We define

$$F = \frac{-1}{n}Y - \xi, \quad X_1 = -C_1Y - Y_1, \quad P = -a_2Y + N + C_1X_1 - \frac{1}{n}F. \tag{6.17}$$

Let $K = 2a_2 + C_1^2 - \frac{1}{n^2}$. By direct computations we have

$$\begin{aligned}
&\langle F, X_1\rangle_2 = 0, \quad \langle F, P\rangle_2 = 0, \quad \langle X_1, P\rangle_2 = 0, \\
&\langle F, F\rangle_2 = -1, \quad \langle X_1, X_1\rangle_2 = 1, \quad \langle P, P\rangle_2 = -K.
\end{aligned} \tag{6.18}$$

From Lemma 4.1, (6.9), and the structure equations of f we derive that

$$\begin{aligned}
&E_1(F) = X_1, && E_\alpha(F) = 0, \\
&E_1(X_1) = P - F, && E_\alpha(X_1) = 0, \\
&E_1(P) = C_1P + KX_1, && E_\alpha(P) = 0.
\end{aligned} \tag{6.19}$$

Thus, subspace $V = span\{F, X_1, P\}$ is fixed along M^n. From (6.9) we get that

$$E_1(K) = 2C_1K, \quad E_\alpha(K) = 0. \tag{6.20}$$

Using the theory of linear first-order differential equations for K, formula (6.20) implies that $K \equiv 0$ or $K \neq 0$ on an open subset $U \subset M^n$. Therefore, we have to consider the following three cases:
Case 1 $K = 0$ on M^n; **Case 2** $K < 0$ on M^n; **Case 3** $K > 0$ on M^n.

Theorem 1.2 is proved by the following three propositions, treating them case by case.

Proposition 6.3 *Let $f : M^n \to \mathbb{R}_1^{n+1}$ $(n \geq 3)$ be a conformally flat Willmore spacelike hypersurface without umbilical point. If $K = 2a_2 + C_1^2 - \frac{1}{n^2} = 0$, then f is conformally equivalent to a cylinder over an n-hyperelastic spacelike curve in $\mathbb{R}_1^2$.*

Proof Since $K = 0$, then $\langle P, P\rangle_2 = 0$. From (6.19), we know that P is of fixed direction. From (6.18), up to a conformal transformation we can write

$$\begin{aligned} P &= \nu(1, 0, \cdots, 0, 1), \quad \nu \in C^\infty(U), \\ V &= span\{F, X_1, P\} \\ &= span\{(1, 0, \cdots, 0, 1), (0, 1, 0, \cdots, 0), (0, 0, 1, 0, \cdots, 0)\} = R_0^3. \end{aligned}$$

Let $(k_1, k_2, \cdots, k_2)$ be the principal curvatures of $f : M^n \to \mathbb{R}_1^{n+1}$, and then

$$b_1 = e^{-\tau}(k_1 - H) = \frac{n-1}{n}, \quad b_2 = e^{-\tau}(k_2 - H) = \frac{-1}{n}, \cdots, b_n = e^{-\tau}(k_2 - H) = \frac{-1}{n}.$$

From (2.10), we have

$$\begin{aligned} F &= b_2 Y - \xi \\ &= k_2\left(\frac{\langle f, f\rangle_1 + 1}{2}, f, \frac{\langle f, f\rangle_1 - 1}{2}\right) - (\langle f, e_{n+1}\rangle_1, e_{n+1}, \langle f, e_{n+1}\rangle_1). \end{aligned} \tag{6.21}$$

From (6.18),

$$\langle P, F\rangle_2 = \langle (1, 0, \cdots, 0, 1), F\rangle_2 = 0, \quad \langle X_1, P\rangle_2 = 0.$$

Thus,

$$k_2 = 0, \quad C_1 + E_1(\tau) = 0, i.e. \;\; E_1(\tau) = -C_1. \tag{6.22}$$

From the definition of F, X_1, and P, we get that $Y_\alpha \perp V$; thus, $< P, Y_\alpha >= 0$, and

$$E_\alpha(\tau) = 0. \tag{6.23}$$

Let $\{e_i = e^\tau E_i, 1 \leq i \leq n\}$. Then $\{e_1, \cdots, e_n\}$ is a local orthonormal basis with respect to the first fundamental form $\langle df, df\rangle_1$. Let $\{\tilde{\omega}_1, \cdots, \tilde{\omega}_n\}$ be the dual basis and $\{\tilde{\omega}_{ij}\}$ connection forms with respect to the basis $\{\tilde{\omega}_1, \cdots, \tilde{\omega}_n\}$. Then, from Lemma 6.1, (6.22), and (6.23), we get

$$\tilde{\omega}_{1\alpha} = 0. \tag{6.24}$$

Therefore, the spacelike hypersurface $f : M^n \to \mathbb{R}_1^{n+1}$ is conformally equivalent to a spacelike hypersurface given by Example 5.1. Since f is a Willmore spacelike hypersurface, from Proposition 5.2 we finish the proof of Proposition 6.3. □

Proposition 6.4 *Let $f : M^n \to \mathbb{R}_1^{n+1}$ $(n \geq 4)$ be a conformally flat Willmore spacelike hypersurface without umbilical points. If $K = 2a_2 + C_1^2 + \frac{1}{n^2} < 0$, then f is conformally equivalent to a cone over an n-hyperelastic spacelike curve in S_1^2.*

Proof of Proposition (6.4) Since $K < 0$, then $\langle P, P\rangle_2$ is positive. From (6.18), up to a conformal transformation we can write

$$\begin{aligned} V &= span\{F, X_1, P\} \\ &= span\{(0,1,0,\cdots,0),(0,0,1,0,\cdots,0),(0,0,0,1,0,\cdots,0)\} = \mathbb{R}_1^3. \end{aligned}$$

Thus,

$$e = (1, 0, \cdots, 0, 1) \perp V.$$

Let $(k_1, k_2, \cdots, k_2)$ be the principal curvatures of $f : M^n \to \mathbb{R}_1^{n+1}$, and then

$$b_1 = e^{-\tau}(k_1 - H) = \frac{n-1}{n}, \quad b_2 = e^{-\tau}(k_2 - H) = \frac{-1}{n}, \cdots, b_n = e^{-\tau}(k_2 - H) = \frac{-1}{n}.$$

From (2.10), we have

$$\begin{aligned} F &= b_2 Y - \xi \\ &= k_2\left(\frac{\langle f,f\rangle_1 + 1}{2}, f, \frac{\langle f,f\rangle_1 - 1}{2}\right) - (\langle f, e_{n+1}\rangle_1, e_{n+1}, \langle f, e_{n+1}\rangle_1). \end{aligned} \tag{6.25}$$

Since $\langle e, F\rangle_2 = \langle e, X_1\rangle_2 = 0$, we have

$$k_2 = 0, \quad C_1 + E_1(\tau) = 0, \quad i.e., E_1(\tau) = -C_1. \tag{6.26}$$

Setting

$$T = -a_2 Y - N + C_1 Y_1 - \frac{1}{n}\xi, \quad \bar{P} = \frac{P}{\sqrt{-K}}, \quad \theta = \frac{T}{\sqrt{-K}},$$

then

$$\begin{aligned} &\langle \bar{P}, \bar{P}\rangle_2 = 1, \quad \langle \theta, \theta\rangle_2 = -1, \\ &\theta \perp V = R^3, \quad \langle \theta, Y_\alpha\rangle_2 = 0. \end{aligned} \tag{6.27}$$

From Lemma 4.1, (6.19), and the structure equations of f we derive that

$$E_1(\theta) = 0, \quad E_\alpha(\theta) = \sqrt{-K} Y_\alpha. \tag{6.28}$$

Since $P + T = -KY$, so

$$Y = \frac{1}{\sqrt{-K}}(\bar{P}, \theta) \in \mathbb{R}_1^{n+3} = \mathbb{R}_1^3 \times \mathbb{R}_1^n.$$

Since the distribution $span\{E_2, \cdots, E_n\}$ is integrable, from (6.19), (6.27), and (6.28), the map Y factors through a conformal diffeomorphism θ from the space of leaves V of this foliation to $\mathbb{H}^{n-1}$. Thus,

$$\bar{P} : I \to S_1^2 \subset R^3, \quad \theta : \mathbb{H}^{n-1} \to \mathbb{R}_1^n.$$

Thus, the spacelike hypersurface $f : M^n \to \mathbb{R}^{n+1}_1$ is conformal equivalent to a spacelike hypersurface given by Example 5.3. Since f is a Willmore spacelike hypersurface, from Proposition 5.4 we finish the proof of Proposition 6.4.

Proposition 6.5 *Let $f : M^n \to \mathbb{R}^{n+1}_1$ $(n \geq 4)$ be a conformally flat Willmore spacelike hypersurface without umbilical points. If $K = 2a_2 + C_1^2 + \frac{1}{n^2} > 0$, then f is conformally equivalent to a rotational hypersurface over an n-hyperelastic spacelike curve in $\mathbb{R}^2_{1+}$.*

Proof of Proposition (6.5), Since $K > 0$, then $\langle P, P\rangle_2$ is negative. From (6.18), up to a conformal transformation we can write

$$\begin{aligned} V &= span\{F, X_1, P\} \\ &= span\{(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), (0, 0, 1, 0, \cdots, 0)\} = \mathbb{R}^3_2. \end{aligned}$$

Thus, $e = (1, 0, \cdots, 0, 1) \in V$ and $\langle e, Y_\alpha\rangle_2 = 0$, and we get that

$$E_\alpha(\tau) = 0. \tag{6.29}$$

Setting

$$T = -a_2 Y - N + C_1 Y_1 - \frac{1}{n}\xi, \quad \bar{P} = \frac{P}{\sqrt{-K}}, \quad \theta = \frac{T}{\sqrt{-K}},$$

then

$$\begin{aligned} &\langle \bar{P}, \bar{P}\rangle_2 = -1, \quad \langle \theta, \theta\rangle_2 = 1, \\ &\theta \perp V = R^3_1, \quad \langle \theta, Y_\alpha\rangle_2 = 0. \end{aligned} \tag{6.30}$$

From Lemma 4.1, (6.19), and the structure equations of f we derive that

$$E_1(\theta) = 0, \quad E_\alpha(\theta) = -\sqrt{K} Y_\alpha. \tag{6.31}$$

Since $P + T = -KY$, so

$$Y = \frac{1}{-\sqrt{K}}(\bar{P}, \theta) \in R^{n+3}_1 = \mathbb{R}^3_2 \times \mathbb{R}^n.$$

Since the distribution $span\{E_2, \cdots, E_n\}$ is integrable, from (6.19), (6.30), and (6.31), the map Y factors through a conformal diffeomorphism θ from the space of leaves V of this foliation to $\mathbb{S}^{n-1}$. Thus,

$$\bar{P} : I \to \mathbb{H}^2_1 \subset \mathbb{R}^3_1, \quad \theta : \mathbb{S}^{n-1} \to \mathbb{R}^n.$$

Writing $\bar{P} = (u_1, u_2, u_3) \in \mathbb{H}^2_1$, then

$$Y = \frac{u_1 - u_3}{-\sqrt{K}}\left(\frac{u_1}{u_1 - u_3}, \frac{u_2}{u_1 - u_3}, \frac{u_3}{u_1 - u_3}, \frac{1}{u_1 - u_3}\theta\right). \tag{6.32}$$

Then hypersurface $f : I \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}_1$, and

$$f = \left(\frac{u_2}{u_1 - u_3}, \frac{1}{u_1 - u_3}\theta\right).$$

Using $\varphi^{-1}:\mathbb{H}_1^2\to\mathbb{R}_{1+}^2$, we know that the spacelike hypersurface $f:M^n\to\mathbb{R}_1^{n+1}$ is conformal equivalent to a spacelike hypersurface given by Example 5.5. Since f is a Willmore spacelike hypersurface, from Proposition 5.6 we finish the proof of Proposition 6.5.

Combining Proposition 6.3, Proposition 6.4, and Proposition 6.5, we finish the proof of Theorem 1.3.

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