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Vector fields and planes in $\mathbb{E}^4$ which play the role of Darboux vector

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Abstract: In this paper, we define some new vector fields along a space curve with nonvanishing curvatures in Euclidean 4-space. By using these vector fields we determine some new planes, curves, and ruled hypersurfaces. We show that the determined new planes play the role of the Darboux vector. We also show that, contrary to their definitions, osculating curves of the first kind and rectifying curves in Euclidean 4-space can be considered as space curves whose position vectors always lie in a two-dimensional subspace. Furthermore, we construct developable and nondevelopable ruled hypersurfaces associated with the new vector fields in which the base curve is always a geodesic on the developable one.

Key words: Darboux vector, vector field, ruled hypersurface, geodesic curve

1. Introduction

Vector fields have always been used for studying differential geometry of curves and surfaces not only in 3-space but also in higher dimensional spaces. The most common known vector fields are natural vector fields in space, Frenet vector fields along curves, the Darboux vector field of a curve in 3-space, normal and tangent vector fields of surfaces and hypersurfaces, etc. These vector fields determine most geometric properties of the related object. Frenet vector fields along a curve constitute an orthonormal frame. This frame is called the Frenet frame and it includes all the information about the curve. The rate of change of the Frenet frame is given by Frenet formulas. These formulas can be rewritten as vector products by means of the Darboux vector field which determines the instantaneous axis of rotation of Frenet frame. Besides, by considering the Darboux vector field of a space curve, we can construct a special ruled surface (called rectifying developable) on which the base curve is always a geodesic [6]. Therefore,the Darboux vector field plays an important role for space curves in Euclidean 3-space $\mathbb{E}^3$. This importance of the Darboux vector led us to look for such vector fields along a space curve in $\mathbb{E}^4$. In the literature, we can find a generalized Darboux vector in $\mathbb{E}^n$ and as a special case in $\mathbb{E}^4$ [2]. However, it does not serve us as we desired.

The first purpose of this paper is to look for vector fields along a space curve in $\mathbb{E}^4$ which enable us to rewrite the Frenet formulas as vector products. For this purpose, along a space curve with nonvanishing curvatures in $\mathbb{E}^4$, we introduce four special vector fields that will serve us as we desired. Later, by using the introduced vector fields, we define some new planes, curves, and ruled hypersurfaces. We show that the determined new planes play the role of the Darboux vector. We characterize the new curves and study singular points of the obtained ruled hypersurfaces. We also show that, contrary to their definitions, osculating curves of

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the first kind and rectifying curves in $\mathbb{E}^4$ can be considered as space curves whose position vectors always lie in a two-dimensional subspace. Furthermore, we study the developability of the ruled hypersurfaces associated with the new vector fields. It is also shown that the base curve is always a geodesic on the obtained new developable ruled hypersurface.

This paper is organized as follows:

Section 2 presents the Frenet formulas for curves in $\mathbb{E}^3$ and $\mathbb{E}^4$, and also includes the definitions of rectifying curve, osculating curve of the first kind and ternary product of vectors in $\mathbb{E}^4$. In Section 3, by defining some new vector fields along a space curve with nonzero curvatures in $\mathbb{E}^4$, we introduce some planes and curves, and give some characterizations. Finally, we define two new ruled hypersurfaces associated with the introduced vector fields and show their developability in Section 4.

2. Preliminaries

2.1. Curves in $\mathbb{E}^3$ and $\mathbb{E}^4$

Let α be a unit speed curve in $\mathbb{E}^3$, and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ denote the Frenet frame of α. The Frenet formulas are given by

$$\mathbf{t}' = \kappa\mathbf{n}, \quad \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}, \quad \mathbf{b}' = -\tau\mathbf{n}, \tag{2.1}$$

where κ and τ denote the curvature and the torsion of α, respectively. On the other hand, by using the Darboux vector field $\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}$ of the curve, we can rewrite the Frenet formulas as [7]

$$\mathbf{t}' = \mathbf{d} \times \mathbf{t}, \quad \mathbf{n}' = \mathbf{d} \times \mathbf{n}, \quad \mathbf{b}' = \mathbf{d} \times \mathbf{b}. \tag{2.2}$$

Similarly, for a unit speed curve β with its Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ in $\mathbb{E}^4$, the Frenet formulas are given by

$$\mathbf{T}' = \kappa_1\mathbf{N}, \quad \mathbf{N}' = -\kappa_1\mathbf{T} + \kappa_2\mathbf{B}_1, \quad \mathbf{B}_1' = -\kappa_2\mathbf{N} + \kappa_3\mathbf{B}_2, \quad \mathbf{B}_2' = -\kappa_3\mathbf{B}_1, \tag{2.3}$$

where $\mathbf{T}, \mathbf{N}, \mathbf{B}_1$, and $\mathbf{B}_2$ denote the tangent, principal normal, first binormal and second binormal vector fields, respectively, and κ_i, $i = 1, 2, 3$ denotes the i-th curvature function of the curve.

Definition 2.1 *Let γ be a curve in $\mathbb{E}^4$. γ is called as rectifying curve [5] if its position vector lies always in the orthogonal complement of its principal normal vector field, and γ is called as osculating curve of the first kind [8] if its position vector lies always in the orthogonal complement of its first binormal vector field.*

Definition 2.2 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis of $\mathbb{R}^4$. The vector*

$$\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

is called the ternary (triple) product of the vectors $\mathbf{a} = \sum_{i=1}^{4} a_i\mathbf{e_i}$, $\mathbf{b} = \sum_{i=1}^{4} b_i\mathbf{e_i}$, and $\mathbf{c} = \sum_{i=1}^{4} c_i\mathbf{e_i}$ [9].

3. New special curves in $\mathbb{E}^4$ and their characterizations

In this section, we define some new special vector fields along a regular curve in $\mathbb{E}^4$. By using these vector fields we define some new special curves, and we give their characterizations. Also, we associate these curves with rectifying curves and osculating curve of the first kind.

It is known that the Darboux vector field which determines the instantaneous axis of rotation of the Frenet frame plays an important role. Inspired by the equations given in (2.2), it is natural to ask the following question:

Question 1: Are there any vector fields that enable us to rewrite the Frenet formulas (2.3) as ternary products of related Frenet vectors?

First of all, it is clear from the ternary product that we need at least two vector fields. Unfortunately, it is impossible to rewrite (2.3) as ternary products by using two specific vector fields.

Let β be a unit speed curve in $\mathbb{E}^4$ with nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$, and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ denotes its Frenet frame. Let us now introduce the following vector fields defined along β:

$$\mathcal{D}_1 = \mathbf{B}_2, \quad \mathcal{D}_2 = \kappa_2\mathbf{T} + \kappa_1\mathbf{B}_1, \quad \mathcal{D}_3 = \kappa_3\mathbf{N} + \kappa_2\mathbf{B}_2, \quad \mathcal{D}_4 = \mathbf{T}. \tag{3.1}$$

Note that $\mathcal{D}_1$ and $\mathcal{D}_4$ are the Frenet vectors of the curve, and $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4\}$ is linearly independent along the curve. Moreover, it is clear that $\{\mathcal{D}_1, \mathcal{D}_2\}$, $\{\mathcal{D}_3, \mathcal{D}_4\}$, and $\{\mathcal{D}_2, \mathcal{D}_3\}$ are orthogonal sets. We denote the subspaces spanned by $\{\mathcal{D}_1, \mathcal{D}_2\}$, $\{\mathcal{D}_3, \mathcal{D}_4\}$, and $\{\mathcal{D}_2, \mathcal{D}_3\}$ as $\mathcal{D}_1\mathcal{D}_2$-plane, $\mathcal{D}_3\mathcal{D}_4$-plane, and $\mathcal{D}_2\mathcal{D}_3$-plane, respectively. Thus, by using these vector fields, as an answer to the above question, we may rewrite (2.3) as follows:

$$\begin{array}{rcl} \mathbf{T}' &=& \mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \mathbf{T}, \\ \mathbf{N}' &=& \mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \mathbf{N}, \\ \mathbf{B}_1' &=& \mathcal{D}_3 \otimes \mathcal{D}_4 \otimes \mathbf{B}_1, \\ \mathbf{B}_2' &=& \mathcal{D}_3 \otimes \mathcal{D}_4 \otimes \mathbf{B}_2. \end{array} \tag{3.2}$$

It is seen from (3.2) that the Frenet vectors $\mathbf{T}$ and $\mathbf{N}$ rotate around the $\mathcal{D}_1\mathcal{D}_2$-plane, and the Frenet vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ rotate around the $\mathcal{D}_3\mathcal{D}_4$-plane. These two planes play the role that the Darboux vector $\mathbf{d}$ plays in 3-space. Thus, considering these planes and inspired by the question of Chen [3] we may ask the following questions:

Question 2: When does the position vector of a space curve in $\mathbb{E}^4$ always lie in its

- **a)** $\mathcal{D}_1\mathcal{D}_2$-plane, or
- **b)** $\mathcal{D}_3\mathcal{D}_4$-plane, or
- **c)** $\mathcal{D}_2\mathcal{D}_3$-plane?

For simplicity, we call such curves as $\mathcal{D}_1\mathcal{D}_2$-curve, $\mathcal{D}_3\mathcal{D}_4$-curve, and $\mathcal{D}_2\mathcal{D}_3$-curve, respectively. The following theorems characterize such curves.

Theorem 3.1 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with nonvanishing curvatures $\kappa_1, \kappa_2, \kappa_3$, and let s denote its arc-length parameter. Then, β is a $\mathcal{D}_1\mathcal{D}_2$-curve if and only if the curvatures of β satisfy*

$$\left\{\frac{1}{\kappa_3(s)}\left(\frac{\kappa_1(s)(s+c)}{\kappa_2(s)}\right)'\right\}' + \frac{\kappa_1(s)\kappa_3(s)(s+c)}{\kappa_2(s)} = 0, \tag{3.3}$$

where c is a constant.

Proof Let β be a $\mathcal{D}_1\mathcal{D}_2$-curve with nonvanishing curvatures $\kappa_1, \kappa_2, \kappa_3$ in $\mathbb{E}^4$. Then, by the definition of $\mathcal{D}_1\mathcal{D}_2$-curve, we may write

$$\beta(s) = \lambda(s)\mathcal{D}_1(s) + \mu(s)\mathcal{D}_2(s) \tag{3.4}$$

for some functions $\lambda(s)$ and $\mu(s)$. If we take the derivative of (3.4) according to s and use the Frenet formulas, we obtain

$$\begin{aligned} &\big(\mu(s)\kappa_2(s)\big)' - 1 = 0, \\ &\big(\mu(s)\kappa_1(s)\big)' - \lambda(s)\kappa_3(s) = 0, \\ &\lambda'(s) + \mu(s)\kappa_1(s)\kappa_3(s) = 0. \end{aligned} \tag{3.5}$$

The first and second equations above yield

$$\mu(s) = \frac{s+c}{\kappa_2(s)}, \quad \lambda(s) = \frac{1}{\kappa_3(s)}\left(\frac{\kappa_1(s)(s+c)}{\kappa_2(s)}\right)',$$

where c is an integration constant. Thus, substituting these results into the third equation of (3.5) gives the desired result.

Conversely, we assume that the equation in (3.3) holds. Let us consider the vector

$$X(s) = \beta(s) - \frac{1}{\kappa_3(s)}\left(\frac{\kappa_1(s)(s+c)}{\kappa_2(s)}\right)'\mathcal{D}_1(s) - \frac{s+c}{\kappa_2(s)}\mathcal{D}_2(s).$$

Differentiating $X(s)$ with respect to s yields zero vector which implies that $X(s)$ is a constant vector. Thus, $\beta(s)$ is congruent to a $\mathcal{D}_1\mathcal{D}_2$-curve. □

If we use Theorem 3.1 given in [5], we may give the following corollary:

Corollary 3.2 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with nonvanishing curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, β is a $\mathcal{D}_1\mathcal{D}_2$-curve if and only if β is a rectifying curve in $\mathbb{E}^4$.*

Remark 3.3 *According to Definition 2.1, position vector of a rectifying curve in $\mathbb{E}^4$ lies always in the orthogonal complement of its principal normal vector field, i.e. it lies always in subspace spanned by $\{\mathbf{T}, \mathbf{B}_1, \mathbf{B}_2\}$. However, as a result of the above corollary, a rectifying curve in $\mathbb{E}^4$ can be considered as a space curve whose position vector lies always in a two-dimensional subspace which we called $\mathcal{D}_1\mathcal{D}_2$-plane.*

Remark 3.4 *Let us reconsider the characterization (3.3) of a $\mathcal{D}_1\mathcal{D}_2$-curve. If we substitute*

$$p = \frac{\kappa_1(s)(s+c)}{\kappa_2(s)}$$

and change the independent variable by using the transformation $t = \int \kappa_3(s)ds$ in (3.3), we obtain

$$\frac{d^2p}{dt^2} + p = 0$$

which has the general solution $p = c_1 \cos t + c_2 \sin t$, c_1, c_2 are constants. Thus, the solution of (3.3) is obtained as

$$\frac{\kappa_1(s)(s+c)}{\kappa_2(s)} = c_1 \cos\left(\int \kappa_3(s)ds\right) + c_2 \sin\left(\int \kappa_3(s)ds\right).$$

Theorem 3.5 *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with nonvanishing curvatures $\kappa_1, \kappa_2, \kappa_3$, and let s denote its arc-length parameter. Then, α is a $\mathcal{D}_3\mathcal{D}_4$-curve if and only if the curvatures of α satisfy*

$$c\left\{\frac{1}{\kappa_1(s)}\left(\frac{\kappa_3(s)}{\kappa_2(s)}\right)'\right\}' + \frac{c\kappa_1(s)\kappa_3(s)}{\kappa_2(s)} + 1 = 0, \tag{3.6}$$

where c is a constant.

Proof Let α be a $\mathcal{D}_3\mathcal{D}_4$-curve with nonvanishing curvatures $\kappa_1, \kappa_2, \kappa_3$ in $\mathbb{E}^4$. Then, by the definition of $\mathcal{D}_3\mathcal{D}_4$-curve, we may write

$$\alpha(s) = \nu(s)\mathcal{D}_3(s) + \eta(s)\mathcal{D}_4(s) \tag{3.7}$$

for some functions $\eta(s)$ and $\nu(s)$. If we take the derivative of (3.7) with respect to s and use the Frenet formulas, we obtain

$$\begin{aligned} &\big(\nu(s)\kappa_2(s)\big)' = 0, \\ &\big(\nu(s)\kappa_3(s)\big)' + \eta(s)\kappa_1(s) = 0, \\ &\eta'(s) - \nu(s)\kappa_1(s)\kappa_3(s) - 1 = 0. \end{aligned} \tag{3.8}$$

The first and second equations above yield

$$\nu(s) = \frac{c}{\kappa_2(s)}, \quad \eta(s) = -\frac{c}{\kappa_1(s)}\left(\frac{\kappa_3(s)}{\kappa_2(s)}\right)',$$

where c is an integration constant. Thus, substituting these results into the third equation of (3.8) gives the desired result.

Conversely, we assume that the curvatures of a unit speed curve α satisfy the equation in (3.6). Let us consider the vector field

$$Y(s) = \alpha(s) - \frac{c}{\kappa_2(s)}\mathcal{D}_3(s) + \frac{c}{\kappa_1(s)}\left(\frac{\kappa_3(s)}{\kappa_2(s)}\right)'\mathcal{D}_4(s).$$

Differentiating $Y(s)$ according to s yields zero vector which implies that $Y(s)$ is a constant vector. Thus, $\alpha(s)$ is congruent to a $\mathcal{D}_3\mathcal{D}_4$-curve. □

If we use Lemma 1 given in [8], we may give the following corollary:

Corollary 3.6 *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with nonvanishing curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, α is a $\mathcal{D}_3\mathcal{D}_4$-curve if and only if α is an osculating curve of the first kind in $\mathbb{E}^4$.*

Remark 3.7 *According to Definition 2.1, position vector of an osculating curve of the first kind in $\mathbb{E}^4$ lies always in the orthogonal complement of its first binormal vector field, i.e. it lies always in subspace spanned by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_2\}$. However, as a result of the above corollary, an osculating curve of the first kind in $\mathbb{E}^4$ can be considered as a space curve whose position vector lies always in a two-dimensional subspace which we called $\mathcal{D}_3\mathcal{D}_4$-plane.*

Remark 3.8 *Let us reconsider the characterization (3.6) of a $\mathcal{D}_3\mathcal{D}_4$-curve. If we substitute*

$$q = \frac{\kappa_3(s)}{\kappa_2(s)}$$

and change the independent variable by using the transformation $t = \int \kappa_1(s)ds := f(s)$ *in (3.6), we obtain the nonhomogeneous second-order differential equation*

$$\frac{d^2q}{dt^2} + q = g(t), \quad g(t) := \frac{-1}{c\kappa_1\left(f^{-1}(t)\right)}.$$

The general solution of the above differential equation is $q = \left(C_1 - \int g(t)\sin t dt\right)\cos t + \left(C_2 + \int g(t)\cos t dt\right)\sin t$, *where* C_1, C_2 *are constants. Thus, the solution of (3.6) is obtained as*

$$\frac{\kappa_3(s)}{\kappa_2(s)} = \left(C_1 + \frac{1}{c}\int \sin\left(\int \kappa_1(s)ds\right)ds\right)\cos\left(\int \kappa_1(s)ds\right) + \left(C_2 - \frac{1}{c}\int \cos\left(\int \kappa_1(s)ds\right)ds\right)\sin\left(\int \kappa_1(s)ds\right).$$

Theorem 3.9 *Let* $\gamma : I \subset \mathbb{R} \to \mathbb{E}^4$ *be a unit speed curve with nonvanishing curvatures* $\kappa_1, \kappa_2, \kappa_3$, *and let* s *denote its arc-length parameter. Then,* γ *is a* $\mathcal{D}_2\mathcal{D}_3$*-curve if and only if the curvatures of* γ *satisfy*

$$c_2\left(\frac{\kappa_2(s)}{\kappa_1(s)}\right)' - c_1\kappa_1(s) - 1 = 0, \quad c_1\left(\frac{\kappa_2(s)}{\kappa_3(s)}\right)' + c_2\kappa_3(s) = 0, \tag{3.9}$$

where c_1, c_2 *are constants.*

Proof Let γ be a $\mathcal{D}_2\mathcal{D}_3$-curve with nonvanishing curvatures $\kappa_1, \kappa_2, \kappa_3$ in $\mathbb{E}^4$. Then, by the definition of $\mathcal{D}_2\mathcal{D}_3$-curve, we may write

$$\gamma(s) = \rho(s)\mathcal{D}_2(s) + \delta(s)\mathcal{D}_3(s) \tag{3.10}$$

for some functions $\rho(s)$ and $\delta(s)$. If we take the derivative of (3.10) with respect to s and use the Frenet formulas, we obtain

$$\begin{aligned} &\left(\delta(s)\kappa_3(s)\right)' = 0, \\ &\left(\rho(s)\kappa_1(s)\right)' = 0, \\ &\left(\rho(s)\kappa_2(s)\right)' - \delta(s)\kappa_1(s)\kappa_3(s) - 1 = 0, \\ &\left(\delta(s)\kappa_2(s)\right)' + \rho(s)\kappa_1(s)\kappa_3(s) = 0. \end{aligned} \tag{3.11}$$

The first and second equations of (3.11) yield

$$\delta(s) = \frac{c_1}{\kappa_3(s)}, \quad \rho(s) = \frac{c_2}{\kappa_1(s)},$$

where c_1, c_2 are integration constants. Thus, substituting these results into the third and fourth equations of (3.11) give the desired result.

Conversely, we assume that the curvatures of γ satisfy the equations in (3.9). Let us consider the vector

$$Z(s) = \gamma(s) - \frac{c_2}{\kappa_1(s)}\mathcal{D}_2(s) - \frac{c_1}{\kappa_3(s)}\mathcal{D}_3(s).$$

Differentiating $Z(s)$ according to s yields zero vector which implies that $Z(s)$ is a constant vector. Thus, $\gamma(s)$ is congruent to a $\mathcal{D}_2\mathcal{D}_3$-curve. □

4. Special ruled hypersurfaces in $\mathbb{E}^4$ and their developability

In this section we introduce two ruled hypersurfaces associated to a space curve in $\mathbb{E}^4$. A ruled hypersurface in $\mathbb{E}^4$ is represented (locally) by the map

$$\varphi : I \times \mathbb{R}^2 \to \mathbb{E}^4,\ \varphi(t, v_1, v_2) = \beta(t) + v_1\mathbf{e}_1(t) + v_2\mathbf{e}_2(t),$$

where $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ denotes the base curve with unit tangent vector $\mathbf{e}_0$, and $\{\mathbf{e}_1(t), \mathbf{e}_2(t)\}$ denotes an orthonormal basis of generating plane along β. Let

$$\operatorname{rank}[\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1', \mathbf{e}_2'] = 4 - k. \tag{4.1}$$

If $k = 0$ in (4.1), then the ruled hypersurface is called nondevelopable. If $k = 1$ in (4.1), then the ruled hypersurface is called developable [1].

Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$ and arc-length function s. Let us consider the vector fields $\mathcal{D}_i, 1 \leq i \leq 4$ defined in (3.1). Since $\{\mathcal{D}_1, \mathcal{D}_2\}$ and $\{\mathcal{D}_2, \mathcal{D}_3\}$ are orthogonal, by normalizing these vectors we obtain the orthonormal frames $\{\mathcal{D}_1, \bar{\mathcal{D}}_2\}$ and $\{\bar{\mathcal{D}}_2, \bar{\mathcal{D}}_3\}$, where

$$\bar{\mathcal{D}}_2(s) = \frac{\mathcal{D}_2(s)}{||\mathcal{D}_2(s)||} = \frac{1}{\sqrt{\kappa_1^2(s) + \kappa_2^2(s)}} \left\{\kappa_2(s)\mathbf{T}(s) + \kappa_1(s)\mathbf{B}_1(s)\right\}, \tag{4.2}$$

$$\bar{\mathcal{D}}_3(s) = \frac{\mathcal{D}_3(s)}{||\mathcal{D}_3(s)||} = \frac{1}{\sqrt{\kappa_2^2(s) + \kappa_3^2(s)}} \left\{\kappa_3(s)\mathbf{N}(s) + \kappa_2(s)\mathbf{B}_2(s)\right\}. \tag{4.3}$$

If we differentiate these normalized vector fields with respect to s, we obtain

$$\bar{\mathcal{D}}_2'(s) = \sigma_1(s)\left\{\kappa_1(s)\mathbf{T}(s) - \kappa_2(s)\mathbf{B}_1(s)\right\} + \sigma_2(s)\mathbf{B}_2(s), \tag{4.4}$$

$$\bar{\mathcal{D}}_3'(s) = -\sigma_3(s)\mathbf{T}(s) + \sigma_4(s)\left[\kappa_2(s)\mathbf{N}(s) - \kappa_3(s)\mathbf{B}_2(s)\right], \tag{4.5}$$

where

$$\sigma_1(s) = \left[\left(\frac{\kappa_2}{\kappa_1}\right)' \frac{\kappa_1^2}{(\kappa_1^2 + \kappa_2^2)^{3/2}}\right](s), \quad \sigma_2(s) = \frac{\kappa_1\kappa_3}{\sqrt{\kappa_1^2 + \kappa_2^2}}(s),$$

$$\sigma_4(s) = \left[\left(\frac{\kappa_3}{\kappa_2}\right)' \frac{\kappa_2^2}{(\kappa_2^2 + \kappa_3^2)^{3/2}}\right](s), \quad \sigma_3(s) = \frac{\kappa_1\kappa_3}{\sqrt{\kappa_2^2 + \kappa_3^2}}(s).$$

Thus, by considering the orthonormal frames $\{\mathcal{D}_1(s), \bar{\mathcal{D}}_2(s)\}$ and $\{\bar{\mathcal{D}}_2(s), \bar{\mathcal{D}}_3(s)\}$, we define the ruled hypersurfaces

$$\varphi(s, u, v) = \beta(s) + u\mathcal{D}_1(s) + v\bar{\mathcal{D}}_2(s), \quad s \in I, \quad u, v \in \mathbb{R}, \tag{4.6}$$

$$\xi(s, u, v) = \beta(s) + u\bar{\mathcal{D}}_2(s) + v\bar{\mathcal{D}}_3(s), \quad s \in I, \quad u, v \in \mathbb{R}, \tag{4.7}$$

and call them as $\mathcal{D}_1\bar{\mathcal{D}}_2$-ruled hypersurface and $\bar{\mathcal{D}}_2\bar{\mathcal{D}}_3$-ruled hypersurface of $\beta(s)$, respectively.

Proposition 4.1 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with arc-length parameter s. Thus,*
i) *(s_0, u_0, v_0) is a singular point of $\mathcal{D}_1\bar{\mathcal{D}}_2$-ruled hypersurface of $\beta(s)$ if and only if it satisfies*

$$\kappa_1(s_0) + u_0\left(\kappa_2\kappa_3\right)(s_0) + v_0\left\{\sigma_1\left(\kappa_1^2+\kappa_2^2\right)\right\}(s_0) = 0.$$

ii) *β is a geodesic curve on $\mathcal{D}_1\bar{\mathcal{D}}_2$-ruled hypersurface of $\beta(s)$.*

Proof **i)** The partial derivatives of $\varphi(s,u,v)$ are obtained as

$$\varphi_s = (1 + v\sigma_1(s)\kappa_1(s))\mathbf{T}(s) - (u\kappa_3(s) + v\sigma_1(s)\kappa_2(s))\mathbf{B}_1(s) + v\sigma_2(s)\mathbf{B}_2(s),$$

$$\varphi_u = \mathbf{B}_2(s), \quad \varphi_v = \left(\frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}}\right)(s)\mathbf{T}(s) + \left(\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}\right)(s)\mathbf{B}_1(s).$$

Then, we have

$$\varphi_s \otimes \varphi_u \otimes \varphi_v = \frac{\kappa_1(s) + u\left(\kappa_2\kappa_3\right)(s) + v\left\{\sigma_1\left(\kappa_1^2+\kappa_2^2\right)\right\}(s)}{\left(\sqrt{\kappa_1^2+\kappa_2^2}\right)(s)}\mathbf{N}(s). \tag{4.8}$$

We know that (s_0,u_0,v_0) is a singular point of $\varphi(s,u,v)$ if and only if $(\varphi_s \otimes \varphi_u \otimes \varphi_v)(s_0,u_0,v_0) = \mathbf{0}$. Thus, the assertion is clear from (4.8).

ii) We have $u = v = 0$ for the points of β. Thus, $\beta(s)$ is a regular point of $\varphi(s,u,v)$ for all $s \in I$. Therefore, by using (4.8) the unit normal vector field of the hypersurface $\varphi(s,u,v)$ along β is given by $\mathcal{N}(s) = \mathbf{N}(s)$. Since principal normal of the curve is parallel to the hypersurface normal vector, β is a geodesic curve on $\mathcal{D}_1\bar{\mathcal{D}}_2$-ruled hypersurface of $\beta(s)$. □

Proposition 4.2 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with arc-length parameter s. The $\mathcal{D}_1\bar{\mathcal{D}}_2$-ruled hypersurface associated with β is developable.*

Proof We have

$$\operatorname{rank}[\mathbf{T}, \mathcal{D}_1, \bar{\mathcal{D}}_2, \mathcal{D}_1', \bar{\mathcal{D}}_2'] = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}} & 0 & \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}} & 0 \\ 0 & 0 & -\kappa_3 & 0 \\ \sigma_1\kappa_1 & 0 & -\sigma_1\kappa_2 & \sigma_2 \end{bmatrix} = 3.$$

Then, according to (4.1), $\mathcal{D}_1\bar{\mathcal{D}}_2$-ruled hypersurface associated with β is developable. □

Remark 4.3 *Note that, similar to the rectifying developable of a curve in $\mathbb{E}^3$, the $\mathcal{D}_1\bar{\mathcal{D}}_2$-ruled hypersurface of a space curve in $\mathbb{E}^4$ is developable, and its base curve is always a geodesic on it.*

Proposition 4.4 *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with arc-length parameter s. Thus, (s_0,u_0,v_0) is a singular point of $\bar{\mathcal{D}}_2\bar{\mathcal{D}}_3$-ruled hypersurface of $\beta(s)$ if and only if it satisfies the followings:*

$$\begin{cases} \kappa_1(s_0) + u_0\left\{\sigma_1\left(\kappa_1^2+\kappa_2^2\right)\right\}(s_0) - v_0\left(\kappa_1\sigma_3\right)(s_0) &= 0, \\ v_0\left\{\sigma_4\left(\kappa_2^2+\kappa_3^2\right)\right\}(s_0) - u_0\left(\kappa_3\sigma_2\right)(s_0) &= 0. \end{cases}$$

Proof The partial derivatives of $\xi(s,u,v)$ are obtained as

$$\begin{aligned}\xi_s &= (1+u\sigma_1(s)\kappa_1(s)-v\sigma_3(s))\mathbf{T}(s)+v\sigma_4(s)\kappa_2(s)\mathbf{N}(s)\\ &\quad -u\sigma_1(s)\kappa_2(s)\mathbf{B}_1(s)+(u\sigma_2(s)-v\sigma_4(s)\kappa_3(s))\mathbf{B}_2(s),\end{aligned}$$

$$\xi_u = \left(\frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}}\right)(s)\mathbf{T}(s)+\left(\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}\right)(s)\mathbf{B}_1(s),$$

$$\xi_v = \left(\frac{\kappa_3}{\sqrt{\kappa_2^2+\kappa_3^2}}\right)(s)\mathbf{N}(s)+\left(\frac{\kappa_2}{\sqrt{\kappa_2^2+\kappa_3^2}}\right)(s)\mathbf{B}_2(s).$$

Then, we have

$$\begin{aligned}\xi_s\otimes\xi_u\otimes\xi_v = &\ \frac{v\left\{\sigma_4\left(\kappa_2^2+\kappa_3^2\right)\right\}(s)-u(\kappa_3\sigma_2)(s)}{\sqrt{\kappa_1^2+\kappa_2^2}\sqrt{\kappa_2^2+\kappa_3^2}}\Big(\kappa_1(s)\mathbf{T}(s)-\kappa_2(s)\mathbf{B}_1(s)\Big)\\ &-\frac{\kappa_1(s)+u\left\{\sigma_1\left(\kappa_1^2+\kappa_2^2\right)\right\}(s)-v(\kappa_1\sigma_3)(s)}{\sqrt{\kappa_1^2+\kappa_2^2}\sqrt{\kappa_2^2+\kappa_3^2}}\Big(\kappa_2(s)\mathbf{N}(s)-\kappa_3(s)\mathbf{B}_2(s)\Big).\end{aligned}\tag{4.9}$$

Thus, the assertion is clear from (4.9). □

Corollary 4.5 *$\beta(s)$, $\forall s\in I$, is a regular point of the $\bar{\mathcal{D}}_2\bar{\mathcal{D}}_3$-ruled hypersurface.*

Proposition 4.6 *Let $\beta : I\subset\mathbb{R}\to\mathbb{E}^4$ be a unit speed curve with arc-length parameter s. The $\bar{\mathcal{D}}_2\bar{\mathcal{D}}_3$-ruled hypersurface associated with β is nondevelopable.*

Proof We have

$$\operatorname{rank}[\mathbf{T},\bar{\mathcal{D}}_2,\bar{\mathcal{D}}_3,\bar{\mathcal{D}}_2',\bar{\mathcal{D}}_3'] = \operatorname{rank}\begin{bmatrix}1 & 0 & 0 & 0\\ \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}} & 0 & \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}} & 0\\ 0 & \frac{\kappa_3}{\sqrt{\kappa_2^2+\kappa_3^2}} & 0 & \frac{\kappa_2}{\sqrt{\kappa_2^2+\kappa_3^2}}\\ \sigma_1\kappa_1 & 0 & -\sigma_1\kappa_2 & \sigma_2\\ -\sigma_3 & \sigma_4\kappa_2 & 0 & -\sigma_4\kappa_3\end{bmatrix} = 4.$$

Then, according to (4.1), $\bar{\mathcal{D}}_2\bar{\mathcal{D}}_3$-ruled hypersurface associated with β is nondevelopable. □

Example 4.7 *Let us consider the unit speed curve given by [4]*

$$\alpha(s) = (\cos(\ell_1 s),\sin(\ell_1 s),\cos(\ell_2 s),\sin(\ell_2 s)),$$

where $\ell_1=\sqrt{\frac{2}{3}}$, $\ell_2=\sqrt{\frac{1}{3}}$. The curvatures of this curve are given by $\kappa_1(s)=\frac{\sqrt{5}}{3}$, $\kappa_2(s)=\frac{1}{3}\sqrt{\frac{2}{5}}$, $\kappa_3(s)=\sqrt{\frac{2}{5}}$. Since these curvatures satisfy (3.6) with $c=-\frac{1}{\sqrt{5}}$, α is a $\mathcal{D}_3\mathcal{D}_4$-curve in $\mathbb{E}^4$. Thus, since $\mathcal{D}_3(s)=\sqrt{\frac{2}{5}}\left(\mathbf{N}(s)+\frac{1}{3}\mathbf{B}_2(s)\right)$, by using the Frenet vectors given in [4] it is easy to verify that we can write $\alpha(s)=-\frac{3}{\sqrt{2}}\mathcal{D}_3(s)+0.\mathcal{D}_4(s)$.

On the other hand, if we consider (4.6), we obtain the developable $\mathcal{D}_1\bar{\mathcal{D}}_2$-ruled hypersurface as

$$\begin{aligned}\varphi(s,u,v) &= \alpha(s) + u\mathbf{B}_2(s) + v\left(\tfrac{\sqrt{6}}{9}\mathbf{T}(s) + \tfrac{5\sqrt{3}}{9}\mathbf{B}_1(s)\right)\\ &= \Big(\left(1+\tfrac{u}{\sqrt{5}}\right)\cos(\ell_1 s) + \tfrac{1}{3}v\sin(\ell_1 s), \left(1+\tfrac{u}{\sqrt{5}}\right)\sin(\ell_1 s) - \tfrac{1}{3}v\cos(\ell_1 s),\\ &\quad \left(1-\tfrac{2u}{\sqrt{5}}\right)\cos(\ell_2 s) - \tfrac{2\sqrt{2}}{3}v\sin(\ell_2 s), \left(1-\tfrac{2u}{\sqrt{5}}\right)\sin(\ell_2 s) + \tfrac{2\sqrt{2}}{3}v\cos(\ell_2 s)\Big)\end{aligned}$$

whose base curve α is a geodesic. Moreover, the point (s_0, u_0, v_0) is a singular point on $\mathcal{D}_1\bar{\mathcal{D}}_2$-ruled hypersurface if and only if $u_0 = -\frac{5\sqrt{5}}{2}$, $s_0, v_0 \in \mathbb{R}$.

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