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Asymptotic properties of solutions to second-order difference equations

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Abstract: In this paper the second-order difference equations of the form

$$\Delta^2 x_n = a_n f(n, x_{\sigma(n)}) + b_n$$

are considered. We establish sufficient conditions for the existence of solutions with prescribed asymptotic behavior. In particular, we present conditions under which there exists an asymptotically linear solution. Moreover, we study the asymptotic behavior of solutions.

Key words: Difference equation, second-order, asymptotic behavior, approximative solution, approximation of solution

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively. In this paper we consider the second-order difference equations of the form

$$\Delta^2 x_n = a_n f(n, x_{\sigma(n)}) + b_n, \tag{1.1}$$

$$n \in \mathbb{N}, \quad a_n, b_n \in \mathbb{R}, \quad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \quad \sigma : \mathbb{N} \to \mathbb{N}, \quad \lim \sigma(n) = \infty.$$

By a solution of equation (1.1), we mean a sequence x which satisfies equality (1.1) for large n.

The study of asymptotic properties of solutions of differential and difference equations is of immense importance and hence attracts many researchers. For second-order equations, asymptotically linear solutions are most often studied. Asymptotically linear solutions of differential equations are considered, for example, in papers [2, 5, 10–12, 14]. Asymptotically linear solutions of difference equations are studied, for example, in papers [1, 3, 4, 13, 15, 16].

In this paper we mainly deal with two problems. Firstly, we establish conditions under which for a given solution y of the equation $\Delta^2 y_n = b_n$ and a given number $s \in (-\infty, 0]$ there exists a solution x of (1.1) such that

$$x_n = y_n + \mathrm{o}(n^s). \tag{1.2}$$

Secondly, we establish conditions under which for a given solution x of (1.1) and a given number $s \in (-\infty, 0]$ there exists a solution y of the equation $\Delta^2 y_n = b_n$ such that the condition (1.2) is satisfied. The following two known results relate to the first problem.

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Theorem 1.1 *Assume* $y:\mathbb{N}\to\mathbb{R}$, $s\in(-\infty,0]$, $\lambda\in\mathbb{R}$,

$$\Delta^2 y_n = b_n, \quad \lim_{n\to\infty} y_n = \infty, \quad \sum_{n=1}^{\infty} n^{1-s}|a_n| < \infty,$$

and f *is continuous and bounded on* $\mathbb{N}\times[\lambda,\infty)$. *Then there exists a solution* x *of* (1.1) *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof [9, Theorem 4.1] □

Theorem 1.2 *Assume* $y:\mathbb{N}\to\mathbb{R}$, $s,\alpha\in(-\infty,0]$, $g:[0,\infty)\to[0,\infty)$,

$$\Delta^2 y_n = b_n, \quad y(\sigma(n)) = \mathrm{O}(n^{-\alpha}), \quad |f(n,t)| \le g(|t|n^{\alpha}), \quad \sum_{n=1}^{\infty} n^{1-s}|a_n| < \infty,$$

f *is continuous, and* g *is locally bounded. Then there exists a solution* x *of* (1.1) *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof The assertion is a consequence of [7, Theorem 5.1]. □

Our first main result, Theorem 3.1, also applies to this problem. It covers a number of cases that are not covered by Theorems 1.1 and 1.2 (see Remark 3.2).

The paper is organized as follows. In Section 2, we introduce some notation and terminology. Moreover, in Lemma 2.1 we present the basic tool that will be used in the main part of the paper. In Section 3, we present our main results concerning the existence of solutions with prescribed asymptotic behavior. In Section 4, we establish some results concerning approximations of solutions. Section 5 is devoted to the "autonomous" case of equation (1.1). In Section 6 we present some additional results.

2. Preliminaries

The space of all sequences $x:\mathbb{N}\to\mathbb{R}$ we denote by $\mathbb{R}^{\mathbb{N}}$. If $x,y\in\mathbb{R}^{\mathbb{N}}$, then xy and $|x|$ denote the sequences defined by $xy(n)=x_n y_n$ and $|x|(n)=|x_n|$, respectively. Moreover,

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|.$$

A function $g:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ is said to be uniformly equicontinuous on a subset Z of $\mathbb{R}$ if for every $\varepsilon>0$ there exists $\delta>0$ such that $|g(n,z_1)-g(n,z_2)|<\varepsilon$ for any $n\in\mathbb{N}$ and any $z_1,z_2\in Z$ such that $|z_1-z_2|<\delta$.

Lemma 2.1 *Assume* $\lambda\in[1,\infty)$, *a function* $f:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ *is uniformly equicontinuous on* $[\lambda,\infty)$, *and the sequence* $(f(n,\lambda))$ *is bounded. Then there exists a positive constant* K *such that* $|f(n,t)|\le Kt$ *for any* $(n,t)\in\mathbb{N}\times[\lambda,\infty)$.

Proof Let

$$Q = \sup_{n\in\mathbb{N}}\{|f(n,\lambda)|\}.$$

By assumption there exists $\delta>0$ such that $|f(n,s)-f(n,t)|<1$ for every $n\in\mathbb{N}$ and every pair $s,t\in[\lambda,\infty)$ such that $|s-t|<\delta$. Let

$$K = \delta^{-1} + Q + 1.$$

Let $t \in [\lambda, \infty)$, $n \in \mathbb{N}$. If $t - \lambda < \delta$ then $|f(n,t) - f(n,\lambda)| < 1$ and

$$|f(n,t)| \leq |f(n,\lambda)| + |f(n,t) - f(n,\lambda)| \leq Q + 1 < K.$$

Since $t \geq \lambda \geq 1$ we obtain

$$|f(n,t)| \leq Kt \tag{2.1}$$

for $t \in [\lambda, \lambda + \delta)$. If $t - \lambda \geq \delta$, choose $p \in \mathbb{N}$ such that

$$(p-1)\delta \leq t - \lambda < p\delta.$$

For $k = 0, 1, \dots, p$ let $t_k = \lambda + kp^{-1}(t - \lambda)$. Then $t_0 = \lambda$, $t_p = t$ and

$$t_k - t_{k-1} = (t - \lambda)p^{-1} < \delta$$

for $k = 1, \dots, p$. Hence,

$$|f(n,t)| = |f(n,t_0) + f(n,t_1) - f(n,t_0) + \cdots + f(n,t_p) - f(n,t_{p-1})|$$

$$\leq |f(n,t_0)| + p = |f(n,\lambda)| + p \leq Q + p.$$

Since $(p-1)\delta \leq t - \lambda$, we have $p \leq (t-\lambda)\delta^{-1} + 1$. Hence,

$$|f(n,t)| \leq Q + p \leq Q + (t-\lambda)\delta^{-1} + 1 \leq Q + t\delta^{-1} + 1 \leq Kt. \tag{2.2}$$

By (2.1) and (2.2)

$$|f(n,t)| \leq Kt$$

for $(n,t) \in \mathbb{N} \times [\lambda, \infty)$. □

Lemma 2.2 *Assume $y, \rho : \mathbb{N} \to \mathbb{R}$, $\lim_{n\to\infty} \rho_n = 0$, and $X = \{x \in \mathbb{R}^{\mathbb{N}} : \ |x - y| \leq |\rho|\}$. Define a metric d on X by the formula*

$$d(x,z) = \|x - z\|. \tag{2.3}$$

Then any continuous map $H : X \to X$ has a fixed point.

Proof This lemma is a consequence of the proof of [6, Theorem 1]. □

Lemma 2.3 *Assume $s \in (-\infty, 0]$, $u : \mathbb{N} \to \mathbb{R}$, and*

$$\sum_{k=1}^{\infty} n^{1-s}|u_n| < \infty, \quad \textit{then} \quad \sum_{k=n}^{\infty}\sum_{j=k}^{\infty} |u_j| = \mathrm{o}(n^s).$$

Proof This lemma is a consequence of [8, Lemma 3.1 (16)]. □

3. Approximative solutions

In this section we deal with the problem of the existence of solutions with prescribed asymptotic behavior. In Theorem 3.1 we present our first main result. Moreover, in Theorem 3.3 we establish conditions under which there exists an asymptotically linear solution of (1.1).

Theorem 3.1 *Assume* $y : \mathbb{N} \to \mathbb{R}$, $s \in (-\infty, 0]$, $\tau \in (0, \infty)$, $\lambda \in [1, \infty)$,

$$\Delta^2 y_n = b_n, \quad \lim_{n\to\infty} y_n = \infty, \quad y_n = \mathrm{O}(n^\tau), \quad \sigma(n) = \mathrm{O}(n),$$

$$\sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty, \tag{3.1}$$

the sequence $(f(n,\lambda))$ *is bounded, and* f *is uniformly equicontinuous on* $[\lambda, \infty)$. *Then there exists a solution* x *of* (1.1) *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof There exist positive constants L, Q such that

$$|y_n| \le Qn^\tau, \qquad \sigma(n) \le Ln$$

for $n \in \mathbb{N}$. By Lemma 2.1, there exists a positive constant K such that

$$|f(n,t)| \le Kt$$

for $(n,t) \in \mathbb{N} \times [\lambda, \infty)$. Define a constant M and sequences α, ρ by

$$M = (2QL^\tau)K, \quad \alpha_n = \sum_{k=n}^{\infty} k^\tau |a_k|, \quad \rho_n = \sum_{k=n}^{\infty} \alpha_k.$$

By Lemma 2.3, $\rho_n = \mathrm{o}(n^s)$. There exists an index $p_1 \in \mathbb{N}$ such that

$$M\rho_n \le 1, \quad \text{and} \quad y_n \ge \lambda + 1$$

for $n \ge p_1$. Let

$$S = \{x \in \mathbb{R}^{\mathbb{N}} : \ x_n = y_n \text{ for } n < p_1, \ |x_n - y_n| \le M\rho_n \text{ for } n \ge p_1\}.$$

There exists $p_2 \ge p_1$ such that $\sigma(n) \ge p_1$ for $n \ge p_2$. If $x \in S$, $n \ge p_1$, then

$$|x_n - y_n| \le M\rho_n \le 1, \quad y_n \ge \lambda + 1.$$

Hence, $x_n \ge y_n - 1 \ge \lambda + 1 - 1 = \lambda$. Therefore,

$$x_{\sigma(n)} \ge \lambda \tag{3.2}$$

for any $x \in S$ and any $n \ge p_2$. Let $x \in S$. If $n \ge p_2$, then

$$|f(n, x_{\sigma(n)})| \le Kx_{\sigma(n)} = K(x_{\sigma(n)} - y_{\sigma(n)} + y_{\sigma(n)}) \le K(|x_{\sigma(n)} - y_{\sigma(n)}| + |y_{\sigma(n)}|)$$

$$\le K(M\rho_{\sigma(n)} + |y_{\sigma(n)}|) \le K(1 + |y_{\sigma(n)}|) \le K(2|y_{\sigma(n)}|)$$

$$\leq 2KQ(\sigma(n)^{\tau}) \leq 2KQ(Ln)^{\tau} = 2KQL^{\tau}n^{\tau} = Mn^{\tau}.$$

Hence,

$$|f(n, x_{\sigma(n)})| \leq Mn^{\tau} \tag{3.3}$$

for any $x \in S$ and any $n \geq p_2$. Let $x \in S$. For $n \in \mathbb{N}$ let

$$x_n^* = \sum_{j=n}^{\infty}(a_j f(j, x_{\sigma(j)})).$$

Then

$$|x_n^*| \leq \sum_{k=n}^{\infty} |a_k||f(k, x_{\sigma(k)})| \leq M \sum_{k=n}^{\infty} k^{\tau}|a_k| = M\alpha_n$$

for $n \geq p_2$. Define a sequence $H(x)$ by

$$H(x)(n) = \begin{cases} y_n & \text{for} \quad n < p_2 \\ y_n + \sum_{j=n}^{\infty} x_j^* & \text{for} \quad n \geq p_2. \end{cases}$$

If $n \geq p_2$, then

$$|H(x)(n) - y_n| \leq \sum_{j=n}^{\infty} |x_j^*| \leq M \sum_{j=n}^{\infty} \alpha_j = M\rho_n.$$

Hence, $H(S) \subset S$.

Let $\varepsilon > 0$. By assumption, there exists $\delta > 0$ such that if $t, s \in [\lambda, \infty)$ and $|t - s| < \delta$ then $|f(n, t) - f(n, s)| < \varepsilon$ for any $n \in \mathbb{N}$. Choose $z \in S$ such that $||x - z|| < \delta$. Then $|x_k - z_k| < \delta$ for any $k \in \mathbb{N}$. Hence, $|f(n, x_k) - f(n, z_k)| < \varepsilon$ for all $n, k \in \mathbb{N}$. Moreover,

$$\|H(x) - H(z)\| = \sup_{n \geq p_2} \left| \sum_{k=n}^{\infty} x_k^* - \sum_{k=n}^{\infty} z_k^* \right| \leq \sum_{k=p_2}^{\infty} |x_k^* - z_k^*|.$$

Since

$$|x_k^* - z_k^*| = \left| \sum_{j=k}^{\infty} a_j f(j, x_{\sigma(j)}) - \sum_{j=k}^{\infty} a_j f(j, z_{\sigma(j)}) \right|$$

$$\leq \sum_{j=k}^{\infty} |a_j||f(j, x_{\sigma(j)}) - f(j, z_{\sigma(j)})| \leq \sum_{j=k}^{\infty} \varepsilon|a_j| \leq \varepsilon \sum_{j=k}^{\infty} j^{\tau}|a_j| = \varepsilon\alpha_k,$$

we obtain

$$\|H(x) - H(z)\| \leq \varepsilon\rho_{p_2}.$$

Hence, the mapping $H : S \to S$ is continuous.

By Lemma 2.2, there exists $x \in S$ such that $H(x) = x$. Then

$$x_n = y_n + \sum_{j=n}^{\infty} x_j^*,$$

$$\Delta^2 x_n = \Delta^2 y_n + \Delta^2 \left(\sum_{j=n}^{\infty} x_j^* \right) = b_n - \Delta x_n^* = a_n f(n, x_{\sigma(n)}) + b_n$$

for $n \geq p_2$. Moreover, since $x \in S$ and $\rho_n = \mathrm{o}(n^s)$, we get

$$x_n = y_n + \mathrm{o}(n^s).$$

□

Remark 3.2 *Let y be a solution of the equation $\Delta^2 y_n = b_n$ such that $\lim_{n\to\infty} y_n = \infty$, and let u be a sequence defined by*

$$u_n = f(n, y(\sigma(n))). \tag{3.4}$$

The main difference between Theorems 1.1 and 1.2 and Theorem 3.1 lies in the behavior of the sequence u.

It is clear that if the assumptions of Theorem 1.1 are satisfied, then the sequence (3.4) is bounded. If the assumptions of Theorem 1.2 are satisfied, then there exists a positive constant L such that $|y(\sigma(n))|n^\alpha \leq L$ for any n. Moreover,

$$|f(n, y(\sigma(n)))| \leq g\left(|y(\sigma(n))|n^\alpha\right).$$

Since g is locally bounded, there exists a positive constant M such that $g([0, L]) \subset [0, M]$. Hence,

$$|u_n| = |f(n, y(\sigma(n)))| \leq M$$

for any n. On the other hand, in Theorem 3.1 the sequence (3.4) may be unbounded. For example if α_n is a bounded sequence, $f(n,t) = t + \alpha_n$ for any $(n,t) \in \mathbb{N} \times \mathbb{R}$, $\tau = 1$, $s = 0$, $b_n = 0$, $\sigma(n) = n$, $y_n = n$, and $a_n = n^{-4}$, then the assumptions of Theorem 3.1 are satisfied and the sequence

$$u_n = f(n, y(\sigma(n))) = f(n, n) = n + \alpha_n$$

is unbounded.

In the next theorem we establish conditions under which there exists an asymptotically linear solution of (1.1).

Theorem 3.3 *Assume $s \in (-\infty, 0]$, $\lambda \in \mathbb{R}$, the function f is uniformly equicontinuous on $[\lambda, \infty)$, the sequence $(f(n,\lambda))$ is bounded, $\sigma(n) = \mathrm{O}(n)$, $c \in (0, \infty)$, $d \in \mathbb{R}$,*

$$\sum_{n=1}^{\infty} n^{2-s}|a_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} n^{1-s}|b_n| < \infty. \tag{3.5}$$

Then there exists a solution x of (1.1) such that $x_n = cn + d + \mathrm{o}(n^s)$.

Proof Let $\tau = 1$. Define a sequence w by

$$w_n = \sum_{k=n}^{\infty} \sum_{j=k}^{\infty} b_j.$$

By Lemma 2.3, $w_n = \mathrm{o}(n^s)$. Let y be a sequence defined by

$$y_n = cn + d + w_n.$$

Then

$$\Delta^2 y_n = \Delta^2(cn + d) + \Delta^2 w_n = 0 + b_n = b_n,$$

$y_n \to \infty$, $y_n = \mathrm{O}(n) = \mathrm{O}(n^\tau)$, and

$$\sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| = \sum_{n=1}^{\infty} n^{2-s}|a_n| < \infty.$$

By Theorem 3.1, there exists a solution x of (1.1) such that $x_n = y_n + \mathrm{o}(n^s)$. Then

$$x_n = cn + d + w_n + \mathrm{o}(n^s) = cn + d + \mathrm{o}(n^s).$$

□

4. Approximation of solutions

In this section we present sufficient conditions for a given solution x of equation (1.1) to have an asymptotic property

$$x_n = y_n + \mathrm{o}(n^s),$$

where y is a solution of the equation $\Delta^2 y_n = b_n$ and $s \in (-\infty, 0]$.

Lemma 4.1 *Assume* $b, x, u : \mathbb{N} \to \mathbb{R}$, $s \in (-\infty, 0]$, $\Delta^2 x_n = \mathrm{O}(u_n) + b_n$, *and*

$$\sum_{n=1}^{\infty} n^{1-s}|u_n| < \infty.$$

Then there exists a sequence y *such that* $\Delta^2 y_n = b_n$ *and* $x_n = y_n + \mathrm{o}(n^s)$.

Proof This lemma is a consequence of [7, Lemma 3.11 (a)]. □

Theorem 4.2 *Assume* $s \in (-\infty, 0]$, $\tau \in (0, \infty)$, $\lambda \in [1, \infty)$, *the function* f *is uniformly equicontinuous on* $[\lambda, \infty)$, *the sequence* $(f(n, \lambda))$ *is bounded,* $\sigma(n) = \mathrm{O}(n)$, x *is a solution of* (1.1), $\lim_{n\to\infty} x_n = \infty$, $x_n = \mathrm{O}(n^\tau)$, *and*

$$\sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty.$$

Then there exists a solution y *of the equation* $\Delta^2 y_n = b_n$ *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof By Lemma 2.1 there exists a positive constant K such that

$$|f(n, t)| \le Kt$$

for $(n, t) \in \mathbb{N} \times [\lambda, \infty)$. Choose positive constants P, Q such that $|x_n| \le Pn^\tau$ and $\sigma(n) \le Qn$ for any $n \in \mathbb{N}$. There exists an index p such that $x_n \ge \lambda$ for $n \ge p$. For $n \ge p$ we have

$$|f(n, x_{\sigma(n)})| \le Kx_{\sigma(n)} \le KP(\sigma(n))^\tau \le KPQ^\tau n^\tau.$$

Define a sequence u by $u_n = n^\tau a_n$. Then

$$\Delta^2 x_n = a_n f(n, x_{\sigma(n)}) + b_n = a_n \mathrm{O}(n^\tau) + b_n = \mathrm{O}(u_n) + b_n.$$

Now, the assertion follows from Lemma 4.1. □

Theorem 4.3 *Assume* $s \in (-\infty, 0]$, $\tau \in (0, \infty)$, $\lambda \in [1, \infty)$, *the function* f *is uniformly equicontinuous on* $[\lambda, \infty)$, *the sequence* $(f(n, \lambda))$ *is bounded,* $\sigma(n) = \mathrm{O}(n)$, x *is a solution of* (1.1), $\lim_{n\to\infty} x_n = \infty$, $x_n = \mathrm{O}(n^\tau)$,

$$\sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} n^{1-s}|b_n| < \infty.$$

Then there exist real constants c, d *such that* $x_n = cn + d + \mathrm{o}(n^s)$.

Proof There exist positive constants K, P, Q and an index p such that $|f(n,t)| \le Kt$ for $(n,t) \in \mathbb{N} \times [\lambda, \infty)$, $|x_n| \le Pn^\tau$, $\sigma(n) \le Qn$ for any n, and $x_n \ge \lambda$ for $n \ge p$. Define a sequence u by $u_n = a_n f(n, x_{\sigma(n)}) + b_n$. For large n we have

$$|u_n| \le KP|a_n|\sigma(n)^\tau + |b_n| \le KPQ|a_n|n^\tau + |b_n|.$$

Hence,

$$\sum_{n=1}^{\infty} n^{1-s}|u_n| \le KPQ \sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| + \sum_{n=1}^{\infty} n^{1-s}|b_n| < \infty.$$

Moreover, $\Delta^2 x_n = \mathrm{O}(u_n) + 0$. Therefore, by Lemma 4.1, there is a sequence y such that $\Delta^2 y_n = 0$ and $x_n = y_n + \mathrm{o}(n^s)$. □

5. Autonomous case

In this section we apply our results to the "autonomous case". We assume that $g : \mathbb{R} \to \mathbb{R}$ and consider the equation of the form

$$\Delta^2 x_n = a_n g(x_{\sigma(n)}) + b_n, \tag{E1}$$

which is a special case of (1.1). It is known that if g is uniformly continuous on some half-line $[\lambda, \infty)$, then there exists a positive constant K such that $|g(t)| \le Kt$ for all large t.

Corollary 5.1 *Assume* $y : \mathbb{N} \to \mathbb{R}$, $s \in (-\infty, 0]$, $\tau \in (0, \infty)$, $\lambda \in [1, \infty)$,

$$\Delta^2 y_n = b_n, \quad \lim_{n\to\infty} y_n = \infty, \quad y_n = \mathrm{O}(n^\tau), \quad \sigma(n) = \mathrm{O}(n), \quad \sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty,$$

and g *is uniformly continuous on* $[\lambda, \infty)$. *Then there exists a solution* x *of* (E1) *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof Define a function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by $f(n,t) = g(t)$, and apply Theorem 3.1. □

Corollary 5.2 *Assume* $s \in (-\infty, 0]$, $\lambda \in \mathbb{R}$, $c \in (0, \infty)$, $d \in \mathbb{R}$, $\sigma(n) = \mathrm{O}(n)$,

$$\sum_{n=1}^{\infty} n^{2-s}|a_n| < \infty, \qquad \sum_{n=1}^{\infty} n^{1-s}|b_n| < \infty, \tag{5.1}$$

and g *is uniformly continuous on* $[\lambda, \infty)$. *Then there exists a solution* x *of* (E1) *such that* $x_n = cn + d + \mathrm{o}(n^s)$.

Proof The assertion is a consequence of Theorem 3.3. □

Corollary 5.3 *Assume* $s \in (-\infty, 0]$, $\tau \in (0, \infty)$, $\lambda \in [1, \infty)$, x *is a solution of* (E1),

$$\sigma(n) = \mathrm{O}(n), \quad \lim_{n\to\infty} x_n = \infty, \quad x_n = \mathrm{O}(n^\tau), \quad \sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty.$$

and g *is uniformly continuous on* $[\lambda, \infty)$. *Then there exists a solution* y *of the equation* $\Delta^2 y_n = b_n$ *such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof The assertion follows from Theorem 4.2. □

Corollary 5.4 *Assume* $s \in (-\infty, 0]$, $\tau \in (0, \infty)$, $\lambda \in [1, \infty)$, g *is uniformly continuous on* $[\lambda, \infty)$, $\sigma(n) = \mathrm{O}(n)$, x *is a solution of* (E1), $\lim_{n\to\infty} x_n = \infty$, $x_n = \mathrm{O}(n^\tau)$,

$$\sum_{n=1}^{\infty} n^{1+\tau-s}|a_n| < \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} n^{1-s}|b_n| < \infty.$$

Then there exist real constants c, d *such that* $x_n = cn + d + \mathrm{o}(n^s)$.

Proof The assertion is a consequence of Theorem 4.3. □

In the next theorem, we present the assumptions under which condition (5.1) is necessary for the existence of an asymptotically linear solution. In the proof of this theorem, we will need the following lemma.

Lemma 5.5 *If* $z_n = \mathrm{o}(1)$ *and the sequence* $\Delta^2 z_n$ *is nonoscillatory, then*

$$\sum_{n=1}^{\infty} n|\Delta^2 z_n| < \infty.$$

Proof This lemma is a consequence of [7, Lemma 4.1 (f) and (b)]. □

Theorem 5.6 *Assume* $K, L, c \in (0, \infty)$, $d \in \mathbb{R}$,

$$\sigma(n) \geq Ln, \quad a_n \geq 0, \quad b_n \geq 0 \quad \textit{for large } n, \quad g(t) \geq Kt \quad \textit{for large } t,$$

and there exists a solution x *of* (E1) *such that* $x_n = cn + d + \mathrm{o}(1)$. *Then*

$$\sum_{n=1}^{\infty} n^2|a_n| < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} n|b_n| < \infty.$$

Proof There exists a positive ε such that $x_n \geq \varepsilon n$ for large n. Choose $p_1 \in \mathbb{N}$ such that

$$a_n \geq 0, \quad b_n \geq 0, \quad \sigma(n) \geq Ln, \quad x_n \geq \varepsilon n$$

for $n \geq p_1$. Choose $p_2 \geq p_1$ such that $\sigma(n) \geq p_1$ for $n \geq p_2$. For $n \in \mathbb{N}$ let

$$u_n = a_n g(x_{\sigma(n)}) + b_n.$$

Then

$$u_n = \Delta^2 x_n = \Delta^2(cn + d + \mathrm{o}(1)) = \Delta^2(\mathrm{o}(1))$$

and u_n is nonoscillatory. By Lemma 5.5

$$\sum_{n=1}^{\infty} n|u_n| < \infty. \tag{5.2}$$

Since $a_n g(x_{\sigma(n)}) \geq 0$ for large n, we have $b_n \leq u_n$ for large n. By (5.2) we obtain

$$\sum_{n=1}^{\infty} n|b_n| < \infty.$$

If $n \geq p_2$, then

$$g(x_{\sigma(n)}) \geq K x_{\sigma(n)} \geq K(\varepsilon\sigma(n)) \geq K(\varepsilon(Ln)) = K\varepsilon Ln.$$

Hence, there exists a positive constant δ such that

$$g(x_{\sigma(n)}) \geq \delta n$$

for $n \geq p_2$. If $n \geq p_2$, then we get

$$n|a_n| \leq \delta^{-1}|a_n| g(x_{\sigma(n)}) \leq \delta^{-1}|u_n|.$$

Now, by (5.2), we get

$$\sum_{n=1}^{\infty} n^2|a_n| < \infty.$$

□

6. Additional result

In this section we show that if $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, $\sum_{n=1}^{\infty} |a_n| < \infty$, and x is a solution of the equation

$$\Delta^2 x_n = a_n F(x)(n) + b_n$$

such that the sequence $F(x)$ is bounded, then the asymptotic behavior of the sequence x_n/n is closely related to the asymptotic behavior of the sequence $u_n = b_1 + b_2 + \cdots + b_n$. In particular, we get additional results concerning the asymptotic properties of solutions of equation (1.1).

Lemma 6.1 *Assume* $a, b, s \in \mathbb{R}^{\mathbb{N}}$,

$$s_n = b_1 + b_2 + \cdots + b_n, \quad \sum_{n=1}^{\infty} |a_n| < \infty, \quad F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}},$$

and x *is a solution of the equation* $\Delta^2 x_n = a_n F(x)(n) + b_n$ *such that the sequence* $F(x)$ *is bounded. Then there exists a real number* α *such that*

$$\liminf_{n\to\infty} s_n \le \liminf_{n\to\infty} \left(\frac{x_n}{n} - \alpha\right) \le \limsup_{n\to\infty} \left(\frac{x_n}{n} - \alpha\right) \le \limsup_{n\to\infty} s_n.$$

Proof Let $M = \|F(x)\|$. For $n \in \mathbb{N}$ let

$$g_n = a_n F(x)(n), \quad t_n = g_1 + \cdots + g_n, \quad u_n = \Delta x_n.$$

Since $|g_n| \le M|a_n|$, the series $\sum_{n=1}^{\infty} g_n$ is absolutely convergent. Let

$$\gamma = \sum_{n=1}^{\infty} g_n, \quad \alpha = u_1 + \gamma.$$

Since $\Delta u_n = \Delta^2 x_n = g_n + b_n$, we have

$$\Delta u_1 + \Delta u_2 + \cdots + \Delta u_{n-1} = t_{n-1} + s_{n-1}.$$

Hence, $u_n - u_1 = t_{n-1} + s_{n-1}$ and we obtain

$$\frac{\Delta x_n}{\Delta n} = \Delta x_n = u_n = u_1 + t_{n-1} + s_{n-1}, \quad u_1 + t_{n-1} \to \alpha. \tag{6.1}$$

By the Stolz–Cesáro lemma, we have

$$\liminf_{n\to\infty} \frac{\Delta x_n}{\Delta n} \le \liminf_{n\to\infty} \frac{x_n}{n} \le \limsup_{n\to\infty} \frac{x_n}{n} \le \limsup_{n\to\infty} \frac{\Delta x_n}{\Delta n}.$$

By (6.1) we have

$$\liminf_{n\to\infty} \frac{\Delta x_n}{\Delta n} = \alpha + \liminf_{n\to\infty} s_n, \quad \limsup_{n\to\infty} \frac{\Delta x_n}{\Delta n} = \alpha + \limsup_{n\to\infty} s_n.$$

□

The following theorem is an immediate consequence of Lemma 6.1.

Theorem 6.2 *Assume* $a, b, s \in \mathbb{R}^{\mathbb{N}}$,

$$s_n = b_1 + b_2 + \cdots + b_n, \quad \sum_{n=1}^{\infty} |a_n| < \infty, \quad F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}},$$

and x *is a solution of the equation* $\Delta^2 x_n = a_n F(x)(n) + b_n$ *such that the sequence* $F(x)$ *is bounded. Then*

(*a*) *if* (s_n) *is bounded from above, then* (x_n/n) *is bounded from above,*

(b) *if* (s_n) *is bounded from below, then* (x_n/n) *is bounded from below,*

(c) *if* (s_n) *is bounded, then* (x_n/n) *is bounded,*

(d) *if* (s_n) *is convergent, then* (x_n/n) *is convergent,*

(e) *if* $\lim_{n\to\infty} s_n = \infty$, *then* $\lim_{n\to\infty} x_n/n = \infty$,

(f) *if* $\lim_{n\to\infty} s_n = -\infty$, *then* $\lim_{n\to\infty} x_n/n = -\infty$.

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