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Direct and inverse approximation theorems in the weighted Orlicz-type spaces with a variable exponent

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Abstract: In weighted Orlicz-type spaces $\mathcal{S}_{\mathbf{p},\,\mu}$ with a variable summation exponent, the direct and inverse approximation theorems are proved in terms of best approximations of functions and moduli of smoothness of fractional order. It is shown that the constant obtained in the inverse approximation theorem is the best in a certain sense. Some applications of the results are also proposed. In particular, the constructive characteristics of functional classes defined by such moduli of smoothness are given. Equivalence between moduli of smoothness and certain Peetre K-functionals is shown in the spaces $\mathcal{S}_{\mathbf{p},\,\mu}$.

Key words: Best approximation, modulus of smoothness, direct approximation theorem, inverse approximation theorem, weighted space, Orlicz-type spaces, K-functionals

1. Introduction

Let $C^r(\mathbb{T})$ ($\mathbb{T} := [0, 2\pi]$, $r \in \mathbb{N}_0 := \{0, 1, \ldots\}$) denote the space of 2π-periodic r-times continuously differentiable functions with the usual max-norm $\|f\| = \max_{x\in\mathbb{T}} |f(x)|$. Let also $E_n(f) = \inf\limits_{\tau_n} \|f - \tau_n\|$ be the best approximation of function $f \in C(\mathbb{T})$ by trigonometric polynomials τ_n of degree n, $n \in \mathbb{N}_0$. The classical theorem of Jackson (1912) says that *i) if* $f \in C^r(\mathbb{T})$, *then the following inequality holds:* $E_n(f) \le K_r n^{-r}\omega(f^{(r)}, n^{-1})$, $n = 1, 2, \ldots$, *where* $\omega(f, t) := \sup\limits_{|h|\le t} \|f(\cdot + h) - f(\cdot)\|$ *is the modulus of continuity of* f. This assertion is a direct approximation theorem, which asserts that smoothness of the function f implies a quick decrease to zero of its error of approximation by trigonometric polynomials.

On the other hand, the following inverse theorem of Bernstein (1912) with the opposite implication is well-known: *ii) if for some* $0 < \alpha < 1$, $E_n(f) \le K_r n^{-r-\alpha}$, $n = 1, 2, \ldots$, *then* $\omega(f^{(r)}, t) = \mathcal{O}(t^\alpha)$, $t \to 0+$. In ideal cases, these two theorems correspond to each other. For example, it follows from i) and ii) that the relation $E_n(f) = \mathcal{O}(n^{-\alpha})$, $0 < \alpha < 1$, is equivalent to the condition $\omega(f, t) = \mathcal{O}(t^\alpha)$, $t \to 0+$. Such theorems have been of great interest to researchers and constitute the classics of modern approximation theory (see, for example the monographs [1, 5, 10, 11, 29, 30]).

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In recent decades, the topics related to the direct and inverse approximation theorems have been actively investigated in the Orlicz spaces and in the Lebesgue spaces with a variable exponent. In particular, for the Lebesgue functional spaces with variable exponent, similar results are contained in the papers of Guven and Israfilov [12], Akgün [2], Akgün and Kokilashvili [3], Chaichenko [7], Jafarov [15, 16] and others. Continuous analogues of the problems considered are also studied in the following papers: [13, 14, 19, 21]. The latest results related to the Lebesgue spaces with variable exponent, and their applications are described in the monograph [9]. We also note the papers by Nekvinda [17, 18] devoted to the investigations of the discrete weighted Lebesgue spaces with a variable exponent.

In 2000, Stepanets [25] considered the spaces S^p of 2π-periodic Lebesgue summable functions f $(f \in L)$ with the finite norm

$$\|f\|_{S^p} := \|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_p(\mathbb{Z})} = \Big(\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^p\Big)^{1/p},$$

where $\widehat{f}(k) := [f]\widehat{\ }(k) = (2\pi)^{-1}\int_0^{2\pi} f(t)\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t$, $k \in \mathbb{Z}$, are the Fourier coefficients of the function f, and investigated some approximation characteristics of these spaces, including in the context of direct and inverse theorems. Stepanets and Serdyuk [26] introduced the notion of kth modulus of smoothness in S^p and established the direct and inverse theorems on approximation in terms of these moduli of smoothness and the best approximations of functions. This topic was also investigated actively in [20, 27, 28, 30, 31] and others.

In [23] and [22], some results for the spaces S^p were extended to the Orlicz spaces l_M and to the spaces $l_{\mathbf{p}}$ with a variable summation exponent. In particular, in these spaces, the authors found the exact values of the best approximations and Kolmogorov's widths of certain sets of images of the diagonal operators. In this paper, we combine the above mentioned studies and prove the direct and inverse approximation theorems in the weighted spaces $\mathcal{S}_{\mathbf{p},\,\mu}$ of the Orlicz-type with a variable summation exponent. Furthermore, we also find an explicit constant in the inverse approximation theorem and show that this constant is the best in a certain sense.

2. Preliminaries

Let $\mathbf{p} = \{p_k\}_{k=-\infty}^{\infty}$ be a sequence of positive numbers such that

$$1 \le p_k \le K, \quad k = 0, \pm 1, \pm 2, \dots, \tag{2.1}$$

where K is a positive number, and $\mu = \{\mu_k\}_{k=-\infty}^{\infty}$ be a sequence of nonnegative numbers. Let $\mathcal{S}_{\mathbf{p},\,\mu}$ be the space of all functions $f \in L$ such that the following quantity (which is also called the Luxemburg norm of f) is finite:

$$\|f\|_{\mathbf{p},\,\mu} := \|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_{\mathbf{p},\,\mu}(\mathbb{Z})} = \inf\Big\{a > 0 : \sum_{k\in\mathbb{Z}} \mu_k |\widehat{f}(k)/a|^{p_k} \le 1\Big\}. \tag{2.2}$$

By definition, we say that the functions $f \in L$ and $g \in L$ are assumed to be equivalent in $\mathcal{S}_{\mathbf{p},\,\mu}$, when $\|f - g\|_{\mathbf{p},\,\mu} = 0$.

If the sequence $\mathbf{p} = \{p_k\}_{k=-\infty}^{\infty}$ satisfies condition (2.1) and $\mu = \{\mu_k\}_{k=-\infty}^{\infty}$ is a sequence of nonnegative numbers, then

$$\mathcal{S}_{\mathbf{p},\,\mu} = \Big\{f \in L \ : \quad \sum_{k\in\mathbb{Z}} \mu_k |\widehat{f}(k)|^{p_k} < \infty\Big\}.$$

The spaces $\mathcal{S}_{\mathbf{p},\,\mu}$ defined in this way are the Banach spaces. In case when $p_k = p$ and $\mu_k = 1$, $k \in \mathbb{Z}$, $p \geq 1$, they coincide with the above-defined spaces S^p.

Let $\mathcal{T}_n$, $n = 0, 1, \ldots$, be the set of trigonometric polynomials $\tau_n(x) := \sum_{|k|\leq n} c_k \mathrm{e}^{\mathrm{i}kx}$ of the order n, where c_k are arbitrary complex numbers. Denote by

$$E_n(f)_{\mathbf{p},\,\mu} := \inf_{\tau_{n-1}\in\mathcal{T}_{n-1}} \|f - \tau_{n-1}\|_{\mathbf{p},\,\mu} = \inf_{c_k\in\mathbb{C}} \Big\| f - \sum_{|k|\leq n-1} c_k \mathrm{e}^{\mathrm{i}k\cdot} \Big\|_{\mathbf{p},\,\mu}$$

the best approximation of $f \in \mathcal{S}_{\mathbf{p},\,\mu}$ by the trigonometric polynomials $\tau_{n-1} \in \mathcal{T}_{n-1}$ in the space $\mathcal{S}_{\mathbf{p},\,\mu}$.

For a fixed $a > 0$ and arbitrary numbers $c_k \in \mathbb{C}$,

$$\sum_{|k|\leq n-1} \mu_k \Big(|\widehat{f}(k) - c_k|/a\Big)^{p_k} + \sum_{|k|\geq n} \mu_k \Big(|\widehat{f}(k)|/a\Big)^{p_k} \geq \sum_{|k|\geq n} \mu_k \Big(|\widehat{f}(k)|/a\Big)^{p_k}$$

Therefore, for any function $f \in \mathcal{S}_{\mathbf{p},\,\mu}$ we have

$$E_n(f)_{\mathbf{p},\,\mu} = \|f - S_{n-1}(f)\|_{\mathbf{p},\,\mu} = \inf \Big\{ a > 0 : \sum_{|k|\geq n} \mu_k \Big(|\widehat{f}(k)|/a\Big)^{p_k} \leq 1 \Big\}, \tag{2.3}$$

where $S_{n-1}(f,x) = \sum_{|k|\leq n-1} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}$ is the Fourier sum of the function f.

3. Differences and moduli of smoothness of fractional order

Similarly to [6], we define the (right) difference of $f \in L$ of the fractional order $\alpha > 0$ with respect to the increment $h \in \mathbb{R}$ by

$$\Delta_h^\alpha f(x) := \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - jh), \tag{3.1}$$

where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-j+1)}{j!}$, $j \in \mathbb{N}$, $\binom{\alpha}{0} := 1$, and assemble some basic properties of the fractional differences.

Lemma 3.1 *Assume that* $f \in \mathcal{S}_{\mathbf{p},\,\mu}$, $\alpha, \beta > 0$, $x, h \in \mathbb{R}$. *Then*

(i) $\|\Delta_h^\alpha f\|_{\mathbf{p},\,\mu} \leq K(\alpha)\|f\|_{\mathbf{p},\,\mu}$, *where* $K(\alpha) := \sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| \leq 2^{\{\alpha\}}$, $\{\alpha\} := \inf\{k \in \mathbb{N} : k \geq \alpha\}$.

(ii) $[\Delta_h^\alpha f]^\frown(k) = (1 - \mathrm{e}^{-\mathrm{i}kh})^\alpha \widehat{f}(k)$, $k \in \mathbb{Z}$.

(iii) $(\Delta_h^\alpha(\Delta_h^\beta f))(x) = \Delta_h^{\alpha+\beta} f(x)$ *(a. e.).*

(iv) $\|\Delta_h^{\alpha+\beta} f\|_{\mathbf{p},\,\mu} \leq 2^{\{\beta\}} \|\Delta_h^\alpha f\|_{\mathbf{p},\,\mu}$.

(v) $\lim_{h\to 0} \|\Delta_h^\alpha f\|_{\mathbf{p},\,\mu} = 0$.

The proof of Lemma 3.1 and other auxiliary statements of the paper will be given in Section 8.

Based on (3.1), the modulus of smoothness of $f \in \mathcal{S}_{\mathbf{p},\,\mu}$ of the index $\alpha > 0$ is defined by

$$\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu} := \sup_{|h|\leq\delta} \|\Delta_h^\alpha f\|_{\mathbf{p},\,\mu}.$$

Using the standard arguments, it can be shown that the functions $\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}$ possess all the basic properties of ordinary moduli of smoothness. Before formulating them, we give the definition of the ψ-derivative of a function.

Let $\psi=\{\psi_k\}_{k=-\infty}^{\infty}$ be an arbitrary sequence of complex numbers, $\psi_k\neq 0$, $k\in\mathbb{Z}$. If for a given function $f\in L$ with the Fourier series of the form $S[f](x)=\sum_{k\in\mathbb{Z}}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}$, the series $\sum_{k\in\mathbb{Z}\setminus\{0\}}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}/\psi_k$ is the Fourier series of a certain function $g\in L$, then g is called (see, for example, [27, Ch. 9]) ψ-derivative of the function f and is denoted as $g:=f^{\psi}$. It is clear that the Fourier coefficients of functions f and f^{ψ} are related by equality

$$\widehat{f}(k)=\psi_k\widehat{f^{\psi}}(k),\quad k\in\mathbb{Z}\setminus\{0\}. \tag{3.2}$$

In the case $\psi_k=|k|^{-r}$, $r>0$, $k\in\mathbb{Z}\setminus\{0\}$, we use the notation $f^{\psi}=:f^{(r)}$.

Lemma 3.2 *Assume that* $f,g\in\mathcal{S}_{\mathbf{p},\,\mu}$, $\alpha\geq\beta>0$ *and* $\delta,\delta_1,\delta_2>0$. *Then*

(i) $\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}$ *is a nonnegative increasing continuous function on* $(0,\infty)$ *such that* $\lim\limits_{\delta\to 0+}\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}=0$.

(ii) $\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}\leq 2^{\{\alpha-\beta\}}\omega_\beta(f,\delta)_{\mathbf{p},\,\mu}$.

(iii) $\omega_\alpha(f+g,\delta)_{\mathbf{p},\,\mu}\leq\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}+\omega_\alpha(g,\delta)_{\mathbf{p},\,\mu}$.

(iv) $\omega_1(f,\delta_1+\delta_2)_{\mathbf{p},\,\mu}\leq\omega_1(f,\delta_1)_{\mathbf{p},\,\mu}+\omega_1(f,\delta_2)_{\mathbf{p},\,\mu}$.

(v) $\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}\leq 2^{\{\alpha\}}\|f\|_{\mathbf{p},\,\mu}$.

(vi) *if there exists* $f^{(\beta)}\in\mathcal{S}_{\mathbf{p},\,\mu}$, *then* $\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}\leq\delta^{\beta}\omega_{\alpha-\beta}(f^{(\beta)},\delta)_{\mathbf{p},\,\mu}$.

(vii) $\omega_\alpha(f,p\delta)_{\mathbf{p},\,\mu}\leq p^{\alpha}\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}\qquad(\alpha\in\mathbb{N},\quad p\in\mathbb{N})$.

(viii) $\omega_\alpha(f,\eta)_{\mathbf{p},\,\mu}\leq\delta^{-\alpha}(\delta+\eta)^{\alpha}\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}\qquad(\alpha\in\mathbb{N})$.

4. Direct approximation theorem.

Proposition 4.1 *Let* $\psi=\{\psi_k\}_{k=-\infty}^{\infty}$ *be an arbitrary sequence of complex numbers such that* $\psi_k\neq 0$ *and* $\lim\limits_{|k|\to\infty}|\psi_k|=0$. *If for a function* $f\in\mathcal{S}_{\mathbf{p},\,\mu}$ *there exists a derivative* $f^{(\psi)}\in\mathcal{S}_{\mathbf{p},\,\mu}$, *then the following inequality holds:*

$$E_n(f)_{\mathbf{p},\,\mu}\leq\varepsilon_n E_n(f^{\psi})_{\mathbf{p},\,\mu},\quad\text{ where }\quad\varepsilon_n=\max_{|k|\geq n}|\psi_k|.$$

Proof According to (2.3) and (3.2), we have

$$\begin{aligned}E_n(f)_{\mathbf{p},\,\mu}&=\inf\left\{a>0:\sum_{|k|\geq n}\mu_k\Big(|\psi_k\widehat{f^{\psi}}(k)|/a\Big)^{p_k}\leq 1\right\}\\&\leq\inf\left\{a>0:\sum_{|k|\geq n}\mu_k\Big(\varepsilon_n|\widehat{f^{\psi}}(k)|/a\Big)^{p_k}\leq 1\right\}\leq\varepsilon_n E_n(f^{\psi})_{\mathbf{p},\,\mu}.\end{aligned}$$

□

Note that if $\varepsilon_n=\max\limits_{|k|\geq n}|\psi_k|=|\psi_{k_0}|$, where k_0 is an integer, $|k_0|\geq n$, then for an arbitrary polynomial $\tilde{\tau}_{k_0}(x):=c\,\mathrm{e}^{\mathrm{i}k_0x}$, $c\neq 0$, obviously, the equality holds:

$$E_n(\tilde{\tau}_{k_0})_{\mathbf{p},\,\mu}=\varepsilon_n E_n(\tilde{\tau}_{k_0}^{\psi})_{\mathbf{p},\,\mu}.$$

Theorem 4.2 *Assume that* $\mathbf{p}=\{p_k\}_{k=-\infty}^{\infty}$ *and* $\mu=\{\mu_k\}_{k=-\infty}^{\infty}$ *are sequences of nonnegative numbers such that* $1<p_k\le K$, $k\in\mathbb{Z}$, *and the function* $f\in\mathcal{S}_{\mathbf{p},\,\mu}$. *Then for any numbers* $\alpha>0$ *and* $n\in\mathbb{N}$, *the following inequality holds:*

$$E_n(f)_{\mathbf{p},\,\mu}\le C(\alpha)\,\omega_\alpha(f;n^{-1})_{\mathbf{p},\,\mu}.$$

where $C=C(\alpha)$ *is a constant that does not depend on* f *and* n.

Let us use the proof scheme from [24], where the similar estimates were obtained in the spaces $C^r(\mathbb{T})$. In order to adapt this scheme in accordance with the properties of the spaces $\mathcal{S}_{\mathbf{p},\,\mu}$, before proving, we formulate the auxiliary Lemma 4.3. This assertion establishes the equivalence of the Luxembourg norm (2.2) and the Orlicz norm, where the latter is defined as follows.

For given sequences $\mathbf{p}=\{p_k\}_{k=-\infty}^{\infty}$ and $\mu=\{\mu_k\}_{k=-\infty}^{\infty}$ of nonnegative numbers such that $1<p_k\le K$, $k\in\mathbb{Z}$, consider the sequence $\mathbf{q}=\{q_k\}_{k\in\mathbb{Z}}$ defined by the equalities $1/p_k+1/q_k=1$, $k\in\mathbb{Z}$, and the set $\Lambda=\Lambda(\mathbf{p},\mu)$ of all numerical sequences $\lambda=\{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}}\mu_k|\lambda_k|^{q_k}\le 1$. For any function $f\in\mathcal{S}_{\mathbf{p},\,\mu}$, define its Orlicz norm by the equality

$$\|f\|^*_{\mathbf{p},\,\mu}:=\sup\Big\{\sum_{k\in\mathbb{Z}}\mu_k\lambda_k|\widehat{f}(k)|:\quad \lambda\in\Lambda\Big\}. \tag{4.1}$$

Lemma 4.3 *Assume that* $\mathbf{p}=\{p_k\}_{k=-\infty}^{\infty}$ *and* $\mu=\{\mu_k\}_{k=-\infty}^{\infty}$ *are sequences of nonnegative numbers such that* $1<p_k\le K$, $k\in\mathbb{Z}$. *Then for any function* $f\in\mathcal{S}_{\mathbf{p},\,\mu}$,

$$\|f\|_{\mathbf{p},\,\mu}\le\|f\|^*_{\mathbf{p},\,\mu}\le 2\,\|f\|_{\mathbf{p},\,\mu}. \tag{4.2}$$

Proof of Theorem 4.2. Let $\{K_n(t)\}_{n=1}^{\infty}$ be a sequence of kernels (where $K_n(t)$ is a trigonometric polynomial of order not greater than n), satisfying for all $n=1,2,\ldots$ the conditions:

$$\int_{-\pi}^{\pi}K_n(t)\;\mathrm{d}t=1, \tag{4.3}$$

$$\int_{-\pi}^{\pi}|t|^r|K_n(t)|\;\mathrm{d}t\le C(r)(n+1)^{-r},\quad r=0,1,2,\ldots \tag{4.4}$$

In the role of such kernels, in particular, we can take the well-known Jackson kernels of sufficiently great order, that is,

$$K_n(t)=b_p\Big(\frac{\sin pt/2}{\sin t/2}\Big)^{2k_0},$$

where k_0 is an integer that does not depend on n, $2k_0\ge r+2$, the positive integer p is determined from the inequality $n/(2k_0)<p\le n/(2k_0)+1$, and the constant b_p is chosen due to the normalization condition (4.3).

It was shown in [24] that for any sequence of kernels $\{K_n(t)\}$ satisfying conditions (4.3)–(4.4), the following estimate holds:

$$\int_{-\pi}^{\pi}(|t|+n^{-1})^r\;|K_n(t)|\;\mathrm{d}t\le C(r)n^{-r},\quad (r,n=1,2,\ldots). \tag{4.5}$$

Let us first consider the case of $\alpha \in \mathbb{N}$. Set

$$\sigma_{n-1}(x) = (-1)^{\alpha+1} \int_{-\pi}^{\pi} K_{n-1}(t) \sum_{j=1}^{\alpha} (-1)^j \binom{\alpha}{j} f(x - jt) \, \mathrm{d}t.$$

It is clear that $\sigma_{n-1}(x)$ is a trigonometric polynomial which order does not exceed n. Further, in view of (4.3), we have

$$f(x) - \sigma_{n-1}(x) = (-1)^{\alpha} \int_{-\pi}^{\pi} K_{n-1}(t) \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} f(x - jt) \, \mathrm{d}t = (-1)^{\alpha} \int_{-\pi}^{\pi} K_{n-1}(t) \Delta_t^{\alpha} f(x) \, \mathrm{d}t.$$

Hence, taking into account relations (4.1)–(4.2) and the definition of the set Λ, we obtain

$$\begin{aligned} E_n(f)_{\mathbf{p},\mu} &\leq \|f - \sigma_{n-1}\|_{\mathbf{p},\mu} \leq \|f - \sigma_{n-1}\|^*_{\mathbf{p},\mu} = \left\| (-1)^{\alpha} \int_{-\pi}^{\pi} K_{n-1}(t) \Delta_t^{\alpha} f \, \mathrm{d}t \right\|^*_{\mathbf{p},\mu} \\ &= \sup \Big\{ \sum_{k \in \mathbb{Z}} \mu_k \lambda_k \Big| \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\int_{-\pi}^{\pi} K_{n-1}(t) \Delta_t^{\alpha} f(x) \, \mathrm{d}t \Big) \, \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x \Big| : \ \lambda \in \Lambda \Big\}. \end{aligned}$$

Applying now the Fubini theorem and again using estimate (4.2), we find

$$\begin{aligned} E_n(f)_{\mathbf{p},\mu} &\leq \int_{-\pi}^{\pi} |K_{n-1}(t)| \sup \Big\{ \sum_{k \in \mathbb{Z}} \mu_k \lambda_k \Big| \frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta_t^{\alpha} f(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x \Big| : \lambda \in \Lambda \Big\} \mathrm{d}t \leq 2 \int_{-\pi}^{\pi} |K_{n-1}(t)| \, \|\Delta_t^{\alpha} f(x)\|^*_{\mathbf{p},\mu} \mathrm{d}t \\ &\leq 2 \int_{-\pi}^{\pi} |K_{n-1}(t)| \, \|\Delta_t^{\alpha} f(x)\|_{\mathbf{p},\mu} \mathrm{d}t \leq 2 \int_{-\pi}^{\pi} |K_{n-1}(t)| \omega_{\alpha}(f; |t|)_{\mathbf{p},\mu} \, \mathrm{d}t. \end{aligned} \tag{4.6}$$

To estimate the integral on the right-hand side of relation (4.6), we use the property (viii) of Lemma 3.2. Setting $\eta = |t|$, $\delta = n^{-1}$, we see that $\omega_{\alpha}(f; |t|)_{\mathbf{p},\mu} \leq n^{\alpha} (|t| + n^{-1})^{\alpha} \omega_{\alpha}(f; n^{-1})_{\mathbf{p},\mu}$. Using this inequality and (4.5), we get

$$\int_{-\pi}^{\pi} |K_{n-1}(t)| \omega_{\alpha}(f; |t|)_{\mathbf{p},\mu} \mathrm{d}t \leq n^{\alpha} \omega_{\alpha}(f; n^{-1})_{\mathbf{p},\mu} \int_{-\pi}^{\pi} (|t| + n^{-1})^{\alpha} \ |K_{n-1}(t)| \mathrm{d}t \leq C(\alpha) \omega_{\alpha}(f; n^{-1})_{\mathbf{p},\mu}.$$

Thus, in the case of $\alpha \in \mathbb{N}$, the theorem is proved.

If $\alpha > 0$, $\alpha \notin \mathbb{N}$, then we denote by β an arbitrary positive integer satisfying the condition $\beta - 1 < \alpha < \beta$. Due to property (ii) of Lemma 3.2, we obtain

$$E_n(f)_{\mathbf{p},\mu} \leq C(\beta) \ \omega_{\beta}(f; n^{-1})_{\mathbf{p},\mu} \leq C(\beta) \ \omega_{\alpha}(f; n^{-1})_{\mathbf{p},\mu}.$$

□

5. Inverse approximation theorem.

Before proving the inverse approximation theorem, let us formulate the known Bernstein inequality in which the norm of the derivative of a trigonometric polynomial is estimated in terms of the norm of this polynomial (see e.g., [29, Ch. 4]), [30, Ch. 4]).

Proposition 5.1 *Let* $\psi = \{\psi_k\}_{k=-\infty}^{\infty}$ *be an arbitrary sequence of complex numbers,* $\psi_k \neq 0$*. Then for any* $\tau_n \in \mathcal{T}_n$*,* $n \in \mathbb{N}$*, the following inequality holds:*

$$\|\tau_n^{\psi}\|_{\mathbf{p},\,\mu} \le \frac{1}{\epsilon_n}\|\tau_n\|_{\mathbf{p},\,\mu}, \quad \epsilon_n := \min_{0<|k|\le n}|\psi_k|,$$

Proof Let $\tau_n(x) = \sum_{|k|\le n} c_k \mathrm{e}^{\mathrm{i}(k,x)}$, $c_k \in \mathbb{C}$. By the definition of the ψ-derivative and equalities (3.2), we get

$$\begin{aligned}\|\tau_n^{\psi}\|_{\mathbf{p},\,\mu} &= \inf\Big\{a>0:\sum_{0<|k|\le n}\mu_k\Big(|c_k|/|a\psi_k|\Big)^{p_k}\le 1\Big\}\\ &\le \max_{0<|k|\le n}|\psi_k|^{-1}\inf\Big\{a>0:\sum_{0<|k|\le n}\mu_k\Big(|c_k|/a\Big)^{p_k}\le 1\Big\} = \frac{1}{\epsilon_n}\|\tau_n\|_{\mathbf{p},\,\mu}.\end{aligned}$$

□

Note that if $\min_{0<|k|\le n}|\psi_k| = |\psi_{k_0}|$, then for arbitrary polynomials of the form $\tilde{\tau}_{k_0}(x) := c\,\mathrm{e}^{\mathrm{i}k_0 x}$, $c \neq 0$, we have

$$\|\tilde{\tau}_{k_0}^{\psi}\|_{\mathbf{p},\,\mu} = \inf\Big\{a>0:\ \mu_{k_0}\Big(|c_{k_0}|/|a\psi_{k_0}|\Big)^{p_k}\le 1\Big\} = \frac{1}{|\psi_{k_0}|}\inf\Big\{a>0:\ \mu_{k_0}\Big(|c_{k_0}|/a\Big)^{p_k}\le 1\Big\} = \frac{1}{\epsilon_n}\|\tau_{k_0}\|_{\mathbf{p},\,\mu}.$$

Corollary 5.2 *Let* $\psi = \{\psi_k\}_{k=-\infty}^{\infty}$ *be an arbitrary sequence of complex numbers such that* $|\psi_{-k}| = |\psi_k| \ge |\psi_{k+1}| > 0$*. Then for any* $\tau_n \in \mathcal{T}_n$*,* $n \in \mathbb{N}$*,*

$$\|\tau_n^{\psi}\|_{\mathbf{p},\,\mu} \le \frac{1}{|\psi_n|}\|\tau_n\|_{\mathbf{p},\,\mu}.$$

In particular, if $\psi_k = |k|^{-r}$*,* $r>0$*,* $k \in \mathbb{Z}\setminus\{0\}$*, then*

$$\|\tau_n^{\psi}\|_{\mathbf{p},\,\mu} = \|\tau_n^{(r)}\|_{\mathbf{p},\,\mu} \le n^r\|\tau_n\|_{\mathbf{p},\,\mu}.$$

Theorem 5.3 *If* $f \in \mathcal{S}_{\mathbf{p},\,\mu}$*, then for any* $\alpha > 0$ *and* $n \in \mathbb{N}$*, the following inequality is true:*

$$\omega_\alpha\Big(f,\frac{\pi}{n}\Big)_{\mathbf{p},\,\mu} \le \Big(\frac{\pi}{n}\Big)^{\alpha}\sum_{\nu=1}^{n}(\nu^\alpha-(\nu-1)^\alpha)E_\nu(f)_{\mathbf{p},\,\mu}. \tag{5.1}$$

Proof Let us use the proof scheme from [26], modifying it taking into account the peculiarities of the spaces $\mathcal{S}_{\mathbf{p},\,\mu}$. Let $f \in \mathcal{S}_{\mathbf{p},\,\mu}$, $n \in \mathbb{N}$ and $f_{jh}(x) := f(x-jh)$, where $j = 0,1,\ldots$ and $h \in \mathbb{R}$. Then for any $k \in \mathbb{Z}$, we have $\widehat{f}_{jh}(k)= \widehat{f}(k)\mathrm{e}^{-\mathrm{i}kjh}$,

$$[\Delta_h^\alpha f]\widehat{\ }(k) = \Big[\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}f_{jh}\Big]\widehat{\ }(k) = \widehat{f}(k)\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\mathrm{e}^{-\mathrm{i}kjh} = (1-\mathrm{e}^{-\mathrm{i}kh})^\alpha\widehat{f}(k). \tag{5.2}$$

and

$$|[\Delta_h^\alpha f]\widehat{\ }(k)| = |1-\mathrm{e}^{-\mathrm{i}kh}|^\alpha |\widehat{f}(k)| = 2^\alpha \Big|\sin\frac{kh}{2}\Big|^\alpha |\widehat{f}(k)|. \tag{5.3}$$

Since $f\in\mathcal{S}_{\mathbf{p},\,\mu}$, then for any $\varepsilon>0$ there exist a number $N_0=N_0(\varepsilon)\in\mathbb{N}$, $N_0>n$, such that for any $N>N_0$, we have $E_N(f)_{\mathbf{p},\,\mu}=\|f-S_{N-1}(f)\|_{\mathbf{p},\,\mu}<2^{-\alpha}\varepsilon$. Let us set $f_0:=S_{N_0}(f)$. Then in view of (5.3), we see that

$$\|\Delta_h^\alpha f\|_{\mathbf{p},\,\mu} \le \|\Delta_h^\alpha f_0\|_{\mathbf{p},\,\mu} + \|\Delta_h^\alpha (f-f_0)\|_{\mathbf{p},\,\mu} \le \|\Delta_h^\alpha f_0\|_{\mathbf{p},\,\mu} + 2^\alpha E_{N_0+1}(f)_{\mathbf{p},\,\mu} < \|\Delta_h^\alpha f_0\|_{\mathbf{p},\,\mu}+\varepsilon. \tag{5.4}$$

Further, let $S_{n-1}:=S_{n-1}(f_0)$ be the Fourier sum of f_0. Then by virtue of (5.3), for $|h|\le\pi/n$, we have

$$\begin{aligned}\|\Delta_h^\alpha f_0\|_{\mathbf{p},\,\mu} &= \|\Delta_h^\alpha (f_0-S_{n-1})+\Delta_h^\alpha S_{n-1}\|_{\mathbf{p},\,\mu} \le \Big\|2^\alpha (f_0-S_{n-1}) + \sum_{|k|\le n-1} |kh|^\alpha |\widehat{f}(k)|\mathrm{e}^{\mathrm{i}k\cdot}\Big\|_{\mathbf{p},\,\mu} \\ &\le \Big\|2^\alpha (f_0-S_{n-1}) + \Big(\frac{\pi}{n}\Big)^\alpha \sum_{|k|\le n-1} |k|^\alpha |\widehat{f}(k)|\mathrm{e}^{\mathrm{i}k\cdot}\Big\|_{\mathbf{p},\,\mu} = \Big\|2^\alpha \sum_{\nu=n}^{N_0} H_\nu + \Big(\frac{\pi}{n}\Big)^\alpha \sum_{\nu=1}^{n-1} \nu^\alpha H_\nu\Big\|_{\mathbf{p},\,\mu},\end{aligned} \tag{5.5}$$

where $H_\nu(x):=H_\nu(f,x)=|\widehat{f}(\nu)|\mathrm{e}^{\mathrm{i}\nu x}+|\widehat{f}(-\nu)|\mathrm{e}^{-\mathrm{i}\nu x}$, $\nu=1,2,\ldots$

Now we use the following assertion which is proved directly.

Lemma 5.4 *Let $\{c_\nu\}_{\nu=1}^\infty$ and $\{a_\nu\}_{\nu=1}^\infty$ be arbitrary numerical sequences. Then the following equality holds for all natural m, M and N $m\le M<N$:*

$$\sum_{\nu=m}^{M} a_\nu c_\nu = a_m \sum_{\nu=m}^{N} c_\nu + \sum_{\nu=m+1}^{M} (a_\nu - a_{\nu-1}) \sum_{i=\nu}^{N} c_i - a_M \sum_{\nu=M+1}^{N} c_\nu. \tag{5.6}$$

Setting $a_\nu=\nu^\alpha$, $c_\nu=H_\nu(x)$, $m=1$, $M=n-1$ and $N=N_0$ in (5.6), we get

$$\sum_{\nu=1}^{n-1} \nu^\alpha H_\nu(x) = \sum_{\nu=1}^{N_0} H_\nu(x) + \sum_{\nu=2}^{n-1} (\nu^\alpha-(\nu-1)^\alpha) \sum_{i=\nu}^{N_0} H_i(x) - (n-1)^\alpha \sum_{\nu=n}^{N_0} H_\nu(x).$$

Therefore,

$$\begin{aligned}\Big\|2^\alpha \sum_{\nu=n}^{N_0} H_\nu + \Big(\frac{\pi}{n}\Big)^\alpha \sum_{\nu=1}^{n-1} \nu^\alpha H_\nu\Big\|_{\mathbf{p},\,\mu} &\le \Big(\frac{\pi}{n}\Big)^\alpha \Big\| n^\alpha \sum_{\nu=n}^{N_0} H_\nu + \sum_{\nu=1}^{n-1} (\nu^\alpha-(\nu-1)^\alpha) \sum_{i=\nu}^{N_0} H_i - (n-1)^\alpha \sum_{\nu=n}^{N_0} H_\nu \Big\|_{\mathbf{p},\,\mu} \\ &\le \Big(\frac{\pi}{n}\Big)^\alpha \Big\| \sum_{\nu=1}^{n} (\nu^\alpha-(\nu-1)^\alpha) \sum_{i=\nu}^{N_0} H_i \Big\|_{\mathbf{p},\,\mu} \le \Big(\frac{\pi}{n}\Big)^\alpha \sum_{\nu=1}^{n} (\nu^\alpha-(\nu-1)^\alpha) E_\nu(f_0)_{\mathbf{p},\,\mu}.\end{aligned} \tag{5.7}$$

Combining relations (5.4), (5.5), and (5.7) and taking into account the definition of the function f_0, we see that for $|h|\le\pi/n$, the following inequality holds:

$$\|\Delta_h^\alpha f\|_{\mathbf{p},\,\mu} \le \Big(\frac{\pi}{n}\Big)^\alpha \sum_{\nu=1}^{n} (\nu^\alpha-(\nu-1)^\alpha) E_\nu(f)_{\mathbf{p},\,\mu} + \varepsilon$$

which, in view of arbitrariness of ε, gives us (5.1). □

In the spaces $\mathcal{S}^p$, similar results were obtained in [28] and [26]. In the Orlicz-type spaces $\mathcal{S}_M$ of functions $f \in L$ with the finite norm

$$\|f\|_M := \|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_M(\mathbb{Z})} = \inf\Big\{a > 0 : \sum_{k\in\mathbb{Z}} M(|\widehat{f}(k)|/a) \le 1\Big\},$$

where M is an Orlicz function, direct and inverse theorems were proved in [8]. Unlike the results of [8], here we get the constant π^α in inequality (5.1) which is exact in the following sense: for any positive number $\varepsilon > 0$, there exists a function $f^* \in \mathcal{S}_{\mathbf{p},\,\mu}$ such that for all n greater that a certain number n_0, we have

$$\omega_\alpha\Big(f^*, \frac{\pi}{n}\Big)_{\mathbf{p},\,\mu} > \frac{\pi^\alpha - \varepsilon}{n^\alpha} \sum_{\nu=1}^{n} (\nu^\alpha - (\nu-1)^\alpha) E_\nu(f^*)_{\mathbf{p},\,\mu}. \tag{5.8}$$

Indeed, consider the function $f^*(x) = \mathrm{e}^{\mathrm{i}k_0 x}$, where k_0 is an arbitrary positive integer. Then $E_\nu(f^*)_{\mathbf{p},\,\mu} = 1$ for $\nu = 1, 2, \ldots, k_0$, $E_\nu(f^*)_{\mathbf{p},\,\mu} = 0$ for $\nu > k_0$ and

$$\omega_\alpha\Big(f^*, \frac{\pi}{n}\Big)_{\mathbf{p},\,\mu} \ge \|\Delta^\alpha_{\frac{\pi}{n}} f^*\|_{\mathbf{p},\,\mu} \ge 2^\alpha \Big|\sin\frac{k_0\pi}{2n}\Big|^\alpha.$$

Since $\sin t/t$ tends to 1 as $t \to 0$, then for all n greater that a certain number n_0, the inequality $2^\alpha|\sin k_0\pi/(2n)|^\alpha > (\pi^\alpha - \varepsilon)k_0^\alpha/n^\alpha$ holds, which yields (5.8).

Since $\nu^\alpha - (\nu-1)^\alpha \le \alpha\nu^{\alpha-1}$, it follows from inequality (5.1) that

$$\omega_\alpha\Big(f, \frac{\pi}{n}\Big)_{\mathbf{p},\,\mu} \le \frac{\pi^\alpha \alpha}{n^\alpha} \sum_{\nu=1}^{n} \nu^{\alpha-1} E_\nu(f)_{\mathbf{p},\,\mu}. \tag{5.9}$$

This, in particular, yields the following statement:

Corollary 5.5 *Assume that the sequence of the best approximations $E_n(f)_{\mathbf{p},\,\mu}$ of a function $f \in \mathcal{S}_{\mathbf{p},\,\mu}$ satisfies the following relation for some $\beta > 0$:*

$$E_n(f)_{\mathbf{p},\,\mu} = \mathcal{O}(n^{-\beta}).$$

Then, for any $\alpha > 0$, one has

$$\omega_\alpha(f, t)_{\mathbf{p},\,\mu} = \begin{cases} \mathcal{O}(t^\beta) & \text{for } \beta < \alpha, \\ \mathcal{O}(t^\alpha|\ln t|) & \text{for } \beta = \alpha, \\ \mathcal{O}(t^\alpha) & \text{for } \beta > \alpha. \end{cases}$$

For the spaces L_p of 2π-periodic functions integrable to the pth power with the usual norm, inequalities of the type (5.9) were proved by M. Timan (see for example [29, Ch. 6], [30, Ch. 2]).

6. Constructive characteristics of the classes of functions defined by the αth moduli of smoothness

In the following two sections some applications of the obtained results are considered. In particular, in this section we give the constructive characteristics of the classes $\mathcal{S}_{\mathbf{p},\,\mu} H_{\omega_\alpha}$ of functions for which the αth moduli of smoothness do not exceed some majorant.

Let ω be a function defined in interval $[0,1]$. For a fixed $\alpha > 0$, we set

$$\mathcal{S}_{\mathbf{p},\,\mu}H_{\alpha}^{\omega} = \Big\{ f \in \mathcal{S}_{\mathbf{p},\,\mu} : \quad \omega_{\alpha}(f;\delta)_{\mathbf{p},\,\mu} = \mathcal{O}(\omega(\delta)), \quad \delta \to 0+ \Big\}. \tag{6.1}$$

Further, we consider the functions $\omega(t)$, $t \in [0,1]$, satisfying the following conditions 1)–4): **1)** $\omega(t)$ is continuous on $[0,1]$; **2)** $\omega(t) \uparrow$; **3)** $\omega(t) \neq 0$ for any $t \in (0,1]$; **4)** $\omega(t) \to 0$ as $t \to 0$; as well-known condition $(\mathcal{B}_{\alpha})$, $\alpha > 0$ (see, e.g. [4]): $\sum_{v=1}^{n} v^{\alpha-1}\omega(t^{-1}) = \mathcal{O}\Big[n^{\alpha}\omega(n^{-1})\Big]$.

Theorem 6.1 *Assume that* $\mathbf{p} = \{p_k\}_{k=-\infty}^{\infty}$ *and* $\mu = \{\mu_k\}_{k=-\infty}^{\infty}$ *are sequences of nonnegative numbers such that* $1 < p_k \leq K$, $k \in \mathbb{Z}$, $\alpha > 0$ *and* ω *is a function, satisfying conditions* 1) – 4) *and* $(\mathcal{B}_{\alpha})$. *Then a function* $f \in \mathcal{S}_{\mathbf{p},\,\mu}$ *belongs to the class* $\mathcal{S}_{\mathbf{p},\,\mu}H_{\alpha}^{\omega}$ *if and only if*

$$E_n(f)_{\mathbf{p},\,\mu} = \mathcal{O}\Big[\omega(n^{-1})\Big]. \tag{6.2}$$

Proof Let $f \in \mathcal{S}_{\mathbf{p},\,\mu}H_{\alpha}^{\omega}$, by virtue of Theorem 4.2, we have

$$E_n(f)_{\mathbf{p},\,\mu} \leq C(\alpha)\omega_{\alpha}(f,n^{-1})_{\mathbf{p},\,\mu}.$$

Therefore, relation (6.1) yields (6.2). On the other hand, if relation (6.2) holds, then by virtue of (5.9), taking into account the condition $(\mathcal{B}_{\alpha})$, we obtain

$$\omega_{\alpha}(f,n^{-1})_{\mathbf{p},\,\mu} \leq \frac{C(\alpha)}{n^{\alpha}}\sum_{\nu=1}^{n}\nu^{\alpha-1}E_{\nu}(f)_{\mathbf{p},\,\mu} \leq \frac{C_1}{n^{\alpha}}\sum_{\nu=1}^{n}\nu^{\alpha-1}\omega(\nu^{-1}) = \mathcal{O}\Big[\omega(n^{-1})\Big].$$

Thus, the function f belongs to the set $\mathcal{S}_{\mathbf{p},\,\mu}H_{\alpha}^{\omega}$. □

The function $\varphi(t) = t^r$, $r \leq \alpha$, satisfies the condition $(\mathcal{B}_{\alpha})$. Hence, denoting by $\mathcal{S}_{\mathbf{p},\,\mu}H_{\alpha}^{r}$ the class $\mathcal{S}_{\mathbf{p},\,\mu}H_{\alpha}^{\omega}$ for $\omega(t) = t^r$, $0 < r \leq \alpha$, we establish the following statement:

Corollary 6.2 *Assume that* $\mathbf{p} = \{p_k\}_{k=-\infty}^{\infty}$ *and* $\mu = \{\mu_k\}_{k=-\infty}^{\infty}$ *are sequences of nonnegative numbers such that* $1 < p_k \leq K$, $k \in \mathbb{Z}$, *and* $\alpha > 0$, $0 < r \leq \alpha$. *Then a function* $f \in \mathcal{S}_{\mathbf{p},\,\mu}$ *belongs to the class* $\mathcal{S}_{\mathbf{p},\,\mu}H_{\alpha}^{r}$ *if and only if*

$$E_n(f)_{\mathbf{p},\,\mu} = \mathcal{O}(n^{-r}).$$

7. The equivalence between αth moduli of smoothness and K-functionals.

K-functionals were introduced by Lions and Peetre in 1961, and defined in their usual form by Peetre in 1963. Unlike the moduli of continuity expressing the smooth properties of functions, K-functionals express some of their approximative properties. In this section, we prove the equivalence of our moduli of smoothness and certain Peetre K-functionals. This connection is important for studying the properties of the modulus of smoothness and the K-functional, and also for their further application to the problems of approximation theory.

In the space $\mathcal{S}_{\mathbf{p},\,\mu}$, the Petree K-functional of a function f (see e.g., [10, Ch. 6]), generated by its derivative of order $\alpha > 0$, is the following quantity:

$$K_{\alpha}(\delta,f)_{\mathbf{p},\,\mu} := \inf\Big\{\|f-h\|_{\mathbf{p},\,\mu} + \delta^{\alpha}\|h^{(\alpha)}\|_{\mathbf{p},\,\mu} : \ h^{(\alpha)} \in \mathcal{S}_{\mathbf{p},\,\mu}\Big\}, \quad \delta > 0.$$

Theorem 7.1 *Assume that* $\mathbf{p} = \{p_k\}_{k=-\infty}^{\infty}$ *and* $\mu = \{\mu_k\}_{k=-\infty}^{\infty}$ *are sequences of nonnegative numbers such that* $1 < p_k \le K$, $k \in \mathbb{Z}$. *Then for each* $f \in \mathcal{S}_{\mathbf{p},\mu}$ *and* $\alpha > 0$, *there exist constants* $C_1(\alpha)$, $C_2(\alpha) > 0$, *such that for* $\delta > 0$

$$C_1(\alpha)\omega_\alpha(f,\delta)_{\mathbf{p},\mu} \le K_\alpha(\delta, f)_{\mathbf{p},\mu} \le C_2(\alpha)\omega_\alpha(f,\delta)_{\mathbf{p},\mu}. \tag{7.1}$$

Before proving Theorem 7.1, let us formulate the following auxiliary Lemma 7.2, which is used to prove the right-hand side of (7.1).

Lemma 7.2 *Assume that* $\alpha > 0$, $n \in \mathbb{N}$ *and* $0 < h < 2\pi/n$. *Then for any* $\tau_n \in \mathcal{T}_n$

$$\Big(\frac{\sin nh/2}{n/2}\Big)^\alpha \|\tau_n^{(\alpha)}\|_{\mathbf{p},\mu} \le \|\Delta_h^\alpha \tau_n\|_{\mathbf{p},\mu} \le h^\alpha \|\tau_n^{(\alpha)}\|_{\mathbf{p},\mu}. \tag{7.2}$$

Proof of Theorem 7.1. Consider an arbitrary function $h \in \mathcal{S}_{\mathbf{p},\mu}$ such that $h^{(\alpha)} \in \mathcal{S}_{\mathbf{p},\mu}$. Then we have by Lemma 3.2 (iii), (v), and (vi)

$$\omega_\alpha(f,\delta)_{\mathbf{p},\mu} \le \omega_\alpha(f-h,\delta)_{\mathbf{p},\mu} + \omega_\alpha(h,\delta)_{\mathbf{p},\mu} \le 2^{\{\alpha\}}\|f-h\|_{\mathbf{p},\mu} + \delta^\alpha \|h^{(\alpha)}\|_{\mathbf{p},\mu}.$$

Taking the infimum over all $h \in \mathcal{S}_{\mathbf{p},\mu}$ such that $h^{(\alpha)} \in \mathcal{S}_{\mathbf{p},\mu}$, we get the left-hand side of (7.1).

Now let $\delta \in (0, 2\pi)$ and $n \in \mathbb{N}$ such that $\pi/n < \delta < 2\pi/n$. Let also $S_n := S_n(f)$ be the Fourier sum of f. Using Lemma 7.2 with $h = \pi/n$ and property (i) of Lemma 3.1, we obtain

$$\begin{aligned}\|S_n^{(\alpha)}\|_{\mathbf{p},\mu} &\le (n/2)^\alpha \|\Delta_{\pi/n}^\alpha S_n\|_{\mathbf{p},\mu} \le (\pi/\delta)^\alpha \Big(\|\Delta_{\pi/n}^\alpha (S_n - f)\|_{\mathbf{p},\mu} + \|\Delta_{\pi/n}^\alpha f\|_{\mathbf{p},\mu}\Big) \\ &\le (\pi/\delta)^\alpha \Big(2^{\{\alpha\}}\|f - S_n\|_{\mathbf{p},\mu} + \|\Delta_{\pi/n}^\alpha f\|_{\mathbf{p},\mu}\Big).\end{aligned} \tag{7.3}$$

By virtue of (2.3) and Theorem 4.2, we have

$$\|f - S_n\|_{\mathbf{p},\mu} = E_n(f)_{\mathbf{p},\mu} \le C(\alpha)\omega_\alpha(f;\delta)_{\mathbf{p},\mu}. \tag{7.4}$$

Combining (7.3), (7.4), and the definition of modulus of smoothness, we obtain the relation

$$\|S_n^{(\alpha)}\|_{\mathbf{p},\mu} \le C_2(\alpha)\delta^{-\alpha}\omega_\alpha(f;\delta)_{\mathbf{p},\mu},$$

where $C_2(\alpha) := \pi^\alpha(2^{\{\alpha\}}C(\alpha)+1)$, which yields the right-hand side of (7.1):

$$K_\alpha(\delta, f)_{\mathbf{p},\mu} \le \|f - S_n\|_{\mathbf{p},\mu} + \delta^\alpha \|S_n^{(\alpha)}\|_{\mathbf{p},\mu} \le C_2(\alpha)\omega_\alpha(f,\delta)_{\mathbf{p},\mu}.$$

□

8. Proof of auxiliary statements

Proof of Lemma 3.1. By virtue of (5.2), we have

$$\|\Delta_h^\alpha f\|_{\mathbf{p},\mu} = \inf\Big\{a > 0 : \sum_{k\in\mathbb{Z}} \mu_k \Big|\widehat{f}(k)\sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} \mathrm{e}^{-\mathrm{i}kjh}/a\Big|^{p_k} \le 1\Big\}.$$

where for any $a > 0$, the following inequalities hold:

$$\sum_{k\in\mathbb{Z}} \mu_k \Big|\widehat{f}(k) \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} \mathrm{e}^{-\mathrm{i}kjh}/a\Big|^{p_k} \le \sum_{k\in\mathbb{Z}} \mu_k \Big(\sum_{j=0}^{\infty} \Big|\binom{\alpha}{j}\Big| |\widehat{f}(k)|/a \Big)^{p_k} \le \sum_{k\in\mathbb{Z}} \mu_k \Big(2^{\{\alpha\}} |\widehat{f}(k)|/a \Big)^{p_k},$$

and hence property (i) is true. Property (ii) follows from (5.2) and property (iii) is its consequence. Part (iv) follows from (i)–(iii).

To prove (v) we first show that the following relation holds:

$$\lim_{|h|\to 0} \|\Delta_h^\alpha \tau_n\|_{\mathbf{p},\,\mu} = 0 \tag{8.1}$$

where τ_n is a polynomial of the form $\tau_n(x) = \sum_{|k|\le n} c_k \mathrm{e}^{\mathrm{i}kx}$, $n \in \mathbb{N}$, $c_k \in \mathbb{C}$.

Since $\|\tau_n\|_{\mathbf{p},\,\mu} = \inf\{a > 0 : \sum_{|k|\le n} \mu_k (|c_k|/a)^{p_k} \le 1\}$, then by virtue of (5.3), for $a_0 = |nh|^\alpha \|\tau_n\|_{\mathbf{p},\,\mu}$, we obtain

$$\sum_{|k|\le n} \mu_k \Big(|[\Delta_h^\alpha \tau_n]^\wedge(k)|/a_0 \Big)^{p_k} = \sum_{|k|\le n} \mu_k \Big(|1 - \mathrm{e}^{-\mathrm{i}kh}|^\alpha |c_k|/a_0 \Big)^{p_k} = \sum_{|k|\le n} \mu_k \Big(2^\alpha \Big|\sin(kh/2)\Big|^\alpha |c_k|/a_0 \Big)^{p_k}$$

$$\le \sum_{|k|\le n} \mu_k \Big(|kh|^\alpha |c_k|/a_0 \Big)^{p_k} \le \sum_{|k|\le n} \mu_k \Big(|nh|^\alpha \frac{|c_k|}{a_0} \Big)^{p_k} = \sum_{|k|\le n} \mu_k \Big(|c_k|/\|\tau_n\|_{\mathbf{p},\,\mu} \Big)^{p_k} \le 1. \tag{8.2}$$

Therefore, $\|\Delta_h^\alpha \tau_n\|_{\mathbf{p},\,\mu} \le |nh|^\alpha \|\tau_n\|_{\mathbf{p},\,\mu}$. For an arbitrary $\varepsilon > 0$, we set $\delta := \delta(\varepsilon) = \Big(\varepsilon / n^\alpha \|\tau_n\|_{\mathbf{p},\,\mu} \Big)^{1/\alpha}$. Then for all $|h| < \delta$, we have $\|\Delta_h^\alpha \tau_n\|_{\mathbf{p},\,\mu} < \varepsilon$, i.e. relation (8.1) is indeed fulfilled.

Now let f be a function from $\mathcal{S}_M$ and $S_n = S_n(f)$ its Fourier sum. Then for any $\varepsilon > 0$ there exist a number $n_0 = n_0(\varepsilon)$ such that for any $n > n_0$, we have $\|f - S_n\|_{\mathbf{p},\,\mu} < \varepsilon / 2^{\{\alpha\}+1}$. Furthermore, by virtue of (8.1), there exist a number $\delta := \delta(\varepsilon, n)$ such that $\|\Delta_h^\alpha S_n\|_{\mathbf{p},\,\mu} < \frac{\varepsilon}{2}$ when $|h| < \delta$. Then using properties of norm and (i), for $n > n_0$ we get the following relation which yields (v):

$$\|\Delta_h^\alpha f\|_{\mathbf{p},\,\mu} \le \|\Delta_h^\alpha (f - S_n)\|_{\mathbf{p},\,\mu} + \|\Delta_h^\alpha S_n\|_{\mathbf{p},\,\mu} \le 2^{\{\alpha\}} \|f - S_n\|_{\mathbf{p},\,\mu} + \|\Delta_h^\alpha S_n\|_{\mathbf{p},\,\mu} < \varepsilon.$$

□

Proof of Lemma 3.2. Property (iii), nonnegativity and increasing of the function $\omega_\alpha(f,t)_{\mathbf{p},\,\mu}$ follow from the definition of modulus of smoothness. In (i), the convergence to zero as $\delta \to 0+$ follows by (v) of Lemma 3.1. Property (v) is the consequence of Lemma 3.1 (i). According to (i) and (iii) of Lemma 3.1, for arbitrary $0 < \alpha \le \beta$, we have $\|\Delta_h^\alpha f\|_{\mathbf{p},\,\mu} = \|\Delta_h^{\alpha-\beta}(\Delta_h^\beta f)\|_{\mathbf{p},\,\mu} \le 2^{\alpha-\beta} \|\Delta_h^\beta f\|_{\mathbf{p},\,\mu}$, whence passing to the exact upper bound over all $|h| \le \delta$, we obtain (ii). Property (iv) is proved by the usual arguments. In particular, this property yields the continuity of the function $\omega_1(f,\delta)_{\mathbf{p},\,\mu}$, since for arbitrary $\delta_1 > \delta_2 > 0$, $\omega_1(f,\delta_1)_{\mathbf{p},\,\mu} - \omega_1(f,\delta_2)_{\mathbf{p},\,\mu} \le \omega_1(\delta_1 - \delta_2)_{\mathbf{p},\,\mu} \to 0$ as $\delta_1 - \delta_2 \to 0$.

Let us prove the continuity of the function $\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}$ for arbitrary $\alpha > 0$. Let $0 < \delta_1 < \delta_2$ and $h = h_1 + h_2$, where $0 < h_1 \le \delta_1$, $0 < h_2 \le \delta_2 - \delta_1$. Since

$$\Delta_h^\alpha f(\delta) = \Delta_{h_1}^\alpha f(\delta) + \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j \Delta_{jh_2}^1 f(\delta + jh_1)$$

and

$$\Big\|\sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^j\Delta^1_{jh_2}f_{jh_1}\Big\|_{\mathbf{p},\mu} = \inf\Big\{a>0:\sum_{k\in\mathbb{Z}}\mu_k\Big|\Big[\sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^j\Delta^1_{jh_2}f_{jh_1}\Big]^{\widehat{}}(k)/a\Big|^{p_k}\le 1\Big\}$$

$$\le \inf\Big\{a>0:\sum_{k\in\mathbb{Z}}\mu_k\Big(2^{\{\alpha\}}\alpha|[\Delta^1_{h_2}f]^{\widehat{}}(k)|/a\Big)^{p_k}\le 1\Big\}\le 2^{\{\alpha\}}\alpha\|\Delta^1_{h_2}f\|_{\mathbf{p},\mu},$$

then $\|\Delta^\alpha_h f\|_{\mathbf{p},\mu}\le\|\Delta^\alpha_{h_1}f\|_{\mathbf{p},\mu}+2^{\{\alpha\}}\alpha\|\Delta^1_{h_2}f\|_{\mathbf{p},\mu}$ and

$$\omega_\alpha(f,\delta_2)_{\mathbf{p},\mu}\le\omega_\alpha(f,\delta_1)_{\mathbf{p},\mu}+2^{\{\alpha\}}\alpha\,\omega_1(f,\delta_2-\delta_1)_{\mathbf{p},\mu}.$$

Hence, we obtain the necessary relation:

$$\omega_\alpha(f,\delta_2)_{\mathbf{p},\mu}-\omega_\alpha(f,\delta_1)_{\mathbf{p},\mu}\le 2^{\{\alpha\}}\alpha\,\omega_1(f,\delta_2-\delta_1)_{\mathbf{p},\mu}\to 0,\quad \delta_2-\delta_1\to 0.$$

If there exists a derivative $f^{(\beta)}\in\mathcal{S}_{\mathbf{p},\mu}$, $0<\beta\le\alpha$, then by virtue of (5.2) and (3.2), for arbitrary numbers $k\in\mathbb{Z}\setminus\{0\}$ and $h\in[0,\delta]$, we have

$$|[\Delta^\alpha_h f]^{\widehat{}}(k)|=2^\beta|\sin(kh/)2|^\beta|1-\mathrm{e}^{-\mathrm{i}kh}|^{\alpha-\beta}|\widehat{f}(k)|\le\delta^\beta|k|^\beta|1-\mathrm{e}^{-\mathrm{i}kh}|^{\alpha-\beta}|\widehat{f}(k)|\le\delta^\beta|[\Delta^{\alpha-\beta}_h f^{(\beta)}]^{\widehat{}}(k),$$

and therefore property (vi) holds.

If α and p are positive integers, then using the representation

$$\Delta^\alpha_{ph}f(x)=\sum_{k_1=0}^{p-1}\ldots\sum_{k_\alpha=0}^{p-1}\Delta^\alpha_h f(x-(k_1+k_2+\ldots+k_\alpha)h),$$

and the relation

$$\Big|[\Delta^\alpha_h f(x-(k_1+k_2+\ldots+k_\alpha)h)]^{\widehat{}}(k)\Big|=\Big|\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=0}^{\alpha}(-1)^j\binom{\alpha}{j}f_{jh}(x-(k_1+k_2+\ldots+k_\alpha)h)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\Big|$$

$$\le\Big|\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=0}^{\alpha}(-1)^j\binom{\alpha}{j}f_{jh}(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\Big|=\Big|[\Delta^\alpha_h f(x)]^{\widehat{}}(k)\Big|,$$

we see that $\|\Delta^\alpha_{ph}f(x)\|_{\mathbf{p},\mu}\le p^\alpha\|\Delta^\alpha_h f(x)\|_{\mathbf{p},\mu}$:

$$\inf\Big\{a>0:\sum_{k\in\mathbb{Z}}\mu_k\Big(\Big|\sum_{k_1=0}^{p-1}\ldots\sum_{k_\alpha=0}^{p-1}[\Delta^\alpha_h f(x-(k_1+\ldots+k_\alpha)h)]^{\widehat{}}(k)\Big|/a\Big)^{p_k}\le 1\Big\}$$

$$\le\inf\Big\{a>0:\sum_{k\in\mathbb{Z}}\mu_k\Big(p^\alpha\Big|[\Delta^\alpha_h f(x)]^{\widehat{}}(k)\Big|/a\Big)^{p_k}\le 1\Big\}.$$

To prove (viii) it is sufficient to consider the case $\delta<\eta$ (for $\delta\ge\eta$, property (viii) is obvious). Choosing the number p such that $\eta/\delta\le p<\eta/\delta+1$, by virtue (i) and (vii), we obtain

$$\omega_\alpha(f;\eta)_{\mathbf{p},\mu}\le\omega_\alpha(f;p\delta)_{\mathbf{p},\mu}\le p^\alpha\omega_\alpha(f;\delta)_{\mathbf{p},\mu}\le(\eta/\delta+1)^\alpha\omega_\alpha(f,\delta)_{\mathbf{p},\mu}.\qquad\square$$

Proof of Lemma 4.3. The right-hand side of (4.2) is obtained from the Young inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad a \ge 0, \quad b \ge 0,$$

as follows (here in the proof, we exclude the trivial case when $f \equiv \mathrm{const}$)

$$\|f\|^*_{\mathbf{p},\mu}/\|f\|_{\mathbf{p},\mu} = \Big\|f/\|f\|_{\mathbf{p},\mu}\Big\|^*_{\mathbf{p},\mu} = \sup\Big\{\sum_{k\in\mathbb{Z}} \mu_k|\lambda_k\widehat{f}(k)|/\|f\|_{\mathbf{p},\mu} : \lambda \in \Lambda\Big\}$$

$$\le \sup\Big\{\sum_{k\in\mathbb{Z}} \mu_k\Big(\frac{|\widehat{f}(k)/\|f\|_{\mathbf{p},\mu}|^{p_k}}{p_k} + \frac{|\lambda_k|^{q_k}}{q_k}\Big) : \lambda \in \Lambda\Big\} \le \sup\Big\{\sum_{k\in\mathbb{Z}} \mu_k\Big(\Big|\widehat{f}(k)/\|f\|_{\mathbf{p},\mu}\Big|^{p_k} + |\lambda_k|^{q_k}\Big) : \lambda \in \Lambda\Big\} \le 2.$$

To prove the left-hand side of (4.2), let us show that for any function $f \in \mathcal{S}_{\mathbf{p},\mu}$, from the inequality $\|f\|^*_{\mathbf{p},\mu} \le 1$, it follows that $\sum_{k\in\mathbb{Z}} \mu_k|\widehat{f}(k)|^{p_k} \le 1$. Indeed, assume that $\sum_{k\in\mathbb{Z}} \mu_k|\widehat{f}(k)|^{p_k} > 1$. Then take any $\rho > 1$ such that $\sum_{k\in\mathbb{Z}} \mu_k|\widehat{f}(k)/\rho|^{p_k} = 1$ and consider the sequence $\tilde{\lambda} = \{\tilde{\lambda}_k\}_{k\in\mathbb{Z}}$ such that $\tilde{\lambda}_k = (|\widehat{f}(k)|/\rho)^{p_k-1}$, $k \in \mathbb{Z}$. We have

$$\sum_{k\in\mathbb{Z}} \mu_k|\tilde{\lambda}_k|^{q_k} = \sum_{k\in\mathbb{Z}} \mu_k\Big|\widehat{f}(k)/\rho\Big|^{(p_k-1)q_k} = \sum_{k\in\mathbb{Z}} \mu_k\Big|\widehat{f}(k)/\rho\Big|^{p_k} = 1$$

that is, $\tilde{\lambda} \in \Lambda(\mathbf{p},\mu)$. However, by the definition (4.1) of the Orlicz norm, we get

$$\|f\|^*_{\mathbf{p},\mu} \ge \sum_{k\in\mathbb{Z}} \mu_k\tilde{\lambda}_k|\widehat{f}(k)| = \rho\sum_{k\in\mathbb{Z}} \mu_k\Big|\widehat{f}(k)/\rho\Big|^{p_k} = \rho > 1,$$

which is a contradiction. Hence, for any function $f \in \mathcal{S}_{\mathbf{p},\mu}$, the inequality $\|f\|^*_{\mathbf{p},\mu} \le 1$ yields the inequality $\sum_{k\in\mathbb{Z}} \mu_k|\widehat{f}(k)|^{p_k} \le 1$.

Since $\Big\|f/\|f\|^*_{\mathbf{p},\mu}\Big\|^*_{\mathbf{p},\mu} = 1$, then $\sum\limits_{k\in\mathbb{Z}} \mu_k\Big|\widehat{f}(k)/\|f\|^*_{\mathbf{p},\mu}\Big|^{p_k} \le 1$; therefore, $\|f\|_{\mathbf{p},\mu} \le \|f\|^*_{\mathbf{p},\mu}$. □

Proof of Lemma 7.2. Since for any polynomial $\tau_n(x) = \sum_{|k|\le n} c_k \mathrm{e}^{\mathrm{i}kx}$, we have $\|\tau_n^{(\alpha)}\|_{\mathbf{p},\mu} = \inf\{a > 0 : \sum_{|k|\le n} \mu_k(|k|^\alpha|c_k|/a)^{p_k} \le 1\}$, then similarly to (8.2), we get

$$\sum_{|k|\le n} \mu_k\Big(\frac{|[\Delta_h^\alpha\tau_n]\widehat{\ }(k)|}{a_1}\Big)^{p_k} \le \sum_{|k|\le n} \mu_k\Big(\frac{|kh|^\alpha|c_k|}{a_1}\Big)^{p_k} \le \sum_{|k|\le n} \mu_k\Big(\frac{|k|^\alpha|c_k|}{\|\tau_n^{(\alpha)}\|_{\mathbf{p},\mu}}\Big)^{p_k} \le 1,$$

when $a_1 := |h|^\alpha\|\tau_n^{(\alpha)}\|_{\mathbf{p},\mu}$. Therefore, $\|\Delta_h^\alpha\tau_n\|_{\mathbf{p},\mu} \le |h|^\alpha\|\tau_n^{(\alpha)}\|_{\mathbf{p},\mu}$.

In (7.2), the first inequality is trivial in the cases where $h = 0$ or $|h| = 2\pi/n$. So, let $0 < |h| < 2\pi/n$. Since the function $t/\sin t$ increase on $(0,\pi)$ and

$$\|\Delta_h^\alpha\tau_n\|_{\mathbf{p},\mu} = \inf\Big\{a > 0 : \sum_{|k|\le n} \mu_k\Big(2^\alpha\Big|\sin\frac{kh}{2}\Big|^\alpha|c_k|/a\Big)^{p_k} \le 1\Big\},$$

then for $a_2 := \left|\frac{n/2}{\sin(nh/2)}\right|^{\alpha} \|\Delta_h^{\alpha}\tau_n\|_{\mathbf{p},\,\mu}$, we get

$$\begin{aligned}\sum_{|k|\le n} \mu_k(|k|^{\alpha}|c_k|/a_2)^{p_k} &= \sum_{|k|\le n} \mu_k\Big(\Big|\frac{kh/2}{\sin(kh/2)}\Big|^{\alpha}\Big|\frac{\sin(kh/2)}{h/2}\Big|^{\alpha}|c_k|/a_2\Big)^{p_k} \\ &\le \sum_{|k|\le n} \mu_k\Big(2^{\alpha}\Big|\sin\frac{kh}{2}\Big|^{\alpha}|c_k|/\|\Delta_h^{\alpha}\tau_n\|_{\mathbf{p},\,\mu}\Big)^{p_k} \le 1.\end{aligned}$$

Thus, the first inequality in (7.2) also holds. □

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