




Ruled surfaces obtained by bending of curves

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Abstract: We consider a first-order infinitesimal bending of a curve in \mathbb{R}^3 to obtain a ruled surface. This paper investigates this kind of ruled surfaces and their properties. Also, we obtain conditions for ruled surfaces obtained by bending to be developable.

Key words: Ruled surfaces, bending of curves, developable surfaces

1. Introduction

The history of differential geometry dates back to the beginning of the 19th century. Differential geometry studies the geometric properties of curves and surfaces using differential calculus. The origin of the studies is related to C.F. Gauss on Gaussian curvature and B. Riemann on Riemannian manifolds.

Ruled surfaces, being a special type of surface, have an important place in the study of surfaces, because ruled surfaces are the easiest of all to parametrize. In [2, 4], authors studied some properties of ruled surfaces. The ruled surfaces with vanishing Gaussian curvature, which can be transformed into the plane without any deformation and distortion, are called developable surfaces. Developable surfaces form a relatively small subset that contains cylinders, cones, and the tangent surfaces. These surfaces are related to various applications from contemporary architecture to the manufacturing of clothing, since they are suitable for the modeling of surfaces that can be made out of leather, paper, fiber, and sheet metal [1, 3, 5, 6].

In this study, we consider the infinitesimal bending of a curve to obtain a ruled surface. The bending of curves and surfaces is related to biology and medicine [12, 13]. A few suggestions for further reading about bending of curves and surfaces are [7, 8, 10, 11]. The relation between ruled surfaces and bending of curves has motivated us to conduct this study. Therefore, we reveal the relation between them and give conditions to get a developable surface from a ruled surface obtained by bending.

2. Basic notions and properties

In this section, we give some basic concepts on differential geometry of space curves in \mathbb{R}^3 . The inner product of two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ is defined by $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$. Let $r : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a curve, r is called a regular curve if $\frac{dr}{du}(u) = r'(u) \neq 0$ for all $u \in I$, and r is called a unit speed curve if $\|r'(u)\| = 1$ for all $u \in I$, where $\|r'(u)\| = \sqrt{r'(u) \cdot r'(u)}$. Let $r(u)$ be a unit speed curve, and the vector field

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$T(u) = r'(u)$ is called the unit tangent field of r . The curvature of r is defined by $\kappa(u) = \|T'(u)\|$. If $\kappa > 0$, then the unit normal vector field N of r is given by $N(u) = \frac{T'}{\kappa}(u)$. Also, the vector field $B(u) = T(u) \times N(u)$ is called the unit binormal vector field of r . The torsion of r is defined by $\tau(u) = -N(u) \cdot B'(u)$. Therefore, the Frenet–Serret equations are given by:

$$\begin{bmatrix} T'(u) \\ N'(u) \\ B'(u) \end{bmatrix} = \begin{bmatrix} & \kappa(u) & \\ -\kappa(u) & & \\ & -\tau(u) & \tau(u) \end{bmatrix} \begin{bmatrix} T(u) \\ N(u) \\ B(u) \end{bmatrix}.$$

If $r(u)$ is a unit speed curve with $\kappa > 0$, the modified Darboux vector field of r is defined by the vector field $\tilde{D}(u) = \frac{\tau}{\kappa}(u)T(u) + B(u)$ along r and the unit Darboux vector field of r is also denoted by $\bar{D} = \frac{1}{\sqrt{\tau^2 + \kappa^2}}(u)(\tau(u)T(u) + \kappa(u)B(u))$ [5].

A cylindrical helix is a curve $r(u)$ with $\kappa > 0$ such that the tangent lines of r make a constant angle with a fixed direction. It is known that the curve $r(u)$ is a cylindrical helix if and only if $\frac{\tau}{\kappa}(u) = constant$. A circular helix is a curve $r(u)$ with $\kappa > 0$ such that the curvature and the torsion of r are both constant [9].

A ruled surface in \mathbb{R}^3 is the map $\bar{r} : I \times J \rightarrow \mathbb{R}^3$, $\bar{r}(u, v) = r(u) + vz(u)$, where $r : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$, $z : I \subset \mathbb{R} \rightarrow \mathbb{R}^3 - \{0\}$ are smooth mappings and I and J are open subsets of \mathbb{R} . The curves r and z are called the base curve and the director curve, respectively [9]. In addition to this, (u_0, v_0) is called a singular point of $\bar{r}(u, v)$ if and only if $r'(u_0) \times z(u_0) + v_0 z'(u_0) \times z(u_0) = 0$. The set of all singular points of \bar{r} is called the singular locus of the surface \bar{r} [4].

A ruled surface $\bar{r}(u, v) = r(u) + vz(u)$ is called cylindrical if $z \times z' \equiv 0$ and called noncylindrical if $z \times z'$ never vanishes. Let $\bar{r}(u, v)$ be a noncylindrical ruled surface and $\|z\| = 1$, and then a curve $\sigma(u)$ is called the striction curve of $\bar{r}(u, v)$ if $\sigma'(u) \cdot z'(u) = 0$ [4].

The Gaussian curvature of a surface is given by the following formula:

$$K = \frac{LN - M^2}{EG - F^2},$$

where E, F, G and L, M, N are coefficients of the first and second fundamental form, respectively [9]. A ruled surface is called a developable surface if its Gaussian curvature is vanishing everywhere. Let $r(u)$ be a unit speed curve with $\kappa > 0$. The ruled surface $\bar{r}(u, v) = r(u) + v\bar{D}(u)$ is called rectifying developable of r . Likewise, $\bar{r}(u, v) = B(u) + vT(u)$ is called the Darboux developable of r and $\bar{r}(u, v) = \bar{D}(u) + vN(u)$ is called the tangential Darboux developable of r .

3. Ruled surfaces obtained by bending

Let $r : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a regular curve, included in a family of the curves

$$\bar{r}(u, v) = \bar{r}_v(u) = r(u) + vz(u), \tag{3.1}$$

where $u \in I$, $v \rightarrow 0$ and we get $\bar{r}_0(u) = r(u)$ for $v = 0$.

Definition 3.1 [10] *The family of curves \bar{r} is an infinitesimal bending of the curve r if*

$$ds_v^2 - ds^2 = o(v). \tag{3.2}$$

The field $z = z(u)$ is the infinitesimal bending field of the curve r .

Theorem 3.2 [10] The necessary and sufficient condition for $z(u)$ to be an infinitesimal bending field of a curve r is to be

$$dr \cdot dz = 0. \tag{3.3}$$

Theorem 3.3 [10] Infinitesimal bending field for a regular curve r is

$$z(u) = \int [p(u)N(u) + q(u)B(u)]du, \tag{3.4}$$

where $p(u)$, $q(u)$ are arbitrary integrable functions, and vector fields $N(u)$, $B(u)$ are unit principal normal and binormal vector fields of r , respectively.

Definition 3.4 Let r be a regular curve parametrized by arclength with $\kappa > 0$. The ruled surface obtained by bending of r is a map $\bar{r} : I \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3$ defined by $\bar{r}(u, v) = r(u) + vz(u)$, where $z(u)$ is the infinitesimal bending field of the curve r and $u \in I$, $v \in (-\epsilon, +\epsilon)$ for $\epsilon \rightarrow 0$.

We now want to visualize the ruled surface obtained by bending (see Figure 1): Let $r(u) = (\cos(\frac{u}{\sqrt{2}}), \sin(\frac{u}{\sqrt{2}}), \frac{u}{\sqrt{2}})$, $0 < u < 2\pi$. For $p(u) = 0$ and $q(u) = 1$ the infinitesimal bending field of the curve r is $z(u) = (-\cos(\frac{u}{\sqrt{2}}), -\sin(\frac{u}{\sqrt{2}}), \frac{u}{\sqrt{2}})$. The ruled surface obtained by this bending is

$$\bar{r}(u, v) = \left((1 - v)\cos\left(\frac{u}{\sqrt{2}}\right), (1 - v)\sin\left(\frac{u}{\sqrt{2}}\right), (1 + v)\frac{u}{\sqrt{2}} \right), \tag{3.5}$$

for $0 < u < 2\pi$, $-0,3 < v < 0,3$.

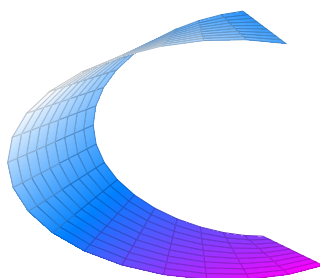


Figure 1. The surface $\bar{r}(u, v)$ obtained by bending of $r(u)$.

Theorem 3.5 Let $r : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a regular curve parametrized by arclength and let $\bar{r}(u, v) = r(u) + vz(u)$ be the ruled surface obtained by bending of r . Then \bar{r} is a cylindrical surface if and only if r is a plane curve and the infinitesimal bending field of r is $z(u) = \int q(u)B(u)du$, where $q(u)$ is an arbitrary differentiable function and $B(u)$ is the unit binormal vector field of r .

Proof Since $\bar{r} = r(u) + vz(u)$ is a cylindrical ruled surface, we have $z(u) \times z'(u) = 0$, where $z'(u) = p(u)N(u) + q(u)B(u)$. Then there exists a real function $k \neq 0$ such that $z(u) = k(u)z'(u)$. In this equation,

using $z'(u) = p(u)N(u) + q(u)B(u)$, and then differentiating the new equation, the following is obtained:

$$z'(u) = -k(u)p(u)\kappa(u)T(u) + ((k(u)p(u))' - k(u)q(u)\tau(u))N(u) + (k(u)p(u)\tau(u) + (k(u)q(u))')B(u).$$

Since $z'(u) = p(u)N(u) + q(u)B(u)$,

$$-k(u)p(u)\kappa(u) = 0, \tag{3.6}$$

$$(k(u)p(u))' - k(u)q(u)\tau(u) = p(u), \tag{3.7}$$

$$k(u)p(u)\tau(u) + (k(u)q(u))' = q(u). \tag{3.8}$$

By equation (3.6), it is obtained that $p = 0$, because of $\kappa > 0$ and $k \neq 0$. Thus, the infinitesimal bending field of r is $z(u) = \int q(u)B(u)du$. By equation (3.7), $\tau = 0$. Thus, the curve r is a plane curve. Conversely, let r be a plane curve and $z(u) = \int q(u)B(u)du$. As $\tau = 0$, the unit binormal vector field of r is a vector field with constant components. Then we have $z(u) = (\int q(u)du)B(u)$ giving $z(u) \times z'(u) = 0$. This completes the proof. \square

Example 3.6 Let $r(u) = (\cos u, \sin u, 0)$, $0 < u < 2\pi$. The unit binormal vector field of r is $B(u) = (0, 0, 1)$. For $q(u) = u$, the ruled surface obtained by bending of r is $\bar{r}(u, v) = (\cos u, \sin u, \frac{u^2 v}{2})$, where $-0.3 \leq v \leq 0.3$. See Figure 2.

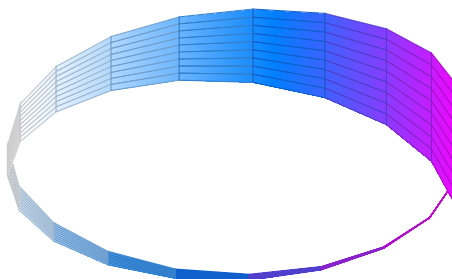


Figure 2. The cylindrical surface $\bar{r}(u, v)$ obtained by bending of $r(u)$.

Theorem 3.7 Let $r : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a regular curve parametrized by arclength and let $\bar{r}(u, v) = r(u) + vz(u)$ be the ruled surface obtained by bending of r . The curve r is the striction curve of $\bar{r}(u, v)$ if and only if $|z| = 1$.

Proof The proof follows from the definition of a striction curve. \square

Theorem 3.8 Let $r : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a regular curve parametrized by arclength with $\kappa > 0$. The tangential Darboux developable surface of r is the ruled surface obtained by bending of the Darboux vector field \bar{D} of r if and only if r is a cylindrical helix.

Proof Let the tangential Darboux developable surface $\bar{r}(u, v) = \bar{D}(u) + vN(u)$ of r be the ruled surface obtained by bending of \bar{D} . Thus, we have $\bar{D}'(u) \cdot N'(u) = 0$. Since $\bar{D}(u) = \frac{1}{\sqrt{\kappa^2(u) + \tau^2(u)}}(\tau(u)T(u) +$

$\kappa(u)B(u)$,

$$\bar{D}'(u) = \left(\frac{\tau(u)}{\sqrt{\kappa^2(u) + \tau^2(u)}} \right)' T(u) + \left(\frac{\kappa(u)}{\sqrt{\kappa^2(u) + \tau^2(u)}} \right)' B(u).$$

Also, as $N'(u) = -\kappa(u)T(u) + \tau(u)B(u)$, the following is obtained:

$$\bar{D}'(u) \cdot N'(u) = \frac{\tau(u)\kappa'(u) - \kappa(u)\tau'(u)}{\sqrt{\kappa^2(u) + \tau^2(u)}} = 0,$$

giving $\frac{\tau}{\kappa}(u) = \text{constant}$. The converse is straightforward. □

Theorem 3.9 *Let $r : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a regular curve parametrized by arclength and let $\bar{r}(u, v)$ be the ruled surface obtained by bending of r . The ruled surface $\bar{r}(u, v)$ is a developable surface if and only if following differential equations hold:*

$$\begin{aligned} a'(u) - b(u)\kappa(u) &= 0, \\ a(u)\kappa(u) + b'(u) - \left(\frac{q}{p}\right)(u)b(u)\tau(u) &= p(u), \\ b(u)\tau(u) + \left(\frac{q}{p}\right)(u)b(u) &= q(u), \end{aligned} \tag{3.9}$$

where a, b are real differentiable functions and $p \neq 0, q$ are the functions determining the infinitesimal bending field of r .

Proof Calculating the Gaussian curvature of $\bar{r}(u, v)$, we have

$$K = \frac{-(z(u) \cdot [q(u)N(u) - p(u)B(u)])^2}{|\bar{r}_u \times \bar{r}_v|^4}.$$

$K = 0$ if and only if $z(u) \cdot [q(u)N(u) - p(u)B(u)] = 0$. Then $z(u)$ is orthogonal to $q(u)N(u) - p(u)B(u)$.

If $z(u) = a(u)T(u) + b(u)N(u) + c(u)B(u)$ for some real integrable functions a, b, c , it is obtained that $b(u)q(u) - c(u)p(u) = 0$ by the orthogonality. As $p \neq 0$, $c(u) = \frac{q}{p}(u)b(u)$. Thus, we have z of the form

$$z(u) = a(u)T(u) + b(u)N(u) + \frac{q}{p}(u)b(u)B(u). \tag{3.10}$$

Differentiating (3.10) and using the equation $z'(u) = p(u)N(u) + q(u)B(u)$, we have the system of differential equations (3.9). This completes the proof. □

Example 3.10 *Let $r(u) = (\cos u, \sin u, 0)$. The bending field of r is*

$$z(u) = \left(-1 - \sin u, \cos u, \frac{\cos u}{1 - \sin u} \right),$$

and the ruled surface obtained by $r(u)$ is the developable surface

$$\bar{r}(u, v) = \left(\cos u - v(1 + \sin u), \sin u + v \cos u, \frac{v \cos u}{1 - \sin u} \right),$$

for $-\frac{\pi}{2} < u < \frac{\pi}{2}$ and $-0.7 \leq v \leq 0.7$. See Figure 3.

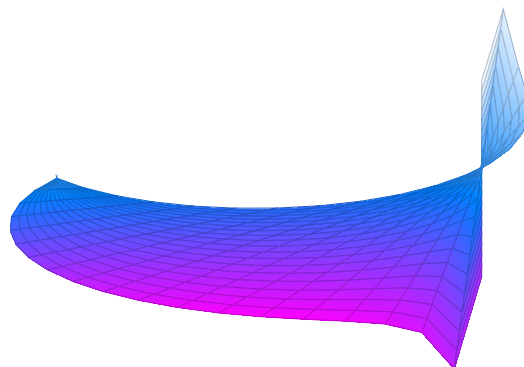


Figure 3. The cylindrical surface $\bar{r}(u, v)$ obtained by bending of $r(u)$.

Corollary 3.11 Let $r : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a regular curve parametrized by arclength and let $\bar{r}(u, v) = r(u) + vz(u)$ be the ruled surface obtained by bending of r such that $\|z(u)\| = 1$. Then \bar{r} is a developable surface if and only if $p = \kappa$ and $q = 0$. In such a case, the ruled surface obtained by bending is $\bar{r}(u, v) = r(u) + vT(u)$.

Example 3.12 Let $\left(\frac{(1+u)^{\frac{3}{2}}}{3}, \frac{(1-u)^{\frac{3}{2}}}{3}, \frac{u}{\sqrt{2}} \right)$. For $p = \kappa$ and $q = 0$ we have that the infinitesimal bending field of r is $z(u) = T(u) = \left(\frac{\sqrt{1+u}}{2}, \frac{\sqrt{1-u}}{2}, \frac{1}{\sqrt{2}} \right)$ and the developable ruled surface (see Figure 4) obtained by bending is

$$\bar{r}(u, v) = \left(\frac{(1+u)^{\frac{3}{2}}}{3} + \frac{v\sqrt{1+u}}{2}, \frac{(1-u)^{\frac{3}{2}}}{3} + \frac{v\sqrt{1-u}}{2}, \frac{u+v}{\sqrt{2}} \right).$$

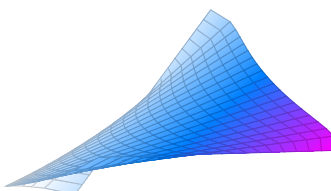


Figure 4. The developable surface $\bar{r}(u, v) = r(u) + vT(u)$.

Theorem 3.13 Let $r : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a regular curve parametrized by arclength and let $\bar{r}(u, v)$ be the developable ruled surface obtained by bending of r . The surface $\bar{r}(u, v)$ is a conical surface if and only if $\left(\frac{b}{ap}\right)'(u) = \left(-\frac{1}{a}\right)(u)$, where the real function $a \neq 0$ is T component of the infinitesimal bending field of r .

Proof Since the surface \bar{r} is developable, z is as (3.10). We find singular points of \bar{r} by the equation $r' \times z + vz' \times z = 0$. Using (3.10), $(b - apv)B - \left(\frac{q}{p}b - aqv\right)N = 0$. This yields $v = \frac{b}{ap}$. Thus, the singular locus of \bar{r} is given by $\sigma(u) = r(u) + \left(\frac{b}{ap}\right)(u)z(u)$. We have $\sigma'(u) = \left(\frac{1}{a}\right)(u)z(u) + \left(\frac{b}{ap}\right)'(u)z(u)$. Therefore, the developable surface \bar{r} is conical; that is, $\sigma'(u) = 0$ if and only if $\left(\frac{b}{ap}\right)'(u) = \left(-\frac{1}{a}\right)(u)$. \square

Remark 3.14 *The developable surface given in Example 3.10 is a conical surface obtained by bending.*

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