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Evaluation of sums of products of Gaussian q-binomial coefficients with rational weight functions

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Abstract: Generalizing earlier results, sums over the products of the Gaussian q-binomial coefficients are computed. Some applications of the results for special choices of q are emphasized. The results are obtained by the elementary technique of partial fraction decomposition.

Key words: Gaussian q-binomial coefficients, Fibonomial coefficients, partial fraction decomposition, sum identities

1. Introduction

Define the second-order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2} \text{ and } V_n = pV_{n-1} + V_{n-2},$$

with initial values $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively.

The Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q}, \qquad V_n = \alpha^n + \beta^n = \alpha^n(1+q^n),$$

where $\alpha, \beta = \left(p \mp \sqrt{\Delta}\right)/2$ with $q = \beta/\alpha = -\alpha^{-2}$ and $\Delta = p^2+4$, so that $\alpha = \mathbf{i}q^{-1/2}$.

In the special instance $p = 1$, the sequences $\{U_n\}$ and $\{V_n\}$ are reduced to the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$, respectively.

For integers n and k such that $n \geq k \geq 0$ and integer m, the generalized Fibonomial coefficients with indices in arithmetic progressions are defined by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} = \frac{U_m U_{2m} \dots U_{nm}}{(U_m U_{2m} \dots U_{km})(U_m U_{2m} \dots U_{(n-k)m})}$$

with $\left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_{U;m} = \left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_{U;m} = 1$ and 0 otherwise. When $m = 1$, we have the generalized Fibonomial coefficients, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{U}$. When $U_n = F_n$, then the generalized Fibonomial coefficients are reduced to the usual Fibonomial coefficients, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{F}$.

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Various sum formulæ including the Fibonomial coefficients with the Fibonacci and Lucas numbers as coefficients have been studied by several authors (for more details, see [9, 10, 13, 16, 17]). For example, Marques and Trojovsky [13] showed that for positive integers m and n,

$$\sum_{j=0}^{4n+2} (-1)^{\binom{j+1}{2}} \left\{ {4m+2 \atop j} \right\}_F L_{2m+1-j} = -\left\{ {4m+2 \atop 4n+3} \right\}_F \frac{F_{4n+3}}{F_{2m+1}}$$

and

$$\sum_{j=0}^{4m} (-1)^{\binom{j}{2}} \left\{ {4m \atop j} \right\}_F F_{n+4m-j} = \frac{1}{2} F_{2m+n} \sum_{j=0}^{4m} (-1)^{\binom{j}{2}} \left\{ {4m \atop j} \right\}_F L_{2m-j}.$$

The Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where $(x;q)_n$ is the q-Pochhammer symbol, $(x;q)_n = (1-x)(1-xq)\ldots(1-xq^{n-1})$ with $(x;q)_0 = 1$.

We recall some useful formulæ [1, 6]:

$$\sum_{k \geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix}_q z^k = \frac{1}{(z;q)_{n+1}}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} z^k = \prod_{k=1}^{n} (1 + zq^k).$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_{U;m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \quad \text{with } q = \beta/\alpha = -\alpha^{-2} \text{ or } \alpha = \mathbf{i} q^{-1/2}.$$

Various sum formulæ including the Gaussian q-binomial coefficients with certain weight functions have been studied by several authors [9, 10]. They also gave some applications by using the link between the generalized Fibonomial and Gaussian q-binomial coefficients to sums including the generalized Fibonomial coefficients.

Melham [14] derived families of identities between sums of powers of the Fibonacci and Lucas numbers. In his work, while deriving these identities, he conjectured a complex identity between the Fibonacci and Lucas numbers. We recall this conjecture:

(a) Let $k, m, n \in \mathbb{Z}$ with $m > 0$ show that

$$\sum_{j=0}^{m-1} \frac{F_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m+3)}{2}} \frac{F_{n-mk}^{m+1}}{\prod_{j=1}^{m} F_{(m+1)k+j}} = F_{(m+1)(n+\frac{m}{2})}.$$

(b) The Lucas counterpart of (a) is given by

$$\sum_{j=0}^{m-1} \frac{L_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m+3)}{2}} \frac{L_{n-mk}^{m+1}}{\prod_{j=1}^{m} F_{(m+1)k+j}} = \begin{cases} 5^{\frac{m+1}{2}} F_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is odd,} \\ 5^{\frac{m}{2}} L_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is even,} \end{cases}$$

where $(F_n)_{(m)}$ is the "falling" factorial, which begins at F_n for $n \neq 0$, and is the product of m Fibonacci numbers excluding F_0. For example $(F_6)_{(5)} = F_6F_5F_4F_3F_2$ and $(F_3)_{(5)} = F_3F_2F_1F_{-1}F_{-2}$. For $m > 0$, define $(F_0)_{(m)} = F_{-1}F_{-2}\dots F_{-m}$ and $(F_0)_{(0)} = 1$.

The authors of [7] converted the sum identities conjectured by Melham into q-form. Recall that the first sum identity (a) takes the form

$$\begin{aligned} &(1-q^{(m+1)k+m}) \begin{bmatrix} (m+1)k+m-1 \\ m-1 \end{bmatrix}_q \\ &\times \sum_{j=0}^{m-1} (-1)^j q^{j(j+1)/2} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \frac{(1-q^{n+k+m-j})^{m+1}}{1-q^{(m+1)k+m-j}} \\ &= \left(1-q^{\frac{(m+1)(2n+m)}{2}}\right) \frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}} - (-1)^m q^{\frac{m(m+1)(2k+1)}{2}} (1-q^{n-mk})^{m+1}. \end{aligned}$$

They used the contour integration method to prove this sum identity.

Quite recently, Li and Chu [12] used the q-derivative operator to prove the same conjecture. There are many kinds of combinatorial sums as well as various proof methods. For example, for the method of integral representation of combinatorial sums as a proof method that is mainly based on the use of residues, we refer to [5].

Recently, Kılıç and Prodinger [10] computed the following sum identity in closed form for any positive integer w, any nonzero real number a, nonnegative integer n, and integers t and r such that $r \geq -1$ and $t \geq -n$:

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{\binom{j+1}{2}+tj} \frac{1}{(aq^j;q^w)_{r+1}}.$$

As a particular consequence of the above sum, Kılıç and Arıkan [8] presented a proof of Clark's conjecture.

More recently, Kılıç and Prodinger [11] presented and proved (using only the elementary partial fraction decomposition method) three sum identities. For any real numbers a and b:

$$\mathtt{SUM}_1 = \sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}-nk} (a-q^k),$$

$$\mathtt{SUM}_2 = \sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}-nk} \frac{1}{q^{-k}-a},$$

and

$$\mathtt{SUM}_3 = \sum_{k=0}^{n} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}-nk} \frac{a-q^{-k}}{b-q^{-k}}.$$

In this paper, inspired by the results of [11], we will present and compute a general sum formula including the Gaussian q-binomial coefficients with a certain rational-parametric weight function. Namely, we consider the sum $S(n;t,a,p,r)$:

$$S(n;t,a,p,r)=\sum_{k=0}^{n}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}+k(t-n)}\frac{1}{(aq^k;p)_{r+1}}.$$

If one chooses specific values of the parameters, many known formulæ can be derived from the above sum.

For example, one can obtain the three sums from [11] from the sum $S(n;t,a,p,r)$ as follows:

- For any values of a and p, setting $r=-1$ in the sum $S(n;t,a,p,r)$, we get $\mathtt{SUM}_1$ as

$$\mathtt{SUM}_1=aS(n;0,\cdot,\cdot,-1)-S(n;1,\cdot,\cdot,-1).$$

- When $r=0$, $t=1$ and $p=q$ in the sum $S(n;t,a,p,r)$, we obtain $\mathtt{SUM}_2$ as

$$\mathtt{SUM}_2=S(n;1,a,q,0).$$

- Since

$$\begin{bmatrix}n+k-1\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q\frac{1-q^n}{1-q^{n+k}}$$

and

$$\frac{1}{(1-q^{n+k})(1-bq^k)}=\frac{1}{(1-bq^{-n})(1-q^{n+k})}+\frac{b}{(b-q^n)(1-bq^k)},$$

we obtain $\mathtt{SUM}_3$ from the sum $S(n;t,a,p,r)$ via

$$\begin{aligned}\mathtt{SUM}_3=&\frac{1-q^n}{1-bq^{-n}}\big[S(n;1,q^n,\cdot,0)-aS(n;2,q^n,\cdot,0)\big]\\&+\frac{b(1-q^n)}{b-q^n}\big[S(n;1,b,\cdot,0)-aS(n;2,b,\cdot,0)\big].\end{aligned}$$

Thus, our results generalize the results of [11].

We use a partial fraction method to prove our claims (for earlier results using this approach, we refer to [2–4, 15]). All identities we will obtain hold for general p and q. Finally, we will present some applications of our results for some special choices of p and q.

Throughout the paper, we assume that a, q, and p are nonzero real numbers; n is a nonnegative integer; and $r\geq-1$ is an integer such that $ap^jq^i\neq1$ for all $i\in\{0,1,\dots,n\}$ and $j\in\{0,1,\dots,r\}$.

2. The main results

Theorem 2.1 *For positive integer t such that $t \leq r+n+2$,*

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(t-n)k} \frac{1}{(aq^k;p)_{r+1}} \tag{2.1}$$

$$= a^{1-t} q^{\binom{n+1}{2}} (-1)^n \sum_{k=0}^{r} (-1)^k p^{\frac{1}{2}k(k-2t+3)} \frac{(ap^k q^{-n};q)_n}{(ap^k;q)_{n+1}(p;p)_k (p;p)_{r-k}}$$

$$- (-1)^{n+t} a^{-r-1} p^{-\binom{r+1}{2}} q^{\binom{n+r-t+3}{2}}$$

$$\times \sum_{k=0}^{t-r-2} \begin{bmatrix} n \\ t-r-2-k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}+k(2+n+r-t)} \sum_{l=0}^{k} \begin{bmatrix} r+l \\ l \end{bmatrix}_p \begin{bmatrix} n+k-l \\ n \end{bmatrix}_q (ap^r)^{-l}.$$

Proof First we rewrite the LHS of the claim as

$$\sum_{k=0}^{n} \frac{(q^{-k}-q)\dots(q^{-k}-q^n)}{(q;q)_k (q;q)_{n-k}} \frac{(-1)^k q^{\binom{k}{2}+k(t-r-1)}}{(q^{-k}-a)(q^{-k}-ap)\dots(q^{-k}-ap^r)}.$$

Define

$$h(z) := \frac{(z-q)\dots(z-q^n)}{(1-z)(1-zq)\dots(1-zq^n)} \frac{z^{r+1}}{(z-a)(z-ap)\dots(z-ap^r)} \frac{1}{z^t}.$$

Then the partial fraction expansion gives us

$$h(z) = \sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}+k(t-n)} \frac{1}{(1-zq^k)(aq^k;p)_{r+1}}$$

$$+ \sum_{k=0}^{r} \frac{A_k}{z-ap^k} + \sum_{k=1}^{t} \frac{B_k}{z^k}.$$

If we multiply both sides of the above equation by z and then let $z \to \infty$, we obtain

$$0 = -\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(t-n)k} \frac{1}{(aq^k;p)_{r+1}} + \sum_{k=0}^{r} A_k + B_1,$$

or

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(t-n)k} \frac{1}{(aq^k;p)_{r+1}} = \sum_{k=0}^{r} A_k + B_1.$$

Now, for $0 \leq k \leq r$, we compute the coefficients A_k and B_1 via

$$A_k = (z-ap^k)h(z)\Big|_{z=ap^k} \quad \text{and} \quad B_1 = [z^{-1}]h(z).$$

First consider the coefficients A_k:

$$\begin{aligned}
A_k &= (z-ap^k)h(z)\Big|_{z=ap^k} \\
&= \frac{(ap^k-q)\dots(ap^k-q^n)}{(1-ap^k)(1-ap^kq)\dots(1-ap^kq^n)}a^{r-t+1}p^{k(r-t+1)} \\
&\times \frac{1}{(ap^k-a)(ap^k-ap)\dots(ap^k-ap^{(k-1)})} \\
&\times \frac{1}{(ap^k-ap^{(k+1)})(ap^k-ap^{(k+2)})\dots(ap^k-ap^r)} \\
&= \frac{a^{r-t+1}p^{k(r-t+1)}}{(ap^k;q)_{n+1}}(-1)^n q^{\binom{n+1}{2}}(ap^kq^{-n};q)_n \\
&\times \frac{(-1)^k}{a^kp^{\binom{k}{2}}(1-p^k)(1-p^{(k-1)})\dots(1-p)} \\
&\times \frac{1}{a^{r-k}p^{k(r-k)}(1-p)(1-p^2)\dots(1-p^{(r-k)})} \\
&= a^{1-t}q^{\binom{n+1}{2}}\frac{(-1)^{n+k}p^{\frac{1}{2}k(k-2t+3)}}{(ap^k;q)_{n+1}}\frac{(ap^kq^{-n};q)_n}{(p;p)_k(p;p)_{r-k}}.
\end{aligned}$$

For the coefficient B_1, consider

$$\begin{aligned}
B_1 &= [z^{-1}]h(z) \\
&= [z^{-2-r+t}]\frac{(z-q)\dots(z-q^n)}{(z;q)_{n+1}}\frac{1}{(z-a)(z-ap)\dots(z-ap^r)} \\
&= [z^{-2-r+t}]z^n\frac{\prod_{k=1}^{n}(1-z^{-1}q^k)}{(z;q)_{n+1}(-1)^{r+1}a^{r+1}p^{\binom{r+1}{2}}(za^{-1}p^{-r};p)_{r+1}} \\
&= \frac{(-1)^{r+1}}{a^{r+1}p^{\binom{r+1}{2}}}[z^{-2-r+t-n}]\sum_{k\geq 0}\begin{bmatrix} n+k \\ k \end{bmatrix}_q z^k \\
&\times \sum_{k\geq 0}\begin{bmatrix} r+k \\ k \end{bmatrix}_p \left(\frac{z}{ap^r}\right)^k \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}}z^{-k} \\
&= \frac{(-1)^{r+1}}{a^{r+1}p^{\binom{r+1}{2}}}[z^{-2-r+t-n}]\sum_{k\geq 0}c_kz^k\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}}z^{-k},
\end{aligned}$$

where

$$c_k = \sum_{l=0}^{k} a_l b_{k-l},$$

with

$$a_l = \begin{bmatrix} r+l \\ l \end{bmatrix}_p \frac{1}{a^lp^{rl}} \text{ and } b_l = \begin{bmatrix} n+l \\ l \end{bmatrix}_q.$$

Finally, for $t \leq n+r+2$, we get

$$B_1 = \frac{(-1)^{r+1}}{a^{r+1}p^{\binom{r+1}{2}}}[z^{-2-r+t-n}]\sum_{k\geq 0} c_k z^k \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}} z^{-k}$$
$$= \frac{(-1)^{r+1}}{a^{r+1}p^{\binom{r+1}{2}}} \sum_{k=0}^{t-r-2} c_k \begin{bmatrix} n \\ t-r-2-k \end{bmatrix}_q (-1)^{k+n+r-t} q^{\binom{k+2+n+r-t}{2}}$$
$$= -(-1)^{n+t} a^{-r-1} q^{\binom{n+r-t+3}{2}} p^{-\binom{r+1}{2}}$$
$$\times \sum_{k=0}^{t-r-2} (-1)^k q^{\binom{k+1}{2}+k(2+n+r-t)} \begin{bmatrix} n \\ t-r-2-k \end{bmatrix}_q c_k,$$

as claimed. □

In Theorem 2.1, we assumed t to be a positive integer. Now we separately consider the case $t=0$ with the following result.

Theorem 2.2

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}-nk} \frac{1}{(aq^k;p)_{r+1}} = (-1)^n q^{-\binom{n+1}{2}}$$
$$+ a(-1)^n q^{\binom{n+1}{2}} \sum_{k=0}^{r} (-1)^k p^{\frac{1}{2}k(k+3)} \frac{(ap^k q^{-n};q)_n}{(ap^k;q)_{n+1}(p;p)_k(p;p)_{r-k}}.$$

Proof Define

$$h(z) := \frac{(z-q)\dots(z-q^n)}{(1-z)(1-zq)\dots(1-zq^n)} \frac{z^{r+1}}{(z-a)(z-ap)\dots(z-ap^r)}.$$

By partial fraction expansion, we write

$$h(z) = \sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}-nk} \frac{1}{(1-zq^k)(aq^k;p)_{r+1}}$$
$$+ \sum_{k=0}^{r} \frac{A_k}{z-ap^k}.$$

Since $\lim_{z\to\infty} zh(z) = (-1)^{n+1} q^{-\binom{n+1}{2}}$, we write

$$(-1)^{n+1} q^{-\binom{n+1}{2}} = -\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}-nk} \frac{1}{(aq^k;p)_{r+1}} + \sum_{k=0}^{r} A_k,$$

where $A_k = (z-ap^k)h(z)\Big|_{z=ap^k}$ for $0 \leq k \leq r$. Thus, the proof follows. □

Now we present some corollaries of Theorem 2.1.

Corollary 2.3

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k+1}{2}-nk} \frac{1}{(aq^k;p)_{r+1}} = (-1)^n q^{\binom{n+1}{2}} \sum_{k=0}^{r} (-1)^k p^{\binom{k+1}{2}} \frac{(ap^k q^{-n};q)_n}{(ap^k;q)_{n+1}(p;p)_k(p;p)_{r-k}}.$$

Proof If we choose $t=1$, then $\binom{k}{2}+k=\binom{k+1}{2}$ and $-r-1<0$. By Theorem 2.1, the result follows. □

Corollary 2.4 *For any positive integer t such that $t<r+2$,*

$$\sum_{k=0}^{n}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}+(t-n)k}\frac{1}{(aq^k;p)_{r+1}}=a^{1-t}q^{\binom{n+1}{2}}(-1)^n\sum_{k=0}^{r}(-1)^k p^{\frac{1}{2}k(k-2t+3)}\frac{(ap^kq^{-n};q)_n}{(ap^k;q)_{n+1}(p;p)_k(p;p)_{r-k}}.$$

Proof Since $t-r-2<0$, the last sum in the RHS of Eq. (2.1) equals 0 and so the claim follows. □

Corollary 2.5 *For any positive integer t such that $t\le n+1$,*

$$\sum_{k=0}^{n}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}+(t-n)k} = (-1)^{n+t+1}q^{\binom{n-t+2}{2}}\sum_{k=0}^{t-1}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\t-1-k\end{bmatrix}_q(-1)^k q^{\binom{k+1}{2}+k(n-t+1)}.$$

Proof The claim follows from Theorem 2.1 with $r=-1$. □

3. Further corollaries

In this section, we will present several generalized Fibonomial-Fibonacci-Lucas sum identities as corollaries of our results.

Corollary 3.1 *For $n\ge 0$,*

$$\sum_{k=0}^{n}\begin{Bmatrix}n+k\\k\end{Bmatrix}_U\begin{Bmatrix}n\\k\end{Bmatrix}_U(-1)^{nk+\binom{k+1}{2}}\frac{1}{U_{n+k+1}V_{n+k+2}U_{n+k+3}}$$
$$=(-1)^{\binom{n-1}{2}}\left[\begin{Bmatrix}2n\\n\end{Bmatrix}_U^{-1}\frac{1}{U_{2n+1}V_1^2}-\begin{Bmatrix}2n+2\\n+1\end{Bmatrix}_V^{-1}\frac{\Delta}{U_2^2V_1}+\begin{Bmatrix}2n+3\\n+1\end{Bmatrix}_U^{-1}\frac{U_{n+2}}{U_2^3}\right],$$

where $\Delta=p^2+4$ is defined as before.

Proof Note that $\Delta=(\alpha-\beta)^2=-q^{-1}(1-q)^2$. Thus, this sum can be equivalently rewritten in q-form as

$$\frac{(1-q)^2}{\alpha^{3n+4}}\sum_{k=0}^{n}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1)^k q^{\binom{k}{2}+(2-n)k}}{(1-q^{n+k+1})(1+q^{n+k+2})(1-q^{n+k+3})}$$
$$=\begin{bmatrix}2n\\n\end{bmatrix}_q^{-1}\frac{(-1)^{-\frac{n}{2}}q^{\frac{1}{2}n^2+n+1}(1-q)}{(1-q^{2n+1})(1+q)^2}-\begin{bmatrix}2n+2\\n+1\end{bmatrix}_q^{-1}\frac{(-1)^{-\frac{n}{2}}q^{\frac{1}{2}n^2+n+1}(1-q)^4}{(1-q^2)^2(1+q)}$$
$$+\begin{bmatrix}2n+3\\n+1\end{bmatrix}_q^{-1}(-1)^{-\frac{n}{2}+1}q^{\frac{1}{2}n^2+n+2}\frac{(1-q)^2(1-q^{n+2})}{(1-q^2)^3}.$$

Applying Corollary 2.4 to the above sum on LHS for the parameters $t=r=2$, $a=q^{n+1}$, and $p=-q$, it equals

$$(-1)^{-\frac{n}{2}}q^{\frac{1}{2}n^2+n+1}(1-q)^2\sum_{k=0}^{2}\frac{(-1)^k(-q)^{\frac{1}{2}k(k-1)}((-1)^kq^{k+1};q)_n}{((-1)^kq^{n+k+1};q)_{n+1}(-q;-q)_k(-q;-q)_{2-k}}$$
$$=(-1)^{-\frac{n}{2}}q^{\frac{1}{2}n^2+n+1}\left[\frac{(q;q)_n(1-q)}{(q^{n+1};q)_{n+1}(1+q)^2}-\frac{(-q^2;q)_n(1-q)^2}{(-q^{n+2};q)_{n+1}(1+q)^2}-\frac{q(q^3;q)_n(1-q)^2}{(q^{n+3};q)_{n+1}(1+q)(1-q^2)}\right].$$

After some algebraic operations and simplifications, the claim follows. □

Corollary 3.2 *For* $n \geq 0$,

$$\sum_{k=0}^{n} \left\{ {n+k \atop k} \right\}_U \left\{ {n \atop k} \right\}_U (-1)^{nk+\binom{k}{2}} \frac{V_{k+2}}{U_{n+k+1}U_{n+k+3}} = (-1)^{\binom{n+1}{2}+1} \frac{U_{n+2}}{U_2} \left\{ {2n+1 \atop n} \right\}_U^{-1} \left[\frac{V_{n-1}}{U_{n+1}U_{n+2}} - \frac{U_{n+2}}{U_{2n+3}} \right].$$

Proof We rewrite the claim in q-form as follows:

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}+(1-n)k} (-1)^k \frac{1+q^{k+2}}{(1-q^{n+k+1})(1-q^{n+k+3})} \tag{3.1}$$
$$= q^{\binom{n}{2}+1} (-1)^n \frac{1-q^{n+2}}{1-q^2} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q^{-1} \left[\frac{1+q^{n-1}}{(1-q^{n+1})(1-q^{n+2})} - \frac{1-q^{n+2}}{1-q^{2n+3}} \right].$$

Consider the LHS of Eq. (3.1) as

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}+k(1-n)} (-1)^k \frac{1+q^{k+2}}{(1-q^{n+k+1})(1-q^{n+k+3})}$$
$$= S(n;1,q^{n+1},q^2,1) + q^2 S(n;2,q^{n+1},q^2,1).$$

By Theorem 2.1, we have that

$$S(n;1,q^{n+1},q^2,1) = q^{\binom{n+1}{2}} (-1)^n \left[\frac{(q;q)_n}{(q^{n+1};q)_{n+1}(1-q^2)} - q^2 \frac{(q^3;q)_n}{(q^{n+3};q)_{n+1}(1-q^2)} \right]$$

and

$$S(n;2,q^{n+1},q^2,1) = q^{\binom{n}{2}-1} (-1)^n \left[\frac{(q;q)_n}{(q^{n+1};q)_{n+1}(1-q^2)} - \frac{(q^3;q)_n}{(q^{n+3};q)_{n+1}(1-q^2)} \right].$$

The claim follows now after some simplifications. □

Corollary 3.3 *For* $n \geq 1$,

$$\sum_{k=0}^{n} \left\{ {n+k \atop k} \right\}_U \left\{ {n \atop k} \right\}_U \alpha^{-k} (-1)^{nk+\binom{k+1}{2}} U_k U_{k+2} = \Delta^{-1/2} (-1)^{\binom{n}{2}} \left[U_{n^2+n}\alpha^2 - (-1)^n \beta^{n^2+n+2} U_n U_{n+1} \right].$$

Proof If we convert the claim into q-form, then we have to prove that

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}-nk} (1-q^k-q^{k+2}+q^{2k+2}) \tag{3.2}$$
$$= (-1)^n \left[q^{-\binom{n+1}{2}} (1-q^{n^2+n}) - q^{\binom{n}{2}+2} \frac{(1-q^n)(1-q^{n+1})}{1-q} \right].$$

Represent the LHS as

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}-nk} (1-(1+q^2)q^k+q^{2k+2}) = T_0 - (1+q^2)T_1 + q^2 T_2,$$

with

$$T_j=\sum_{k=0}^{n}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}+(j-n)k},$$

for $0\leq j\leq 2$. By Theorem 2.2 and Corollary 2.5, we get

$$T_1=(-1)^n q^{-\binom{n+1}{2}},\quad T_2=(-1)^n q^{\binom{n+1}{2}}$$

and

$$T_3=(-1)^n q^{\binom{n}{2}}\left[\frac{1-q^n}{1-q}-q^n\frac{1-q^{n+1}}{1-q}\right].$$

After some simplifications, we obtain

$$T_0-(1+q^2)T_1+q^2T_2=(-1)^n q^{-\binom{n+1}{2}}(1-q^{n^2+n})-(-1)^n q^{\binom{n}{2}+2}\frac{(1-q^n)(1-q^{n+1})}{1-q},$$

which equals the RHS of Eq. (3.2). Thus, we have the conclusion. □

Corollary 3.4 *For $n\geq 3$,*

$$\sum_{k=0}^{n}\begin{Bmatrix}n+k\\k\end{Bmatrix}_U\begin{Bmatrix}n\\k\end{Bmatrix}_U(-1)^{nk+\binom{k+1}{2}}\frac{1}{U_{k+1}U_{k+2}U_{k+3}}=0.$$

Proof If we convert the LHS of the claim into q-form, we get

$$\frac{(1-q)^3}{\alpha^3}\sum_{k=0}^{n}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}+(2-n)k}\frac{1}{(q^{k+1};q)_3},$$

which, by Corollary 2.4, equals

$$\frac{(1-q)^3}{\alpha^3}q^{\binom{n+1}{2}}(-1)^n\sum_{k=0}^{2}(-1)^k q^{\binom{k}{2}}\frac{(q^{k+1-n};q)_n}{(q^{k+1};q)_{n+1}(q;q)_k(q;q)_{2-k}},$$

which, since $(q^{k+1-n};q)_n=0$ for $k=0,1,2$ and $n\geq 3$, equals 0, as claimed. □

Now we recall an auxiliary result from [15], for which we will prove a q-analogue.

Corollary 3.5 *For $0\leq m\leq n$,*

$$\sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k}(-1)^{n-k}\frac{m+k}{j(j+m+k)}=\frac{1}{j}-\frac{(j+m-1)!^2}{(j+m-n-1)!(j+m+n)!}.$$

Proof Consider the sum

$$s_n=\sum_{k=0}^{n}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{n-k}\frac{1-q^{m+k}}{1-q^{j+m+k}}=(-1)^n\big[S(n;0,q^{j+m},.,0)-q^mS(n;1,q^{j+m},.,0)\big].$$

Since $\lim_{q\to 1}\frac{1-q^n}{1-q}=n$, we observe that

$$\frac{1}{j}\lim_{q\to 1} s_n=\sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k}(-1)^{n-k}\frac{m+k}{j(j+m+k)}.$$

By Theorems 2.1 and 2.2,

$$\begin{aligned}
s_n &= (-1)^n\left[S(n;0,q^{j+m},.,0)-q^m S(n;1,q^{j+m},.,0)\right]\\
&= q^{-\binom{n+1}{2}}+q^{\binom{n+1}{2}+j+m}\frac{(q^{j+m-n};q)_n}{(q^{j+m};q)_{n+1}}-q^{\binom{n+1}{2}+m}\frac{(q^{j+m-n};q)_n}{(q^{j+m};q)_{n+1}}\\
&= q^{-\binom{n+1}{2}}-q^{\binom{n+1}{2}+m}\frac{(q^{j+m-n};q)_n}{(q^{j+m};q)_{n+1}}(1-q^j)\\
&= q^{-\binom{n+1}{2}}-q^{\binom{n+1}{2}+m}\frac{(q;q)^2_{j+m-1}}{(q;q)_{j+m-n-1}(q;q)_{j+m+n}}(1-q^j).
\end{aligned}$$

Multiplying this by $\frac{1}{j}$ and then performing the limit $q\to 1$, we finally have

$$\frac{1}{j}\lim_{q\to 1} s_n=\frac{1}{j}-\frac{(j+m-1)!^2}{(j+m-n-1)!(j+m+n)!},$$

and so the proof is complete. □

The above example is a prototype of how to deduce binomial sum identities from our main results by performing the limit $q\to 1$.

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