Submanifolds of almost poly-Norden Riemannian manifolds*

Selcen YÜKSEL PERKTAŞ[†]
Department of Mathematics, Faculty of Arts and Science, Adıyaman University, Adıyaman, Turkey

Abstract: Our aim in the present paper is to initiate the study of submanifolds in an almost poly-Norden Riemannian manifold, which is a new type of manifold first introduced by Şahin [17]. We give fundamental properties of submanifolds equipped with induced structures provided by almost poly-Norden Riemannian structures and find some conditions for such submanifolds to be totally geodesics. We introduce some subclasses of submanifolds in almost poly-Norden Riemannian manifolds such as invariant and antiinvariant submanifolds. We investigate conditions for a hypersurface of almost poly-Norden Riemannian manifolds to be invariant and totally geodesic, respectively, by using the components of the structure induced by the almost poly-Norden Riemannian structure of the ambient manifold. We also obtain some characterizations for totally umbilical hypersurfaces and give some examples of invariant and noninvariant hypersurfaces.

Key words: Bronze mean, poly-Norden structure, poly-Norden manifold, invariant submanifold, antiinvariant submanifold

1. Introduction

In Riemannian (as well as semi-Riemannian) manifolds, different geometric structures such as almost complex structures, almost product structures, almost contact structures, and almost paracontact structures allow significant results to emerge while investigating differential and geometric properties of submanifolds.

As a generalization of the golden mean, the number $\phi = \frac{1+\sqrt{5}}{2} = 1.618...$ is known as a solution of the equation $x^2 - x - 1 = 0$, Spinadel introduced a family of metallic proportions in [5]. The positive solutions of the equation $x^2 - px - q = 0$ create members of the metallic proportions (or means) family, which are called (p, q) metallic numbers and denoted by

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}. \tag{1.1}$$

The well-known members of the metallic means family are the golden mean, the silver mean, the bronze mean, the copper mean, etc. The members of the metallic means family have important mathematical properties in physics and art.

In recent years, inspired by the golden mean and the metallic mean, the golden structure on a Riemannian manifold and metallic Riemannian manifolds were introduced in [4] and [14], respectively. Golden Riemannian

---

manifolds, which can be viewed as one of the most important subclasses of metallic Riemannian manifolds, and their submanifolds have been studied extensively by many geometers (see [7–9, 12]). Poyraz Önen and Yaşar [16] initiated the study of lightlike geometry in golden semi-Riemannian manifolds by investigating lightlike hypersurfaces of a golden semi-Riemannian manifold. Since the metallic structure on the ambient Riemannian (or semi-Riemannian) manifold provides more general geometric results than the consequences provided by the golden structure on submanifolds, the metallic Riemannian (as well as semi-Riemannian) manifolds have been studied by many authors. Invariant, antiinvariant, semiinvariant, slant, and semislant submanifolds of a metallic Riemannian manifold were studied in [3, 10, 11]. Some special types of lightlike submanifolds on a metallic semi-Riemannian manifold were introduced in [1, 6].

In 2011, by a different approach, Kalia [15] introduced a new bronze mean and studied bronze Fibonacci and Lucas numbers. The author showed the relationship between the convergents of continued fractions of the power of bronze means and the bronze Fibonacci and Lucas numbers. Note that, unlike the bronze mean contained by the metallic means family defined in [5], that new bronze mean given by Kalia [15] cannot be expressed with $\sigma_{p,q}$ for any positive integers p and q.

Considering the study on a Riemannian manifold with the golden structure [4] and the bronze mean introduced by [15], Şahin in [17] defined a new type of manifold equipped with the bronze structure and named it an almost poly-Norden manifold. He gave some important geometric results and investigated the constancy of certain maps.

In the present paper, we initiate the study of submanifolds in almost poly-Norden Riemannian manifolds.

2. Preliminaries
The positive solution of

$$x^2 - mx + 1 = 0$$

is called the bronze mean [15], which is defined by

$$B_m = \frac{m + \sqrt{m^2 - 4}}{2}. \tag{2.1}$$

A family of sequences $(f_{m,n})$ given by the recurrence

$$f_{m,n+2} = mf_{m,n+1} - f_{m,n},$$

where $f_{m,0} = 0$ and $f_{m,1} = 1$, is called the bronze Fibonacci numbers. The bronze Lucas numbers denoted by $(l_{m,n})$ are a sequence family characterized by

$$l_{m,n+2} = ml_{m,n+1} - l_{m,n},$$

where $l_{m,0} = 2$ and $l_{m,1} = m$. For the bronze means, the continued fractions are $\{m - 1; \overline{1, m - 2}\}$. Also, note that $B_m^{n+2} = mB_m^{n+1} - B_m^n$ and the bronze Fibonacci numbers and the bronze Lucas numbers are related by

$$B_m^n = \frac{l_{m,n} + f_{m,n}\sqrt{m^2 - 4}}{2}.$$

For further reading, we refer to [15].

In [17], by using the bronze mean given in (2.1), the author defined a new type of differentiable manifold equipped with a bronze structure. Let $\tilde{M}$ be a smooth manifold. If a tensor field $\tilde{\Phi}$ of type $(1,1)$ satisfies

$$\tilde{\Phi}^2 = m\tilde{\Phi} - I, \tag{2.2}$$

then $\tilde{\Phi}$ is said to be a poly-Norden structure on $\tilde{M}$. Then we say that $\tilde{M}$ is an almost poly-Norden manifold equipped with a poly-Norden structure $\tilde{\Phi}$. An almost poly-Norden structure $\tilde{\Phi}$ is an isomorphism on a tangent space of $\tilde{M}$, which has eigenvalues $\frac{m+\sqrt{m^2-4}}{2}$ and $\frac{m-\sqrt{m^2-4}}{2}$.

Also, if $(\tilde{M}, \tilde{g})$ is a semi-Riemannian manifold endowed with a poly-Norden structure $\tilde{\Phi}$ such that the semi-Riemannian metric $\tilde{g}$ is $\tilde{\Phi}$-compatible, i.e.

$$\tilde{g}(\tilde{\Phi}\tilde{X}, \tilde{\Phi}\tilde{Y}) = m\tilde{g}(\tilde{\Phi}\tilde{X}, \tilde{Y}) - \tilde{g}(\tilde{X}, \tilde{Y}), \tag{2.3}$$

equivalent to

$$\tilde{g}(\tilde{\Phi}\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{\Phi}\tilde{Y}), \tag{2.4}$$

for every $\tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M})$, then $(\tilde{\Phi}, \tilde{g})$ is called an almost poly-Norden semi-Riemannian structure and $(\tilde{M}, \tilde{\Phi}, \tilde{g})$ is named an almost poly-Norden semi-Riemannian manifold [17].

Proposition 2.1 *[17] Every complex structure $\tilde{J}$ on a semi-Riemannian manifold induces two poly-Norden structures on $\tilde{M}$ given by*

$$\tilde{\Phi}_1 = \frac{m}{2}I + \frac{\sqrt{4-m^2}}{2}\tilde{J}, \quad \tilde{\Phi}_2 = \frac{m}{2}I - \frac{\sqrt{4-m^2}}{2}\tilde{J}, \quad -2 < m < 2.$$

Conversely, every poly-Norden structure $\tilde{\Phi}$ on $\tilde{M}$ induces two almost complex structures on this manifold given as follows:

$$\tilde{J}_1 = -\frac{m}{\sqrt{4-m^2}}I + \frac{2}{\sqrt{4-m^2}}\tilde{\Phi}, \quad \tilde{J}_2 = \frac{m}{\sqrt{4-m^2}}I - \frac{2}{\sqrt{4-m^2}}\tilde{\Phi}, \quad -2 < m < 2.$$

Definition 2.2 *[17] An almost poly-Norden semi-Riemannian manifold $(\tilde{M}, \tilde{\Phi}, \tilde{g})$ is called a poly-Norden semi-Riemannian manifold if the almost poly-Norden structure $\tilde{\Phi}$ is parallel with respect to Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$.*

Definition 2.3 *[17] An almost poly-Norden structure $\tilde{\Phi}$ is called integrable if its Nijenhuis tensor field $N_{\tilde{\Phi}}(\tilde{X}, \tilde{Y}) := [\tilde{\Phi}\tilde{X}, \tilde{\Phi}\tilde{Y}] - \tilde{\Phi}[\tilde{\Phi}\tilde{X}, \tilde{Y}] - \tilde{\Phi}[\tilde{X}, \tilde{\Phi}\tilde{Y}] + \tilde{\Phi}^2[\tilde{X}, \tilde{Y}]$ vanishes.*

Note that $N_{\tilde{\Phi}} = 0$ is equivalent to $\tilde{\nabla}\tilde{\Phi} = 0$. It was shown that every almost Norden manifold is an almost poly-Norden manifold with $m = 0$ [17]. Throughout the paper we will consider that $m \neq 0$.

3. Submanifolds of almost poly-Norden Riemannian manifolds

Let M be an n-dimensional submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. We denote the induced Riemannian metric on M by g. For any $X \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$, we put

$$\tilde{\Phi}X = fX + wX, \tag{3.1}$$

$$\tilde{\Phi}U = BU + CU, \tag{3.2}$$

where fX (resp. wX) is the tangential (resp. normal) part of $\tilde{\Phi}X$ and BU (resp. CU) is the tangential (resp. normal) part of $\tilde{\Phi}U$.

From (2.4) and (3.1)-(3.2) one can easily see that

$$g(fX, Y) = g(X, fY), \quad \forall X, Y \in \Gamma(TM), \tag{3.3}$$

$$g(CU, V) = g(U, CV), \quad \forall U, V \in \Gamma(TM^{\perp}). \tag{3.4}$$

Also, the maps w and B are related by

$$g(wX, U) = g(X, BU).$$

The Levi-Civita connections on M and $\tilde{M}$ will be denoted by ∇ and $\tilde{\nabla}$, respectively. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_{\beta=1}^{k} h_\beta(X, Y) N_\beta, \tag{3.5}$$

$$\tilde{\nabla}_X N_\beta = -A_{N_\beta} X + \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X) N_\gamma, \tag{3.6}$$

for any $X, Y \in \Gamma(TM)$ and an orthonormal basis $\{N_1, N_2, ..., N_k\}$ of $TM^{\perp}$, where $\beta, \gamma \in \{1, 2, ..., k\}$. Here, h_β $(1 \leq \beta \leq k)$ are the second fundamental tensors that provide identification of the second fundamental form h of M such that $h(X, Y) = \sum_{\beta=1}^{k} h_\beta(X, Y) N_\beta$ and A_{N_β} denotes the shape operator in the direction of N_β given by $g(A_{N_\beta} X, Y) = h_\beta(X, Y)$. Also, $\sigma_{\beta\gamma}$ $(1 \leq \beta, \gamma \leq k)$ denotes the 1-forms on the submanifold M that satisfy $\nabla_X^{\perp} N_\beta = \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X) N_\gamma$. Note that if we take the covariant derivative of $g(N_\beta, N_\gamma) = \delta_{\beta\gamma}$ on M, then we have $\sigma_{\beta\gamma} = -\sigma_{\gamma\beta}$.

Note that a submanifold M is called:

i) totally geodesic if h (or equivalently the shape operator A) vanishes,

ii) minimal if the mean curvature vector H defined by $H = \frac{1}{n} trace\, h$ of the submanifold vanishes, and

iii) totally umbilical if $A = aI$, for some function a.

From (2.4) we give following.

Lemma 3.1 *Let $(\tilde{M}, \tilde{\Phi}, g)$ be an almost poly-Norden Riemannian manifold. Then we have*

$$g((\tilde{\nabla}_{\tilde{X}} \tilde{\Phi})\tilde{Y}, \tilde{Z}) = g(\tilde{Y}, (\tilde{\nabla}_{\tilde{X}} \tilde{\Phi})\tilde{Z}), \quad \forall \tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(T\tilde{M}). \tag{3.7}$$

Proposition 3.2 *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then we have*

$$g((\nabla_X f)Y, Z) = g(Y, (\nabla_X f)Z), \tag{3.8}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof From (3.5) and (3.6) and by using equations (3.1) and (3.2), we write

$$
\begin{aligned}
(\tilde{\nabla}_X \tilde{\Phi})Y &= \tilde{\nabla}_X \tilde{\Phi}Y - \tilde{\Phi}(\tilde{\nabla}_X Y) \\[2mm]
&= \tilde{\nabla}_X fY + \tilde{\nabla}_X wY - \tilde{\Phi}\nabla_X Y - \sum_{\beta=1}^{k} h_\beta(X,Y)\tilde{\Phi}N_\beta \\[2mm]
&= (\nabla_X f)Y - \sum_{\beta=1}^{k} g(wY, N_\beta)A_{N_\beta} X - \sum_{\beta=1}^{k} h_\beta(X,Y)BN_\beta \\[2mm]
&\quad + \sum_{\beta=1}^{k} \left\{ h_\beta(X, fY) + X\left(g(wY, N_\beta)\right) - g(wY, N_\gamma)\sigma_{\beta\gamma}(X) \right\} N_\beta \\[2mm]
&\quad - \sum_{\beta=1}^{k} h_\beta(X,Y)CN_\beta - w\nabla_X Y.
\end{aligned}
\tag{3.9}
$$

If we use the fact $g(wX, N_\beta) = g(X, \tilde{\Phi}N_\beta)$, for $1 \le \beta \le k$, the last equation above reduces to

$$g((\tilde{\nabla}_X \tilde{\Phi})Y, Z) = g((\nabla_X f)Y, Z) - \sum_{\beta=1}^{k} \left\{ \begin{array}{c} h_\beta(X, Z)g(Y, \tilde{\Phi}N_\beta) \\ +h_\beta(X, Y)g(Z, \tilde{\Phi}N_\beta) \end{array} \right\}. \tag{3.10}$$

By interchanging the roles of Y and Z in (3.10) and using (3.8) we complete the proof. $\quad\square$

Consider that $(\tilde{M}, \tilde{\Phi}, g)$ is an $(n+k)$-dimensional poly-Norden Riemannian manifold and M is a submanifold of codimension k in $\tilde{M}$. For any $X, Y \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X wY = \sum_{\beta=1}^{k} \left\{ \begin{array}{c} X\left(g(wY, N_\beta)\right)N_\beta - g(wY, N_\beta)A_{N_\beta} X \\ +\sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X)g(wY, N_\beta)N_\gamma \end{array} \right\}. \tag{3.11}$$

Since $\tilde{M}$ is a poly-Norden Riemannian manifold, i.e. $\tilde{\nabla}\tilde{\Phi} = 0$, then by using (3.1), (3.2), (3.5), and (3.6), we calculate

$$
\begin{array}{c}
\nabla_X fY + \sum_{\beta=1}^{k} \left\{ \begin{array}{c} h_\beta(X, fY)N_\beta \\ -g(wY, N_\beta)A_{N_\beta} X \end{array} \right\} \\
+\sum_{\beta=1}^{k} \left\{ \begin{array}{c} X\left(g(wY, N_\beta)\right)N_\beta \\ -\sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X)g(wY, N_\gamma)N_\beta \end{array} \right\}
\end{array}
=
\begin{array}{c}
f\nabla_X Y + w\nabla_X Y \\
+\sum_{\beta=1}^{k} \left\{ \begin{array}{c} h_\beta(X, Y)BN_\beta \\ +h_\beta(X, Y)CN_\beta \end{array} \right\}.
\end{array}
\tag{3.12}
$$

On the other hand, for any $X \in \Gamma(TM)$ and $U \in \Gamma(TM^\perp)$, we write

$$\tilde{\nabla}_X U = \sum_{\beta=1}^{k} \left\{ \begin{array}{c} X\left(g(U, N_\beta)\right) N_\beta - g(U, N_\beta) A_{N_\beta} X \\ + \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X) g(U, N_\beta) N_\gamma \end{array} \right\}. \tag{3.13}$$

By applying $\tilde{\Phi}$ to both sides of (3.13) and using the fact that $\tilde{M}$ is a poly-Norden Riemannian manifold, we obtain

$$\begin{array}{c} \nabla_X BU + \sum_{\beta=1}^{k} \left\{ \begin{array}{c} h_\beta(X, BU) \\ + X(g(CU, N_\beta)) \end{array} \right\} N_\beta \\ + \sum_{\beta=1}^{k} \left\{ \begin{array}{c} -g(CU, N_\beta) A_{N_\beta} X \\ + \sum_{\gamma=1}^{k} g(CU, N_\beta) \sigma_{\beta\gamma}(X) N_\gamma \end{array} \right\} \end{array} = \begin{array}{c} \sum_{\beta=1}^{k} \left\{ \begin{array}{c} X(g(U, N_\beta)) BN_\beta \\ + X(g(U, N_\beta)) CN_\beta \\ - g(U, N_\beta) f A_{N_\beta} X \\ - g(U, N_\beta) w A_{N_\beta} X \end{array} \right\} \\ \sum_{\beta,\gamma=1}^{k} \left\{ \begin{array}{c} g(U, N_\beta) \sigma_{\beta\gamma}(X) BN_\gamma \\ + g(U, N_\beta) \sigma_{\beta\gamma}(X) CN_\gamma \end{array} \right\} \end{array}, \tag{3.14}$$

via (3.1), (3.2), (3.5), and (3.6).

In view of (3.12) and (3.14), we get:

Proposition 3.3 *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. In this case, the followings hold:*

$$(\nabla_X f) Y = \sum_{\beta=1}^{k} \left\{ g(wY, N_\beta) A_{N_\beta} X + h_\beta(X, Y) BN_\beta \right\},$$

$$\sum_{\beta=1}^{k} g(wY, N_\beta) \nabla_X^\perp N_\beta = w \nabla_X Y + \sum_{\beta=1}^{k} \left\{ \begin{array}{c} h_\beta(X, Y) CN_\beta \\ - h_\beta(X, fY) N_\beta \\ - X\left(g(wY, N_\beta)\right) N_\beta \end{array} \right\},$$

$$\nabla_X BU = \sum_{\beta=1}^{k} \left\{ \begin{array}{c} g(CU, N_\beta) A_{N_\beta} X - g(U, N_\beta) f(A_{N_\beta} X) \\ + X(g(U, N_\beta)) BN_\beta \\ + \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X) g(U, N_\beta) BN_\gamma \end{array} \right\},$$

$$\sum_{\beta=1}^{k} g(CU, N_\beta) \nabla_X^\perp N_\beta = \sum_{\beta=1}^{k} \left\{ \begin{array}{c} - \left[X(g(CU, N_\beta)) + h_\beta(X, BU) \right] N_\beta \\ - g(U, N_\beta) w\left(A_{N_\beta} X \right) \\ + X(g(U, N_\beta)) CN_\beta \\ + \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X) g(U, N_\beta) CN_\gamma \end{array} \right\}.$$

Let $\{N_1, N_2, ..., N_k\}$ be an orthonormal basis of the normal space $TM^\perp$ of an n-dimensional submanifold M in an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then, for any $X \in \Gamma(TM)$, $\tilde{\Phi}X$ and $\tilde{\Phi}N_\beta$ $(1 \le \beta \le k)$ can be written respectively in the following forms:

$$\tilde{\Phi}X = fX + \sum_{\beta=1}^{k} \upsilon_\beta(X) N_\beta, \tag{3.15}$$

$$\tilde{\Phi} N_\beta = \zeta_\beta + \sum_{\gamma=1}^{k} \theta_{\beta\gamma} N_\gamma, \tag{3.16}$$

where f is a tensor field of type $(1,1)$ on M, which transforms tangent vector field X on M to the tangential component of $\tilde{\Phi} X$, while v_β are real 1-forms and ζ_β vector fields on M. Here, $\theta_{\beta\gamma}$ are differentiable real valued functions on the submanifold M provided a $k \times k$ matrix denoted by $(\theta_{\beta\gamma})_{1\le\beta,\gamma\le k}$.

Since $g(\tilde{\Phi} X, N_\beta) = g(X, \tilde{\Phi} N_\beta)$ and $g(\tilde{\Phi} N_\beta, N_\gamma) = g(N_\beta, \tilde{\Phi} N_\gamma)$, then by using (2.3) and (3.3), we have:

Lemma 3.4 *In a submanifold M of an almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$, we have*

$$v_\beta(X) = g(\tilde{\Phi} X, N_\beta) = g(X, \zeta_\beta), \tag{3.17}$$

$$g(fX, fY) = mg(X, fY) - g(X, Y) - \sum_{\beta=1}^{k} v_\beta(X) v_\beta(Y), \tag{3.18}$$

$$\theta_{\beta\gamma} = \theta_{\gamma\beta}, \tag{3.19}$$

for all $X, Y \in \Gamma(TM)$ where $1 \le \beta, \gamma \le k$.

Proposition 3.5 *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then there exists a structure $(f, g, v_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$ on M induced by the almost poly-Norden structure of $\tilde{M}$, which satisfies*

$$f^2 X = mfX - X - \sum_{\beta=1}^{k} v_\beta(X) \zeta_\beta, \tag{3.20}$$

$$v_\beta(fX) = mv_\beta(X) - \sum_{\gamma=1}^{k} \theta_{\beta\gamma} v_\gamma(X), \tag{3.21}$$

$$v_\gamma(\zeta_\beta) = m\theta_{\beta\gamma} - \delta_{\beta\gamma} - \sum_{\lambda=1}^{k} \theta_{\beta\lambda} \theta_{\lambda\gamma}, \tag{3.22}$$

$$f \zeta_\beta = m\zeta_\beta - \sum_{\gamma=1}^{k} \theta_{\beta\gamma} \zeta_\gamma, \tag{3.23}$$

for all $X \in \Gamma(TM)$.

Proof Applying $\tilde{\Phi}$ to both sides of equation (3.15) and using (2.2), we have

$$m\tilde{\Phi} X - X = \tilde{\Phi}(fX) + \sum_{\beta=1}^{k} v_\beta(X) \tilde{\Phi} N_\beta,$$

which implies

$$
\begin{aligned}
mfX + m\sum_{\beta=1}^{k} v_\beta(X)\,N_\beta - X \;=\;& f^2 X + \sum_{\beta=1}^{k} v_\beta(fX)\,N_\beta \\
& + \sum_{\beta=1}^{k} v_\beta(X)\left(\zeta_\beta + \sum_{\gamma=1}^{k}\theta_{\beta\gamma}\,N_\gamma\right).
\end{aligned}
$$

If we equate the tangential and the normal parts of the last equation we obtain (3.20) and (3.21), respectively. From (3.16) we write

$$
g(\tilde{\Phi}N_\beta, \tilde{\Phi}N_\gamma) = g(\zeta_\beta, \zeta_\gamma) + \sum_{\lambda=1}^{k}\theta_{\beta\lambda}\theta_{\gamma\lambda}. \tag{3.24}
$$

Also, from (2.3), we have

$$
g(\tilde{\Phi}N_\beta, \tilde{\Phi}N_\gamma) = m\theta_{\beta\gamma} - \delta_{\beta\gamma}. \tag{3.25}
$$

Then, by using (3.19), (3.24), and (3.25) we obtain (3.22). Finally, taking $X = \zeta_\beta$ in (3.21) gives (3.23) and we complete the proof. $\qquad\square$

Proposition 3.6 *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then the following equations hold:*

$$
f\,A_{N_\beta}X + \nabla_X\,\zeta_\beta - \sum_{\gamma=1}^{k}\theta_{\beta\gamma}A_{N_\gamma}X - \sum_{\gamma=1}^{k}\sigma_{\beta\gamma}(X)\zeta_\gamma \;=\; 0,
$$

$$
X(\theta_{\beta\lambda}) + h_\lambda(X,\zeta_\beta) + h_\beta(X,\zeta_\lambda) + \sum_{\gamma=1}^{k}\left(\theta_{\beta\gamma}\sigma_{\gamma\lambda}(X) - \theta_{\gamma\lambda}\sigma_{\beta\gamma}(X)\right) \;=\; 0.
$$

Proof From (3.5), (3.6), (3.15), and (3.16), we calculate

$$
\begin{aligned}
(\tilde{\nabla}_X\tilde{\Phi})N_\beta \;=\;& \tilde{\nabla}_X\tilde{\Phi}N_\beta - \tilde{\Phi}\left(\tilde{\nabla}_X N_\beta\right) \\[2mm]
=\;& \tilde{\nabla}_X\left(\zeta_\beta + \sum_{\gamma=1}^{k}\theta_{\beta\gamma}\,N_\gamma\right) - \tilde{\Phi}\left(-A_{N_\beta}X + \sum_{\gamma=1}^{k}\sigma_{\beta\gamma}(X)N_\gamma\right) \\[2mm]
=\;& \nabla_X\zeta_\beta + \sum_{\gamma=1}^{k}X(\theta_{\beta\gamma})N_\gamma + \sum_{\gamma=1}^{k}\theta_{\beta\gamma}\left(-A_{N_\gamma}X + \sum_{\lambda=1}^{k}\sigma_{\gamma\lambda}(X)N_\lambda\right) \\[2mm]
& + f\,A_{N_\beta}X + w\,A_{N_\beta}X - \sum_{\gamma=1}^{k}\sigma_{\beta\gamma}(X)\left(\zeta_\gamma + \sum_{\lambda=1}^{k}\theta_{\gamma\lambda}\,N_\lambda\right) \\[2mm]
=\;& f\,A_{N_\beta}X + \nabla_X\zeta_\beta - \sum_{\gamma=1}^{k}\theta_{\beta\gamma}A_{N_\gamma}X - \sum_{\gamma=1}^{k}\sigma_{\beta\gamma}(X)\zeta_\gamma \\[2mm]
& + \sum_{\gamma=1}^{k}X(\theta_{\beta\gamma})N_\gamma + \sum_{\gamma,\lambda=1}^{k}\theta_{\beta\gamma}\sigma_{\gamma\lambda}(X)N_\lambda + w\,A_{N_\beta}X - \sum_{\gamma,\lambda=1}^{k}\sigma_{\beta\gamma}(X)\theta_{\gamma\lambda}\,N_\lambda.
\end{aligned}
$$

Since $\tilde{M}$ is a poly-Norden manifold, i.e. $(\tilde{\nabla}_X \tilde{\Phi})N_\beta = 0$ for all $X \in \Gamma(TM)$, then by equating the tangential and the normal parts of the last equation above we complete the proof. $\qquad \square$

Theorem 3.7 *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. If ζ_β $(1 \le \beta \le k)$ are linearly independent and f is parallel with respect to the Levi-Civita connection on M, then M is totally geodesic.*

Proof From the first equation given in Proposition 3.3, we can write

$$\sum_{\beta=1}^{k} \left\{ g(wY, N_\beta)g(A_{N_\beta} X, Z) + h_\beta(X, Y)g(BN_\beta, Z) \right\} = 0,$$

for all $Z \in \Gamma(TM)$, which implies

$$\sum_{\beta=1}^{k} \upsilon_\beta(Y)h_\beta(X, Z) = -\sum_{\beta=1}^{k} \upsilon_\beta(Z)h_\beta(X, Y),$$

via (3.17). If we change the roles of X and Z in the last equation, we get

$$\sum_{\beta=1}^{k} \upsilon_\beta(Y)h_\beta(X, Z) = -\sum_{\beta=1}^{k} \upsilon_\beta(X)h_\beta(Y, Z).$$

Hence, $\sum_{\beta=1}^{k} \upsilon_\beta(Y)h_\beta(X, Z)$ is symmetric and also skew-symmetric in X, Y. Then we get

$$\sum_{\beta=1}^{k} \upsilon_\beta(Y)h_\beta(X, Z) = 0,$$

which implies

$$\sum_{\beta=1}^{k} g(Y, h_\beta(X, Z)BN_\beta) = 0.$$

Since $BN_\beta = \zeta_\beta$ $(1 \le \beta \le k)$ are linearly independent, we complete the proof. $\qquad \square$

3.1. Invariant submanifolds of almost poly-Norden Riemannian manifolds

Definition 3.8 *Let M be a submanifold of an almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. If $\tilde{\Phi}(T_pM) \subset T_pM$ for any point $p \in M$ then M is called an invariant submanifold.*

Assume that M is an n-dimensional invariant submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then from (3.15) we have $\upsilon_\beta = 0$ (equivalently, $\zeta_\beta = 0$) for $1 \le \beta \le k$. The converse of the last statement also holds.

Proposition 3.9 *Let M be an n-dimensional invariant submanifold of an $(n + k)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then the matrix $\Theta = (\theta_{\beta\gamma})_{k \times k}$ of the induced structure $(f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k \times k})$ is an almost poly-Norden matrix; that is,*

$$\Theta^2 = m\Theta - I_k,$$

where I_k denotes the unit matrix of order k.

Proof Since M is an n-dimensional invariant submanifold, then from (3.22) we have

$$\sum_{\lambda=1}^{k} \theta_{\beta\lambda}\theta_{\lambda\gamma} = m\theta_{\beta\gamma} - \delta_{\beta\gamma},$$

for $1 \leq \beta, \gamma \leq k$. If we denote the matrix $(\theta_{\beta\gamma})_{k \times k}$ by Θ, we complete the proof. □

Proposition 3.10 *Let $(f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k \times k})$ be the induced structure on an n-dimensional submanifold M of an $(n + k)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then M is an invariant submanifold if and only if the induced structure (f, g) on M is an almost poly-Norden Riemannian structure.*

Proof Assume that M is an invariant submanifold. Since $\upsilon_\beta = 0$ (equivalently, $\zeta_\beta = 0$) for $1 \leq \beta \leq k$, then from (3.18) and (3.20), we see that

$$\begin{aligned} f^2 X &= mfX - X, \\ g(fX, fY) &= mg(X, fY) - g(X, Y), \end{aligned}$$

for all $X, Y \in \Gamma(TM)$, which imply that (f, g) is an almost poly-Norden Riemannian structure on M.

Conversely, if (f, g) is an almost poly-Norden Riemannian structure on M, then from (3.20) we write

$$\sum_{\beta=1}^{k} \upsilon_\beta(X)\zeta_\beta = 0,$$

for all $X \in \Gamma(TM)$, which shows

$$\upsilon_\beta(X) = 0.$$

Hence, M is an invariant submanifold. This completes the proof. □

Example 3.11 *Let R^4 be the 4-dimensional real number space with a coordinate system (x_1, x_2, y_1, y_2). We define*

$$\begin{aligned} \tilde{\Phi}: \quad R^4 &\to R^4 \\ (x_1, x_2, y_1, y_2) &\to \tilde{\Phi}(x_1, x_2, y_1, y_2) = (B_m x_1, B_m x_2, (m - B_m) y_1, (m - B_m) y_2), \end{aligned}$$

where $B_m = \frac{m+\sqrt{m^2-4}}{2}$. Then $(R^4, \tilde{\Phi})$ is an almost poly-Norden manifold [17]. If we consider the usual scalar product $\langle,\rangle$ on R^4, then we see that it is $\tilde{\Phi}$-compatible and $(R^4, \tilde{\Phi}, \langle,\rangle)$ is an almost poly-Norden Riemannian manifold.

Now assume that M is a submanifold of $(R^4, \tilde{\Phi}, \langle,\rangle)$ defined by

$$x_1 = x_2, \quad y_1 = y_2.$$

In this case $\Gamma(TM) = Span\{X, Y\}$, where

$$X_1 = (1, 1, 0, 0), \quad X_2 = (0, 0, 1, 1).$$

It is easy to see that $\tilde{\Phi}X = B_m X$ and $\tilde{\Phi}Y = (m - B_m)Y$, which imply that M is an invariant submanifold of $(R^4, \tilde{\Phi}, \langle,\rangle)$.

3.2. Antiinvariant submanifolds of almost poly-Norden Riemannian manifolds

Definition 3.12 *Let M be a submanifold of an almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. If $\tilde{\Phi}(T_pM) \subset (T_pM)^\perp$ for any point $p \in M$ then M is called an antiinvariant submanifold.*

Assume that M is an n-dimensional antiinvariant submanifold of an $(n + k)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. In this case, from (3.15) and (3.16), for any $X \in \Gamma(TM)$, $\tilde{\Phi}X$ and $\tilde{\Phi}N_\beta$ $(1 \le \beta \le k)$ can be written respectively in the following forms:

$$\tilde{\Phi}X = \sum_{\beta=1}^{k} \upsilon_\beta(X) N_\beta, \tag{3.26}$$

$$\tilde{\Phi}N_\beta = \zeta_\beta + \sum_{\gamma=1}^{k} \theta_{\beta\gamma} N_\gamma. \tag{3.27}$$

As a consequence of Proposition 3.3, we have:

Proposition 3.13 *Let M be an n-dimensional antiinvariant submanifold of an $(n + k)$-dimensional poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. In this case, the followings hold:*

$$\sum_{\beta=1}^{k} g(\tilde{\Phi}Y, N_\beta) A_{N_\beta} X = -\sum_{\beta=1}^{k} h_\beta(X, Y) BN_\beta,$$

$$\sum_{\beta=1}^{k} g(\tilde{\Phi}Y, N_\beta) \nabla_X^\perp N_\beta = w\nabla_X Y + \sum_{\beta=1}^{k} \left\{ \begin{array}{c} h_\beta(X, Y) CN_\beta \\ -X(g(wY, N_\beta)) N_\beta \end{array} \right\}.$$

Proposition 3.14 *Let M be an n-dimensional antiinvariant submanifold of an $(n + k)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then there exists a structure $(g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k \times k})$ on M induced*

by the almost poly- Norden structure of $\tilde{M}$, which satisfies

$$X \quad = \quad -\sum_{\beta=1}^{k} \upsilon_\beta(X)\zeta_\beta,$$

$$\upsilon_\beta(X) \quad = \quad \frac{1}{m}\sum_{\gamma=1}^{k} \theta_{\beta\gamma}\upsilon_\gamma(X),$$

$$\upsilon_\gamma(\zeta_\beta) \quad = \quad m\theta_{\beta\gamma} - \delta_{\beta\gamma} - \sum_{\lambda=1}^{k} \theta_{\beta\lambda}\theta_{\lambda\gamma},$$

$$\zeta_\beta \quad = \quad \frac{1}{m}\sum_{\gamma=1}^{k} \theta_{\beta\gamma}\zeta_\gamma, \tag{3.28}$$

$$g(X,Y) \quad = \quad -\sum_{\beta=1}^{k} \upsilon_\beta(X)\upsilon_\beta(Y),$$

for all $X,Y \in \Gamma(TM)$. Moreover, if $\tilde{M}$ is a poly-Norden Riemannian manifold then we have

$$\nabla_X \zeta_\beta \quad = \quad \sum_{\gamma=1}^{k}(\theta_{\beta\gamma}A_{N_\gamma} X + \sigma_{\beta\gamma}(X)\zeta_\gamma)$$

$$X(\theta_{\beta\lambda}) + h_\lambda(X,\zeta_\beta) + h_\beta(X,\zeta_\lambda) \quad = \quad -\sum_{\gamma=1}^{k}(\theta_{\beta\gamma}\sigma_{\gamma\lambda}(X) - \theta_{\gamma\lambda}\sigma_{\beta\gamma}(X)).$$

4. Hypersurfaces of almost poly-Norden Riemannian manifolds

Suppose that M is a hypersurface in an $(n+1)$-dimensional almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. In this case (3.15) and (3.16) can be written as follows:

$$\tilde{\Phi}X = fX + \upsilon(X)\,N, \tag{4.1}$$

$$\tilde{\Phi}N = \zeta + \theta\,N, \tag{4.2}$$

where $\upsilon(X) = g(X,\zeta)$, for all $X \in \Gamma(TM)$.

From Lemma 3.4 and Proposition 3.5, we have:

Proposition 4.1 *Let $(f,g,\upsilon,\zeta,\theta)$ be the induced structure on a hypersurface M of an almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then for any $X,Y \in \Gamma(TM)$ we have*

$$\begin{aligned}
f^2X \quad &= \quad mfX - X - \upsilon(X)\zeta, \\
\upsilon(fX) \quad &= \quad (m-\theta)\,\upsilon(X), \\
\upsilon(\zeta) \quad &= \quad \theta\,(m-\theta) - 1, \\
f\zeta \quad &= \quad (m-\theta)\,\zeta, \\
g(fX,fY) \quad &= \quad mg(X,fY) - g(X,Y) - \upsilon(X)\upsilon(Y).
\end{aligned} \tag{4.3}$$

For a hypersurface M, the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)N,$$
$$\tilde{\nabla}_X N = -A_N X,$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, respectively.

If $(\tilde{M}, \tilde{\Phi}, g)$ is a poly-Norden Riemannian manifold, by using Proposition 3.3 we get the following:

Proposition 4.2 *Let $(f, g, \upsilon, \zeta, \theta)$ be the induced structure on a hypersurface M of a poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then we have*

$$(\nabla_X f)\, Y = \upsilon(Y)\, A_N X + h(X,Y)\, \zeta, \tag{4.4}$$

$$(\nabla_X \upsilon)\, Y = -h(X, fY) + \theta h(X,Y), \tag{4.5}$$

$$X(\theta) + 2h(X,\zeta) = 0, \tag{4.6}$$

$$f\, A_N X + \nabla_X \zeta - \theta A_N X = 0. \tag{4.7}$$

Corollary 4.3 *Let $(f, g, \upsilon, \zeta, \theta)$ be the induced structure on a hypersurface M of a poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$. Then we have*

$$(\nabla_X \upsilon)\, Y = g(\nabla_X \zeta, Y). \tag{4.8}$$

Corollary 4.4 *Let M be an invariant hypersurface of a poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$ with the induced structure $(f, g, \upsilon, \zeta, \theta)$. Then f is parallel with respect to ∇.*

Theorem 4.5 *A hypersurface M of an almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$ is invariant if and only if the normal vector of M is an eigenvector of $\tilde{\Phi}$ with the eigenvalue θ.*

Proof Assume that M is an invariant hypersurface in $(\tilde{M}, \tilde{\Phi}, g)$. Then we have $\upsilon = 0$ (or equivalently $\zeta = 0$), which implies $\tilde{\Phi} N = \theta N$, via (4.2).

Conversely, let the normal vector N of M be an eigenvector of $\tilde{\Phi}$ with the eigenvalue θ. Hence, we get $\zeta = 0$. This completes the proof. □

Theorem 4.6 *Let M be a hypersurface of an almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$ with the induced structure $(f, g, \upsilon, \zeta, \theta)$. Then M is invariant if and only if either $\theta = m - B_m$ or $\theta = B_m$.*

Proof Since $\upsilon(\zeta) = g(\zeta, \zeta) = \theta\,(m - \theta) - 1$, if M is an invariant hypersurface, then we have

$$\theta^2 - m\theta + 1 = 0,$$

which implies either $\theta = \frac{m - \sqrt{m^2 - 4}}{2}$ or $\theta = \frac{m + \sqrt{m^2 - 4}}{2}$. Conversely, if either $\theta = m - B_m$ or $\theta = B_m$, then we get $\upsilon(\zeta) = g(\zeta, \zeta) = 0$. This completes the proof. □

Theorem 4.7 *Let M be a noninvariant hypersurface of an almost poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$ with the induced structure (f, g, v, ζ, θ). Then the following are satisfied equivalently:*

(i) *M is totally geodesic,*

(ii) *f is parallel on M with respect to ∇,*

(iii) *$\nabla \zeta = 0$,*

(iv) *$\nabla v = 0$.*

Proof   Assume that M is totally geodesic. Then from Proposition 4.2, one can easily see that $\nabla f = 0$, $\nabla \zeta = 0$, and $\nabla v = 0$.

Suppose that f is parallel on M with respect to ∇. Then by using (4.7) in Proposition 4.2, we write

$$-g(Y, \zeta) A_N X = g(A_N X, Y) \zeta, \tag{4.9}$$

which gives

$$g(A_N X, Y) g(Y, \zeta) = 0,$$

for all $X, Y \in \Gamma(TM)$. Putting $Y = \zeta$ in the last equation, since ζ is a nonnull vector, then we have

$$g(A_N X, \zeta) = 0.$$

If we put $Y = A_N X$, in (4.9), we get $g(A_N X, A_N X) = 0$, which implies $A_N X = 0$, for all $X \in \Gamma(TM)$. Thus, we observe that M is totally geodesic. Also, from (4.5) and (4.7) we get $\nabla \zeta = 0$ and $\nabla v = 0$.

Next, let $\nabla \zeta = 0$. Using (4.7) we write

$$f A_N X = \theta A_N X,$$

for all $X \in \Gamma(TM)$. If we apply f to both sides of the last equation, we get

$$f^2 A_N X = \theta^2 A_N X. \tag{4.10}$$

By using the first equation of (4.3) and (4.6) in (4.10), we obtain

$$(\theta(m - \theta) - 1) A_N X = -\frac{1}{2} X(\theta) \zeta. \tag{4.11}$$

Since $v(\zeta) = g(\zeta, \zeta) = \theta(m - \theta) - 1$ on a noninvariant hypersurface, then (4.11) can be written as

$$A_N X = -\frac{1}{2 \|\zeta\|^2} X(\theta) \zeta. \tag{4.12}$$

Applying f again to both sides of (4.12) and using Proposition 4.1, we have

$$(2\theta - m) \left(\frac{1}{2 \|\zeta\|^2} X(\theta) \right) \zeta = 0,$$

which implies that either $X(\theta) = 0$ or $2\theta = m$. For both cases it is obvious that $X(\theta) = 0$, and from (4.12) we conclude that $A_N X = 0$, for any $X \in \Gamma(TM)$. If $A_N X = 0$, then (i), (ii), and (iv) also hold.

Finally, assume that $\nabla v = 0$. From (4.8) we see that $\nabla \zeta = 0$. If ζ is parallel on M, then it is obvious from the above parts of the proof that M is totally geodesic and f is parallel. This completes the proof. □

Let M be a totally umbilical hypersurface of a poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$ endowed with the induced structure (f, g, v, ζ, θ). Then from Proposition 4.2 we have

$$
\begin{aligned}
(\nabla_X f) Y &= a\left(g(X,Y)\zeta + v(Y)X\right), \\
(\nabla_X v) Y &= -a\left(g(X, fY) - \theta g(X,Y)\right), \\
X(\theta) &= -2a\nu(X), \\
\nabla_X \zeta &= -a\left(fX - \theta X\right), \\
\nabla_\zeta \zeta &= -a(m - 2\theta)\zeta,
\end{aligned}
\tag{4.13}
$$

for all $X, Y \in \Gamma(TM)$.

By using the definition of an invariant hypersurface and the fourth equation in (4.13) we get:

Theorem 4.8 *Let M be a totally umbilical invariant hypersurface of a poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$ endowed with the induced structure (f, g, v, ζ, θ). Then $f = \theta I$, where θ is a constant function on M satisfying either $\theta = m - B_m$ or $\theta = B_m$.*

Conversely, suppose that M is a hypersurface of a poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$ such that $f = \theta I$, where (f, g, v, ζ, θ) is the induced structure on M. Then from (4.13) we can easily see that $\nabla_X \zeta = 0$ for any $X \in \Gamma(TM)$, which implies that M is invariant.

Hence, we get the following.

Theorem 4.9 *Let M be a hypersurface of a poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$ endowed with the induced structure (f, g, v, ζ, θ). If $f = \theta I$, then either M is an invariant hypersurface with $\theta = m - B_m$ or $\theta = B_m$, or M is a noninvariant totally geodesic hypersurface of the poly-Norden Riemannian manifold $(\tilde{M}, \tilde{\Phi}, g)$.*

Inspired from the examples constructed for the golden case [13] and the metallic case [3, 10, 14], we give the following examples:

Example 4.10 *Let R^5 be the 5-dimensional real number space with a coordinate system (x_1, x_2, y_1, y_2, z). We define*

$$
\begin{aligned}
\tilde{\Phi}: \quad & R^5 \quad \rightarrow \quad & R^5 \\
& (x_1, x_2, y_1, y_2, z) \quad \rightarrow \quad & (B_m x_1, B_m x_2, (m - B_m) y_1, (m - B_m) y_2, B_m z),
\end{aligned}
$$

where $B_m = \frac{m + \sqrt{m^2 - 4}}{2}$. It is easy to see that $(R^5, \tilde{\Phi})$ is an almost poly-Norden manifold. Since the usual product $\langle, \rangle$ on R^5 is $\tilde{\Phi}$-compatible, then $(R^5, \tilde{\Phi}, \langle, \rangle)$ becomes an almost poly-Norden Riemannian manifold.

Now suppose that M is a hypersurface of $(R^5, \tilde{\Phi}, \langle, \rangle)$ defined by

$$x_1 = z.$$

The tangent bundle of the hypersurface is generated by $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, where

$$\begin{aligned}
\Psi_1 &= (1,0,0,0,1), \quad \Psi_2 = (0,1,0,0,0), \\
\Psi_3 &= (0,0,1,0,0), \quad \Psi_4 = (0,0,0,1,0).
\end{aligned}$$

It can be seen that $\tilde{\Phi}(\Psi_i) = B_m \Psi_i$ and $\tilde{\Phi}(\Psi_j) = (m - B_m)\Psi_j$, $i = 1, 2$, $j = 3, 4$, which imply that M is an invariant hypersurface of $(R^5, \tilde{\Phi}, \langle, \rangle)$.

Example 4.11 Let R^{n+k} be the $(n+k)$-dimensional real number space with a coordinate system $(x_1, ..., x_n, y_1, ..., y_k)$. We define

$$\tilde{\Phi}(x_1, ..., x_n, y_1, ..., y_k) = (B_m x_1, ..., B_m x_n, (m - B_m) y_1, ..., (m - B_m) y_k),$$

where $B_m = \frac{m + \sqrt{m^2 - 4}}{2}$. Then it is easy to verify that $\tilde{\Phi}^2 = m\tilde{\Phi} - I$, where I is the identity operator; that is, $(R^{n+k}, \tilde{\Phi})$ is an almost poly-Norden manifold (see also [17]). Since the usual product $\langle, \rangle$ on R^{n+k} is $\tilde{\Phi}$-compatible, then $(R^{n+k}, \tilde{\Phi}, \langle, \rangle)$ is an almost poly-Norden Riemannian manifold.

Consider the hypersphere $S^{n+k-1}(r)$ of R^{n+k}, which is defined by

$$S^{n+k-1}(r) = \{(x_1, ..., x_n, y_1, ..., y_k) : \sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{k} y_j^2 = r_1^2 + r_2^2 = r^2\} \subset R^{n+k}.$$

The normal vector field of $S^{n+k-1}(r)$ at any point $(x_1, ..., x_n, y_1, ..., y_k) \in S^{n+k-1}(r)$ is given by

$$N = \frac{1}{r}(x_1, ..., x_n, y_1, ..., y_k).$$

Then there exists a tangent vector $(X_1, ..., X_n, Y_1, ..., Y_k)$ on $S^{n+k-1}(r)$ for every point $(x_1, ..., x_n, y_1, ..., y_k) \in S^{n+k-1}(r)$ if and only if

$$\sum_{i=1}^{n} x_i X_i + \sum_{j=1}^{k} y_j Y_j = 0. \tag{4.14}$$

By using (4.1) and (4.2), we write

$$\begin{aligned}
\tilde{\Phi}(X_i, Y_j) &= f(X_i, Y_j) + \upsilon(X_i, Y_j) N, \tag{4.15} \\
\tilde{\Phi} N &= \zeta + \theta N, \tag{4.16}
\end{aligned}$$

where $(X_i, Y_j) = (X_1, ..., X_n, Y_1, ..., Y_k) \in T_{(x_1, ..., x_n, y_1, ..., y_k)} S^{n+k-1}(r)$. Since

$$\tilde{\Phi} N = \frac{1}{r}(B_m x_1, ..., B_m x_n, (m - B_m) y_1, ..., (m - B_m) y_k)$$

and $\theta = \left\langle \tilde{\Phi} N, N \right\rangle$, *then we calculate*

$$\theta = \frac{1}{r^2}(B_m r_1^2 + (m - B_m) r_2^2). \tag{4.17}$$

By using $\zeta = \tilde{\Phi} N - \theta N$ *and the last equation above we get*

$$\zeta = \frac{2B_m - m}{r^3}(r_1^2 x_1, ..., r_1^2 x_n, -r_2^2 y_1, ..., -r_2^2 y_k), \tag{4.18}$$

which implies

$$
\begin{aligned}
\upsilon\left(X_i, Y_j\right) &= \langle (X_i, Y_j), \zeta \rangle \\
&= \frac{2B_m - m}{r^3} \left(r_1^2 \left(\sum_{i=1}^{n} x_i X_i \right) - r_2^2 \left(\sum_{j=1}^{k} y_j Y_j \right) \right).
\end{aligned}
\tag{4.19}
$$

From (4.14)*, if we put* $\sum_{i=1}^{n} x_i X_i = -\sum_{j=1}^{k} y_j Y_j = s$, (4.19) *can be written as*

$$\upsilon\left(X_i, Y_j\right) = \frac{2B_m - m}{r} s. \tag{4.20}$$

Furthermore, by using (4.15) *and* (4.20) *we have*

$$f\left(X_i, Y_j\right) = \left(B_m X_i - \frac{2B_m - m}{r^2} s x_i, (m - B_m) Y_j - \frac{2B_m - m}{r^2} s y_j \right). \tag{4.21}$$

Hence, $S^{n+k-1}(r)$ *is a noninvariant hypersurface of the almost poly-Norden Riemannian manifold* $(R^{n+k}, \tilde{\Phi}, \langle, \rangle)$ *endowed with the induced structure* $(f, \langle, \rangle, \upsilon, \zeta, \theta)$ *given by* (4.17)–(4.21).

Example 4.12 *Consider that* R^{2n+k} *is the* $(2n + k)$-*dimensional real number space with a coordinate system* $(x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_k)$. *We denote*

$$(x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_k) = (x_i, y_i, z_j),$$

where $i \in \{1, ..., n\}$, $j \in \{1, ..., k\}$, *and we define*

$$\tilde{\Phi}(x_i, y_i, z_j) = \left(\frac{m}{2} x_i + \frac{\sqrt{m^2 - 4}}{2} y_i, \frac{m}{2} y_i + \frac{\sqrt{m^2 - 4}}{2} x_i, B_m z_j \right),$$

where $B_m = \frac{m + \sqrt{m^2 - 4}}{2}$. *One can easily see that* (2.2) *is satisfied and the scalar product* $\langle, \rangle$ *on* R^{2n+k} *is* $\tilde{\Phi}$-*compatible. Then* $(R^{2n+k}, \tilde{\Phi}, \langle, \rangle)$ *is an almost poly-Norden Riemannian manifold. Let* $S^{2n+k-1}(r)$ *be the hypersphere of* R^{2n+k}, *which is defined by*

$$S^{2n+k-1}(r) = \{(x_i, y_i, z_j) : \sum_{i=1}^{n}(x_i^2 + y_i^2) + \sum_{j=1}^{k} z_j^2 = r_1^2 + r_2^2 + r_3^2 = r^2\} \subset R^{2n+k}.$$

In this case, the normal vector field of $S^{2n+k-1}(r)$ for any point $(x_i, y_i, z_j) \in S^{2n+k-1}(r)$ is given by

$$N = \frac{1}{r}(x_i, y_i, z_j).$$

A vector field $(X_1, ..., X_n, Y_1, ..., Y_n, Z_1, ..., Z_k)$ on $S^{2n+k-1}(r)$ at each point $(x_i, y_i, z_j) \in S^{n+k-1}(r)$ will be denoted by (X_i, Y_i, Z_j). Since $X \perp N$, then we have

$$\sum_{i=1}^{n}(x_i X_i + y_i Y_i) + \sum_{j=1}^{k} z_j Z_j = 0. \tag{4.22}$$

Applying $\tilde{\Phi}$ to N, we get

$$\tilde{\Phi}N = \frac{1}{r}\left(\frac{m}{2}x_i + \frac{\sqrt{m^2-4}}{2}y_i, \frac{m}{2}y_i + \frac{\sqrt{m^2-4}}{2}x_i, B_m z_j\right).$$

By using $\theta = \left\langle \tilde{\Phi}N, N \right\rangle$, we obtain

$$\theta = \frac{1}{r^2}\left(\frac{m}{2}(r_1^2 + r_2^2) + \sqrt{m^2-4}\sum_{i=1}^{n} x_i y_i + B_m r_3^2\right).$$

From (4.2) and (4.1), we obtain

$$\zeta = \frac{1}{r}\left(\left(\frac{m}{2} - \theta\right)x_i + \frac{\sqrt{m^2-4}}{2}y_i, \left(\frac{m}{2} - \theta\right)y_i + \frac{\sqrt{m^2-4}}{2}x_i, (B_m - \theta)z_j\right),$$

and

$$\upsilon(X_i, Y_i, Z_j) = \frac{\sqrt{m^2-4}}{2r}\left(\sum_{i=1}^{n}(x_i Y_i + y_i X_i) + \sum_{j=1}^{k} z_j Z_j\right).$$

Moreover, we calculate

$$f(X_i, Y_i, Z_j) = \begin{pmatrix} \frac{m}{2}X_i + \frac{\sqrt{m^2-4}}{2}Y_i - \frac{1}{r}\upsilon(X_i, Y_i, Z_j) x_i, \\ \frac{m}{2}Y_i + \frac{\sqrt{m^2-4}}{2}X_i - \frac{1}{r}\upsilon(X_i, Y_i, Z_j) y_i, \\ B_m Z_j - \frac{1}{r}\upsilon(X_i, Y_i, Z_j) z_j \end{pmatrix}.$$

Hence, $S^{2n+k-1}(r)$ is a noninvariant hypersurface of the almost poly-Norden Riemannian manifold $(R^{2n+k}, \tilde{\Phi}, \langle, \rangle)$ endowed with the induced structure $(f, \langle, \rangle, \upsilon, \zeta, \theta)$ given above.

References

[1] Acet BE. Lightlike hypersurfaces of metallic semi-Riemannian manifolds. International Journal of Geometric Methods in Modern Physics 2018; 15: 12. doi: 10.1142/S0219887818502018

[2] Adati T. Submanifolds of an almost product Riemannian manifold. Kodai Mathematical Journal 1981; 4 (2): 327-343.

[3] Blaga AM, Hretcanu CE. Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold. Novi Sad Journal of Mathematics 2018; 48 (2): 57-82. doi: 10.30755/NSJOM.06365

[4] Crasmareanu M, Hretcanu CE. Golden differential geometry. Chaos, Solitons & Fractals 2008; 38 (5): 1229-1238.

[5] De Spinadel VW. The metallic means family and multifractal spectra. Nonlinear Analysis Series B: Real World Applications 1999; 36 (6): 721-745.

[6] Erdoğan FE. Transversal lightlike submanifolds of metallic semi-Riemannian manifolds. Turkish Journal of Mathematics 2018; 42 (6): 3133-3148. doi: 10.3906/mat-1804-88

[7] Erdoğan FE, Yıldırım C. Semi-invariant submanifolds of golden Riemannian manifolds. In: 2nd International Conference on Advances in Natural and Applied Sciences-AIP Conference Proceedings; 2017. doi: 10.1063/1.4981692

[8] Erdoğan FE, Yıldırım C. On a study of the totally umbilical semi-invariant submanifolds of golden Riemannian manifolds. Journal of Polytechnic 2018; 21 (4): 967-970. doi: 10.2339/politeknik.389629

[9] Gezer A, Cengiz N, Salimov A. On integrability of golden Riemannian structures. Turkish Journal of Mathematics 2013; 37(4): 693-703. doi: 10.3906/mat-1108-35

[10] Hretcanu CE, Blaga AM. Submanifolds in metallic Riemannian manifolds. Differential Geometry-Dynamical Systems 2018; 20: 83-97.

[11] Hretcanu CE, Blaga AM. Slant and semi-slant submanifolds in metallic Riemannian manifolds. Journal of Function Spaces 2018; 2018: 2864263. doi: 10.1155/2018/2864263

[12] Hretcanu CE, Crasmareanu M. On some invariant submanifolds in a Riemannian manifold with golden structure. Analele Stiintifice Ale Universitatii Al. I. Cuza Din Iasi (S.N.) Matematica 2007; 53: 199-211.

[13] Hretcanu CE, Crasmareanu M. Applications of the golden ratio on Riemannian manifolds. Turkish Journal of Mathematics 2009; 33 (2): 179-191. doi: 10.3906/mat-0711-29

[14] Hretcanu CE, Crasmareanu M. Metallic structures on Riemannian manifolds. Revista de la Union Matematica Argentina 2013; 54 (2): 15-27.

[15] Kalia S. The generalizations of the golden ratio: their powers, continued fractions, and convergents. Cambridge, MA, USA: MIT. Available at http://math.mit.edu/research/highschool/primes/papers.php.

[16] Önen Poyraz N, Yaşar E. Lightlike hypersurfaces of a golden semi-Riemannian manifold. Mediterranean Journal of Mathematics 2017; 14 (5): 204. doi: 10.1007/s00009-017-0999-2

[17] Şahin B. Almost poly-Norden manifolds. International Journal of Maps in Mathematics 2018; 1 (1): 68-79.