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Weighted composition operators from the Bloch space to nth weighted-type spaces

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Abstract: In this work, we characterize the boundedness of weighted composition operators from the Bloch space and the little Bloch space to nth weighted-type spaces. Some estimates for the essential norm of these operators are also given. As a corollary, we obtain some characterizations for the compactness of weighted composition operators from the Bloch space and the little Bloch space to nth weighted-type spaces.

Key words: Weighted composition operator, Bloch space, essential norm, nth weighted-type space

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$, and $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. The Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$, is the set of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The Bloch space $\mathcal{B}$ is a Banach space with the above norm $\|\cdot\|_{\mathcal{B}}$. The little Bloch space $\mathcal{B}_0$ consists of all $f \in H(\mathbb{D})$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. It is well known that $\mathcal{B}_0$ is the closure of polynomials in $\mathcal{B}$.

Let μ be a weight, which means that μ is a positive and continuous function on $\mathbb{D}$. Let $n \in \mathbb{N}$, the set of all positive integers. The nth weighted-type space, denoted by $\mathcal{W}_\mu^n = \mathcal{W}_\mu^n(\mathbb{D})$, is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{W}_\mu^n} = \sum_{i=0}^{n-1} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)| < \infty.$$

It is easy to check that $\mathcal{W}_\mu^n$ is a Banach space with the above norm. We refer the interested reader to [16–18] for the space $\mathcal{W}_\mu^n$. When $n = 1$, the space $\mathcal{W}_\mu^1$ is called the Bloch-type space. Let $\beta > 0$ and $\mu(z) = (1-|z|^2)^\beta$. The space $\mathcal{W}_\mu^1$ coincides with the Bloch-type space $\mathcal{B}^\beta$. In particular, $\mathcal{B}^1$ is the classical Bloch space $\mathcal{B}$.

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The space $\mathcal{W}_\mu^2$ is also called the Zygmund-type space. For more information about Bloch-type spaces and Zygmund-type spaces, see [2, 7, 9, 21, 22].

Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the set of all analytic self-maps of $\mathbb{D}$. The weighted composition operator, denoted by C_φ^g, induced by φ and g is defined as follows.

$$(C_\varphi^g f)(z) = g(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

When $g \equiv 1$, the operator C_φ^g is denoted by C_φ and called the composition operator. If $\varphi(z) = z$, then C_φ^g is called the multiplication operator and denoted by M_g. Interested readers can refer to [5] for the theory of composition operators and weighted composition operators.

For any $\varphi \in S(\mathbb{D})$, it is widely known that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded. The compactness of $C_\varphi : \mathcal{B} \to \mathcal{B}$ was investigated in [12]. Wulan et al. [19] showed that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{k\to\infty} \|\varphi^k\|_\mathcal{B} = 0$. Zhao in [20] characterized the essential norm of $C_\varphi : \mathcal{B} \to \mathcal{B}$ by using $\|\varphi^k\|_\mathcal{B}$. The operator $C_\varphi^g : \mathcal{B} \to \mathcal{B}$ was studied in [14, 15]. In [3], Colonna investigated $C_\varphi^g : \mathcal{B} \to \mathcal{B}$ by using $\|g\varphi^k\|_\mathcal{B}$. See [6, 8, 10, 11, 13, 23, 24] for more results of the operator $C_\varphi^g : \mathcal{B} \to \mathcal{B}$.

Motivated by [10, 19], in this paper, we give some characterizations for the boundedness of $C_\varphi^g : \mathcal{B} \to \mathcal{W}_\mu^n$. Moreover, we give some estimates for the essential norm of $C_\varphi^g : \mathcal{B} \to \mathcal{W}_\mu^n$. By applying these estimates, some characterizations for the compactness of $C_\varphi^g : \mathcal{B} \to \mathcal{W}_\mu^n$ are obtained.

Throughout this paper, we will use the notation $A \preceq B$ if there exists a constant $C > 0$ such that $A \le CB$. In particular, if $A \preceq B$ and $B \preceq A$, then we write $A \approx B$ and say that A and B are comparable.

2. Boundedness

To investigate the boundedness of $C_\varphi^g : \mathcal{B} \to \mathcal{W}_\mu^n$, we need to state some lemmas. The next lemma was proved in [21].

Lemma 2.1 *For any $f \in \mathcal{B}$ and $n \in \mathbb{N}$,*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| \approx \sum_{i=0}^{n} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{n+1}|f^{(n+1)}(z)|.$$

Lemma 2.2 *([11]) Let $f \in \mathcal{B}$. Then,*

$$|f(z)| \le \frac{1}{\log 2}\|f\|_\mathcal{B} \log\frac{2}{1-|z|^2}, \qquad z \in \mathbb{D}.$$

For any $a \in \mathbb{D}$ and $i \in \{1, 2, ..., n\}$, set

$$f_{i,a}(z) = \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^i, \quad z \in \mathbb{D}. \tag{2.1}$$

It is clear that $f_{i,a} \in \mathcal{B}$ for each $i \in \{1, 2, ..., n\}$. From Lemma 2.3 of [1], we have the following Lemma.

Lemma 2.3 *For any $0 \ne a \in \mathbb{D}$ and $i \in \{1, ..., n\}$, there exist functions $v_{i,a} \in \mathcal{B}$ such that*

$$v_{i,a}^{(k)}(a) = \begin{cases} \frac{\bar{a}^i}{(1-|a|^2)^i}, & k = i, \\ 0, & k \ne i. \end{cases}$$

For $n, k \in \mathbb{N}_0$, the set of all nonnegative integers, with $k \leq n$, the partial Bell polynomials are defined by

$$B_{n,k}(x_1, x_2, ..., x_{n-k+1}) = \sum \frac{n!}{j_1!j_2!...j_{n-k+1}!}(\frac{x_1}{1!})^{j_1}(\frac{x_2}{2!})^{j_2}...(\frac{x_{n-k+1}}{(n-k+1)!})^{j_{n-k+1}},$$

where the sum taken over all $j_1, j_2, ..., j_{n-k+1}$ such that

$$j_1 + j_2 + ... + j_{n-k+1} = k, \quad j_1 + 2j_2 + ... + (n-k+1)j_{n-k+1} = n.$$

where $j_1, j_2, ..., j_{n-k+1} \in \mathbb{N}_0$. For more information about Bell polynomials, see [4, p 134].

The proof of the following lemma is similar to the proof of Lemma 4 in [17], so we omit the details.

Lemma 2.4 *Let $f, h, g \in H(\mathbb{D})$. Then for any $n \in \mathbb{N}_0$,*

$$(C_h^g f)^{(n)}(z) = \sum_{i=0}^{n} f^{(i)}(h(z)) \sum_{l=i}^{n} \binom{n}{l} g^{(n-l)}(z) B_{l,i}(h'(z), ..., h^{(l-i+1)}(z)).$$

Lemma 2.5 *Let $n \in \mathbb{N}$ and $\varphi \in S(\mathbb{D})$. Then for any $a \in \mathbb{D}$, there exists a function $u_a \in \mathcal{B}_0$ such that*

$$u_a(\varphi(a)) = \log \frac{2}{1- \mid \varphi(a) \mid^2} \text{ and } u_a'(\varphi(a)) = u_a''(\varphi(a)) = \ldots = u_a^{(n)}(\varphi(a)) = 0.$$

Proof If $\varphi(a) = 0$, then $u_a(z) = \log 2$. For any $a \in \mathbb{D}$ with $\varphi(a) \neq 0$, set

$$g_{a,k}(z) = (n+k) \log \frac{2}{1 - \overline{\varphi(a)}z} - \frac{(\log \frac{2}{1-\overline{\varphi(a)}z})^{n+k}}{(\log \frac{2}{1-|\varphi(a)|^2})^{n+k-1}}, \quad k \in \{1, 2, \ldots, n\}.$$

Then $g_{a,k} \in \mathcal{B}$ and $\lim_{|z|\to 1}(1-|z|^2)|g'_{a,k}(z)| = 0$. So, $g_{a,k} \in \mathcal{B}_0$. Now we calculate $g_{a,k}^{(s)}(\varphi(a))$ for $s \in \{2, \ldots, n\}$. Set

$$g(z) = 1, \quad h(z) = \log \frac{2}{1 - \overline{\varphi(a)}z} \quad \text{and} \quad f(z) = (n+k)z - \frac{z^{n+k}}{\gamma^{n+k-1}}$$

in Lemma 2.4, where $\gamma = \log \frac{2}{1-|\varphi(a)|^2}$. We get

$$\begin{aligned} g_{a,k}^{(s)}(\varphi(a)) &= (f(h))^{(s)}(\varphi(a)) \\ &= \sum_{i=1}^{s} f^{(i)}(h(\varphi(a))) B_{s,i}\big(h'(\varphi(a)), \ldots, h^{(s-i+1)}(\varphi(a))\big). \end{aligned} \tag{2.2}$$

Also $f'(h(\varphi(a))) = g'_{a,k}(\varphi(a)) = 0$ and for $i \in \{2, \ldots, s\}$,

$$f^{(i)}(h(\varphi(a))) = -p_i^{n+k} \times \gamma^{1-i}, \tag{2.3}$$

where $p_i^{n+k} = \frac{(n+k)!}{(n+k-i)!}$. After a calculation, we have

$$h^{(i)}(\varphi(a)) = \frac{(i-1)!\overline{\varphi(a)}^i}{(1- \mid \varphi(a) \mid^2)^i} = (i-1)!\big(h'(\varphi(a))\big)^i.$$

Hence,

$$\begin{aligned} B_{s,i}\big(h'(\varphi(a)),\dots,h^{(s-i+1)}(\varphi(a))\big) &= \sum \frac{s!}{j_1!j_2!\dots j_{s-i+1}}\Big(\frac{h'(\varphi(a))}{1!}\Big)^{j_1}\Big(\frac{h''(\varphi(a))}{2!}\Big)^{j_2}\dots\Big(\frac{h^{(s-i+1)}(\varphi(a))}{(s-i+1)!}\Big)^{j_{s-i+1}} \\ &= \sum \frac{s!}{j_1!\dots j_{s-i+1}!1^{j_1}\dots(s-i+1)^{j_{s-i+1}}}\big(h'(\varphi(a))\big)^{j_1+\dots+(s-i+1)j_{s-i+1}} \\ &= \underbrace{\sum \frac{s!}{j_1!j_2!\dots j_{s-i+1}!1^{j_1}2^{j_2}\dots(s-i+1)^{j_{s-i+1}}}}_{b_i^s} \times \frac{\overline{\varphi(a)}^s}{(1-|\varphi(a)|^2)^s}. \end{aligned} \tag{2.4}$$

So, from (2.2)–(2.4), for any $s \in \{2,\dots,n\}$, we get

$$g_{a,k}^{(s)}(\varphi(a)) = -\frac{\overline{\varphi(a)}^s}{(1-|\varphi(a)|^2)^s}\sum_{i=2}^{s}\gamma^{1-i}b_i^s p_i^{n+k}.$$

Now for any $a \in \mathbb{D}$ with $\varphi(a) \neq 0$ and coefficients $c_1, c_2, \dots, c_n$, we set

$$d_{a,c_1,c_2,\dots,c_n}(z) = \sum_{k=1}^{n} c_k g_{a,k}(z).$$

It is clear that $d_{a,c_1,c_2,\dots,c_n} \in \mathcal{B}_0$. We consider the system of linear equations

$$\begin{array}{rcccl} d_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) = & \gamma\sum_{k=1}^{n}(n+k-1)c_k = & \gamma \\ d'_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) = & \sum_{k=1}^{n} c_k \times \underbrace{g'_{a,k}(\varphi(a))}_{0} = & 0 \\ d''_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) = & -\frac{\overline{\varphi(a)}^2}{(1-|\varphi(a)|^2)^2}\gamma^{-1}\sum_{k=1}^{n} b_2^2 p_2^{n+k} c_k = & 0 \\ \dots = & \dots = & \dots \\ d^{(s)}_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) = & -\frac{\overline{\varphi(a)}^s}{(1-|\varphi(a)|^2)^s}\sum_{k=1}^{n}\big(\sum_{i=2}^{s}\gamma^{1-i}b_i^s p_i^{n+k}\big)c_k = & 0 \\ \dots = & \dots = & \dots \\ d^{(n)}_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) = & -\frac{\overline{\varphi(a)}^n}{(1-|\varphi(a)|^2)^n}\sum_{k=1}^{n}\big(\sum_{i=2}^{n}\gamma^{1-i}b_i^n p_i^{n+k}\big)c_k = & 0. \end{array} \tag{2.5}$$

Similar to the proof of Lemma 2.3 in [1], we see that the system (2.5) has a unique solution and the solution is independent of choice a and $\varphi(a)$. If $c_1, c_2, \dots, c_n$ is that solution, we get $u_a(z) = d_{a,c_1,c_2,\dots,c_n}(z)$, as desired. □

Let $\varphi \in S(\mathbb{D})$, $i, n \in \mathbb{N}_0$ and $i \leq n$. For simplicity, we set

$$I_i^n(z) := \sum_{l=i}^{n}\binom{n}{l} g^{(n-l)}(z) B_{l,i}(\varphi'(z), \varphi''(z), \dots, \varphi^{(l-i+1)}(z)). \tag{2.6}$$

Theorem 2.6 *Let $n \in \mathbb{N}$, μ be a weight, $g \in \mathcal{W}_\mu^n$ and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent.*

(a) $C_\varphi^g : \mathcal{B} \to \mathcal{W}_\mu^n$ is bounded.

(b) $C_\varphi^g : \mathcal{B}_0 \to \mathcal{W}_\mu^n$ is bounded.

(c) $\sup_{z\in\mathbb{D}} \mu(z)|g^{(n)}(z)| \log\frac{2}{1-|\varphi(z)|^2} < \infty$ and $\sup_{j\geq 0}\|g\varphi^j\|_{\mathcal{W}_\mu^n} < \infty$.

(d) $\sup_{z\in\mathbb{D}}\mu(z)|g^{(n)}(z)|\log\frac{2}{1-|\varphi(z)|^2}<\infty$ *and for each* $i\in\{1,...,n\}$,

$$\sup_{a\in\mathbb{D}}\|C_\varphi^g f_{i,a}\|_{\mathcal{W}_\mu^n}<\infty,\qquad \sup_{z\in\mathbb{D}}\mu(z)|I_i^n(z)|<\infty,$$

where $f_{i,a}$ *are defined in* (2.1).

(e) $\sup_{z\in\mathbb{D}}\mu(z)|g^{(n)}(z)|\log\frac{2}{1-|\varphi(z)|^2}<\infty$ *and for each* $i\in\{1,...,n\}$,

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|I_i^n(z)|}{(1-|\varphi(z)|^2)^i}<\infty.$$

Proof $(a)\Rightarrow(b)$ It is obvious.

$(b)\Rightarrow(c)$ For any $w\in\mathbb{D}$, let u_w be the function defined in Lemma 2.5. Then $L=\sup_{w\in\mathbb{D}}\|u_w\|_{\mathcal{B}}<\infty$. Using Lemmas 2.4 and 2.5, we get

$$\mu(w)|g^{(n)}(w)|\log\frac{2}{1-|\varphi(w)\,|^2}=\mu(w)|g^{(n)}(w)||u_w(\varphi(w))|$$
$$=\mu(w)|(C_\varphi^g u_w)^{(n)}(\varphi(w))|\le\|C_\varphi^g u_w\|_{\mathcal{W}_\mu^n}\le L\|C_\varphi^g\|_{\mathcal{W}_\mu^n}<\infty.$$

Thus, $\sup_{z\in\mathbb{D}}\mu(z)|g^{(n)}(z)|\log\frac{2}{1-|\varphi(z)|^2}<\infty$. Since the sequence $\{z^j\}_0^\infty$ is bounded in $\mathcal{B}_0$ (see [11]), we get $\sup_{j\ge0}\|g\varphi^j\|_{\mathcal{W}_\mu^n}<\infty$ by the boundedness of $C_\varphi^g:\mathcal{B}_0\to\mathcal{W}_\mu^n$.

$(c)\Rightarrow(d)$ It follows from Theorem 3.1 in [1].

$(d)\Rightarrow(e)$ It also follows from Theorem 3.1 in [1].

$(e)\Rightarrow(a)$ For any $f\in\mathcal{B}$, from Lemmas 2.1, 2.2 and 2.4, we get

$$\mu(z)|(C_\varphi^g f)^{(n)}(z)|\le\frac{1}{\log 2}\|f\|_{\mathcal{B}}\sup_{z\in\mathbb{D}}\mu(z)|g^{(n)}(z)|\log\frac{2}{1-|\varphi(z)|^2}+C\|f\|_{\mathcal{B}}\sum_{i=1}^n\sup_{z\in\mathbb{D}}\frac{\mu(z)|I_i^n(z)|}{(1-|\varphi(z)|^2)^i}$$

and for $j=0,1,\cdots,n-1$,

$$|(C_\varphi^g f)^{(j)}(0)|\le\frac{1}{\log 2}|g(0)|\|f\|_{\mathcal{B}}\log\frac{2}{1-|\varphi(0)|^2}+C\frac{\|f\|_{\mathcal{B}}}{\mu(0)}\sum_{i=1}^j\frac{\mu(0)|I_i^j(0)|}{(1-|\varphi(0)|^2)^i}.$$

Thus, $C_\varphi^g:\mathcal{B}\to\mathcal{W}_\mu^n$ is bounded by the assumed condition. The proof is complete. □

3. Essential norm

In this section we give some estimates for $\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n}$, the essential norm of $C_\varphi^g:\mathcal{B}\to\mathcal{W}_\mu^n$. Recall that

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n}=\inf\{\|C_\varphi^g-K\|_{\mathcal{B}\to\mathcal{W}_\mu^n}:K\text{ is compact}\}.$$

We begin with the following lemma.

Lemma 3.1 *Let* $n\in\mathbb{N}$, $\varphi\in S(\mathbb{D})$ *and* $\{a_i\}$ *be a sequence in* $\mathbb{D}$ *such that* $|\varphi(a_i)|\to1$ *as* $i\to\infty$. *Then there exists a bounded sequence* $\{h_i\}$ *in* $\mathcal{B}_0$ *such that,* $\{h_i\}$ *converge to* 0 *uniformly on compact subsets of* $\mathbb{D}$ *and*

$$h_i(\varphi(a_i))=\log\frac{2}{1-|\varphi(a_i)|^2},\ h_i'(\varphi(a_i))=h_i''(\varphi(a_i))=\ldots=h_i^{(n)}(\varphi(a_i))=0.$$

Proof The proof is similar to the proof of Lemma 2.5. Hence, we omit the details. □

Theorem 3.2 *Let* $n \in \mathbb{N}$, μ *be a weight,* $g \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$ *such that* $C_\varphi^g : \mathcal{B} \to \mathcal{W}_\mu^n$ *is bounded. Then*

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \approx \max\{A_i\}_{i=0}^n \approx \max\{B_i\}_{i=0}^n \approx \|C_\varphi^g\|_{e,\mathcal{B}_0\to\mathcal{W}_\mu^n} \approx \max\Big\{\limsup_{j\to\infty}\|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0\Big\},$$

where

$$A_0 = B_0 = \limsup_{|\varphi(z)|\to 1} \mu(z)|g^{(n)}(z)|\log\frac{2}{1-|\varphi(z)|^2},$$

$$A_i = \limsup_{|a|\to 1}\|C_\varphi^g f_{i,a}\|_{\mathcal{W}_\mu^n}, \quad B_i = \limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|I_i^n(z)|}{(1-|\varphi(z)|^2)^i}, i \in \{1,\dots,n\},$$

and $f_{i,a}$ *are defined in* (2.1)*.*

Proof First we show that

$$\|C_\varphi^g\|_{e,\mathcal{B}_0\to\mathcal{W}_\mu^n} \succeq A_0 = B_0. \tag{3.1}$$

Let $\{z_i\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. We assume that for any i, $\varphi(z_i) \neq 0$. Let $\{h_i\}$ be the sequence defined in Lemma 4.1. Then for any compact operator $K : \mathcal{B}_0 \to \mathcal{W}_\mu^n$, $\lim_{i\to\infty}\|Kh_i\|_{\mathcal{W}_\mu^n} = 0$. Thus,

$$\|(C_\varphi^g - K)h_i\|_{\mathcal{B}_0\to\mathcal{W}_\mu^n} \geq \limsup_{i\to\infty}\|C_\varphi^g h_i\|_{\mathcal{W}_\mu^n} - \limsup_{i\to\infty}\|Kh_i\|_{\mathcal{W}_\mu^n} \geq \limsup_{i\to\infty}\mu(z_i)|g^{(n)}(z_i)|\log\frac{2}{1-|\varphi(z_i)|^2},$$

which implies the desired result.

From Theorem 4.2 in [1], we have

$$\|C_\varphi^g\|_{e,H^\infty\to\mathcal{W}_\mu^n} \succeq \max\{A_i\}_{i=1}^n \quad \text{and} \quad \|C_\varphi^g\|_{e,H^\infty\to\mathcal{W}_\mu^n} \succeq \max\{B_i\}_{i=1}^n.$$

So,

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \succeq \max\{A_i\}_{i=1}^n \quad \text{and} \quad \|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \succeq \max\{B_i\}_{i=1}^n.$$

From the last inequality and (3.1) we obtain

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \succeq \max\{A_i\}_{i=0}^n \quad \text{and} \quad \|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \succeq \max\{B_i\}_{i=0}^n. \tag{3.2}$$

Next, we show that

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \preceq \max\{A_i\}_{i=0}^n \quad \text{and} \quad \|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \preceq \max\{B_i\}_{i=0}^n.$$

For $r \in [0,1)$, let $K_r f(z) = f_r(z) = f(rz)$. Then $K_r : \mathcal{B} \to \mathcal{B}$ is compact with $\|K_r\| \leq 1$. It is obvious that f_r uniformly converge to f on compact subsets of $\mathbb{D}$ as $r \to 1$. Let $\{r_j\} \subset (0,1)$ such that $r_j \to 1$ as $j \to \infty$. Then for any $j \in \mathbb{N}$, $C_\varphi^g K_{r_j} : \mathcal{B} \to \mathcal{W}_\mu^n$ is compact. Thus,

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \leq \limsup_{j\to\infty}\|C_\varphi^g - C_\varphi^g K_{r_j}\|. \tag{3.3}$$

Hence, we only need to show that

$$\limsup_{j\to\infty} \|C_\varphi^g - C_\varphi^g K_{r_j}\|_{\mathcal{B}\to\mathcal{W}_\mu^n} \preceq \min\{\max\{A_i\}_{i=0}^n, \max\{B_i\}_{i=0}^n\}.$$

For any $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{B}} \le 1$,

$$\begin{aligned}
\|(C_\varphi^g - C_\varphi^g K_{r_j})f\|_{\mathcal{W}_\mu^n} &= \sum_{t=0}^{n-1}\left|\sum_{k=0}^{t}(f-f_{r_j})^{(k)}(\varphi(0))I_k^t(0)\right| + \sup_{z\in\mathbb{D}}\mu(z)\left|\sum_{k=0}^{n}(f-f_{r_j})^{(k)}(\varphi(z))I_k^n(z)\right| \\
&\le \underbrace{\sum_{t=0}^{n-1}\left|\sum_{k=0}^{t}(f-f_{r_j})^{(k)}(\varphi(0))I_k^t(0)\right|}_{H_1} + \underbrace{\sup_{|\varphi(z)|\le r_N}\mu(z)\left|\sum_{k=0}^{n}(f-f_{r_j})^{(k)}(\varphi(z))I_k^n(z)\right|}_{H_2} \\
&\quad + \underbrace{\sup_{|\varphi(z)|> r_N}\mu(z)\left|\sum_{k=0}^{n}(f-f_{r_j})^{(k)}(\varphi(z))I_k^n(z)\right|}_{H_3}, \qquad (3.4)
\end{aligned}$$

where $N \in \mathbb{N}$ such that $r_j \ge \frac{2}{3}$ for all $j \ge N$. Since for any $s \in \mathbb{N}_0$, $(f-f_{r_j})^{(s)}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, by using Theorem 2.6 (d), we obtain

$$\limsup_{j\to\infty} H_1 = 0 \qquad \text{and} \qquad \limsup_{j\to\infty} H_2 = 0. \qquad (3.5)$$

In addition,

$$H_3 \le \sum_{k=0}^{n}\Big[\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|f^{(k)}(\varphi(z))||I_k^n(z)|}_{M_k} + \underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|r_j^k f^{(k)}(r_j\varphi(z))||I_k^n(z)|}_{N_k}\Big]. \qquad (3.6)$$

First we estimate M_0 and N_0. From Lemma 2.2,

$$M_0 = \sup_{|\varphi(z)|>r_N}\mu(z)|f(\varphi(z))||g^{(n)}(z)| \le \sup_{|\varphi(z)|>r_N}\mu(z)|g^{(n)}(z)|\frac{1}{\log 2}\|f\|_{\mathcal{B}}\log\frac{2}{1-|\varphi(z)|^2} \preceq A_0 = B_0. \qquad (3.7)$$

Similarly,

$$N_0 \preceq A_0 = B_0. \qquad (3.8)$$

For $k \in \{1,\ldots,n\}$, by (2.6) and Lemmas 2.1, 2.3 and 2.4,

$$\begin{aligned}
M_k &= \sup_{|\varphi(z)|>r_N}\mu(z)\frac{(1-|\varphi(z)|^2)^k|f^{(k)}(\varphi(z))|}{|\varphi(z)|^k}\frac{|\varphi(z)|^k|I_k^n(z)|}{(1-|\varphi(z)|^2)^k} \\
&\preceq \|f\|_{\mathcal{B}}\sup_{|\varphi(z)|>r_N}\|C_\varphi^g v_{k,\varphi(z)}\|_{\mathcal{W}_\mu^n} \preceq \sum_{j=1}^{n}|c_j^k|\sup_{|a|>r_N}\|C_\varphi^g f_{j,a}\|_{\mathcal{W}_\mu^n}. \qquad (3.9)
\end{aligned}$$

Taking the above limit as $N \to \infty$, we obtain

$$\limsup_{j\to\infty} M_k \preceq \sum_{i=1}^{n}\underbrace{\limsup_{|a|\to 1}\|C_\varphi^g f_{i,a}\|_{\mathcal{W}_\mu^n}}_{A_i} \preceq \max\{A_i\}_{i=1}^n \quad \text{and} \quad \limsup_{j\to\infty} M_k \preceq B_k. \qquad (3.10)$$

Similarly,

$$\limsup_{j\to\infty} N_k \preceq \max\{A_i\}_{i=1}^n \qquad \text{and} \qquad \limsup_{j\to\infty} N_k \preceq B_k. \tag{3.11}$$

Thus, from (3.4)–(3.8), (3.10), and (3.11), we obtain

$$\limsup_{j\to\infty} \|C_\varphi^g - C_\varphi^g K_{r_j}\|_{\mathcal{B}\to\mathcal{W}_\mu^n} \preceq \min\Big\{ \max\{A_i\}_{i=0}^n, \max\{B_i\}_{i=0}^n \Big\}.$$

Hence, from (3.3),

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \preceq \min\Big\{ \max\{A_i\}_{i=0}^n, \max\{B_i\}_{i=0}^n \Big\}.$$

Therefore,

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \approx \max\{A_i\}_{i=0}^n \approx \max\{B_i\}_{i=0}^n.$$

Finally, we show that

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \approx \|C_\varphi^g\|_{e,\mathcal{B}_0\to\mathcal{W}_\mu^n} \approx \max\Big\{ \limsup_{j\to\infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0 \Big\}. \tag{3.12}$$

Let j be any positive integer and $h_j(z) = z^j$. Then $h_j \in \mathcal{B}_0$, $\|h_j\|_{\mathcal{B}} \approx 1$ and $\{h_j\}_{j\in\mathbb{N}}$ converges to 0 weakly (see [11]). Hence, for any compact operator K from $\mathcal{B}_0$ into $\mathcal{W}_\mu^n$, we have $\lim_{j\to\infty} \|Kh_j\|_{\mathcal{W}_\mu^n} = 0$. Thus,

$$\|C_\varphi^g - K\|_{\mathcal{B}_0\to\mathcal{W}_\mu^n} \succeq \limsup_{j\to\infty} \|(C_\varphi^g - K)h_j\|_{\mathcal{W}_\mu^n} \geq \limsup_{j\to\infty} \|C_\varphi^g h_j\|_{\mathcal{W}_\mu^n} - \limsup_{j\to\infty} \|Kh_j\|_{\mathcal{W}_\mu^n} = \limsup_{j\to\infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}.$$

Hence, $\|C_\varphi^g\|_{e,\mathcal{B}_0\to\mathcal{W}_\mu^n} \succeq \limsup_{j\to\infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}$, which together with (3.1) imply

$$\|C_\varphi^g\|_{e,\mathcal{B}_0\to\mathcal{W}_\mu^n} \succeq \max\Big\{ \limsup_{j\to\infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0 \Big\}. \tag{3.13}$$

From the proof of Theorem 4.3 in [1], we have that $\max\{A_i\}_{i=1}^n \preceq \limsup_{j\to\infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}$; hence,

$$\max\{A_i\}_{i=0}^n \preceq \max\Big\{ \limsup_{j\to\infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0 \Big\}.$$

Since $\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \approx \max\{A_i\}_{i=0}^n$, we obtain

$$\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} \preceq \max\Big\{ \limsup_{j\to\infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0 \Big\}. \tag{3.14}$$

Since $\|C_\varphi^g\|_{e,\mathcal{B}_0\to\mathcal{W}_\mu^n} \leq \|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n}$, from (3.13) and (3.14), we get (3.12). The proof is complete. □

It is well known that $\|C_\varphi^g\|_{e,\mathcal{B}\to\mathcal{W}_\mu^n} = 0$ if and only if $C_\varphi^g : \mathcal{B} \to \mathcal{W}_\mu^n$ is a compact operator. Hence, we have the following result.

Corollary 3.3 *Let $n \in \mathbb{N}$, μ be a weight, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $C_\varphi^g : \mathcal{B} \to \mathcal{W}_\mu^n$ is bounded. Then the following statements are equivalent.*

(a) $C_\varphi^g : \mathcal{B} \to \mathcal{W}_\mu^n$ *is compact.*

(b) $C_\varphi^g : \mathcal{B}_0 \to \mathcal{W}_\mu^n$ *is compact.*

(c) $\limsup_{j\to\infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n} = 0$ *and* $\limsup_{|\varphi(z)|\to 1} \mu(z)|g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} = 0.$

(d) *For* $i \in \{1, \dots, n\}$*,* $\limsup_{|a|\to 1} \|C_\varphi^g f_{i,a}\|_{\mathcal{W}_\mu^n} = 0$ *and* $\limsup_{|\varphi(z)|\to 1} \mu(z)|g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} = 0.$

(e) *For* $i \in \{1, \dots, n\}$*,* $\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|I_i^n(z)|}{(1-|\varphi(z)|^2)^i} = 0$ *and* $\limsup_{|\varphi(z)|\to 1} \mu(z)|g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} = 0.$

Remark 3.4 *Putting* $n = 1$ *and* $\mu(z) = (1-|z|^2)^\beta$ *in Theorem 2.6 and Corollary 3.3, we get some characterizations for the boundedness and compactness of* $C_\varphi^g : \mathcal{B} \to \mathcal{B}^\beta$ *(see Theorems 2.1 and 3.1 in [15]). Moreover, we obtain some estimates for the essential norm of* $C_\varphi^g : \mathcal{B} \to \mathcal{B}^\beta$ *(see [8, 10, 11]).*

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