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Unimodality and linear recurrences associated with rays in the Delannoy triangle

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Abstract: In this paper, we study the unimodality of sequences located in the infinite transversals of the Delannoy triangle. We establish recurrence relations associated with the sum of elements laying along the finite transversals of the cited triangle and we give the generating function of the established sum. Moreover, new identities for the odd and even terms of the Tribonacci sequence are given. Finally, we define a q-analogue for the Delannoy numbers and we propose a q-deformation of the Tribonacci sequence.

Key words: Delannoy triangle, log-concavity, unimodality, recurrence relations, q-Delannoy numbers.

1. Introduction

It is well known that the Delannoy numbers are defined by the following recurrence relation, see [2, 14],

$$D(n,k) = D(n-1,k) + D(n,k-1) + D(n-1,k-1), \tag{1.1}$$

with $D(n,0) = D(0,k) = 1$.

Alladi and Hoggat [1] have represented these numbers as a Pascal-shaped triangle, see Table 1 below.

Let $\binom{n}{k}_{[2]}$ be the element in the n^{th} row and k^{th} column of the Delannoy triangle. This triangle can be obtained by the following recurrence relation

$$\binom{n}{k}_{[2]} = \binom{n-1}{k}_{[2]} + \binom{n-1}{k-1}_{[2]} + \binom{n-2}{k-1}_{[2]}, \tag{1.2}$$

where $\binom{n}{0}_{[2]} = \binom{n}{n}_{[2]} = 1$. We use the convention $\binom{n}{k}_{[2]} = 0$ for $k \notin \{0, \dots, n\}$.

The recurrence relation (1.2) has already been used by Kuhapatanakul [16] in order to give a connection between the Delannoy triangle and the Fibonacci sequence.

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Table . Delannoy triangle.

n\k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	3	1							
3	1	5	5	1						
4	1	7	13	7	1					
5	1	9	25	25	9	1				
6	1	11	41	63	41	11	1			
7	1	13	61	129	129	61	13	1		
8	1	15	85	231	321	231	85	15	1	
9	1	17	113	377	681	681	377	113	17	1

The relation between the Delannoy numbers $D(n,k)$ and the coefficients of the Delannoy triangle $\binom{n}{k}_{[2]}$ is given by the following formula

$$\binom{n}{k}_{[2]} = D(n-k,k), \tag{1.3}$$

while the generating function of the Delannoy triangle coefficients is

$$F_k(x) := \sum_{n\geq 0} \binom{n}{k}_{[2]} x^n = \frac{(x+x^2)^k}{(1-x)^{k+1}}. \tag{1.4}$$

The Delannoy triangle is also called Tribonacci triangle and it is a well known triangle in the theory of Riordan matrices, see [18], represented as

$$\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right).$$

Alladi and Hoggat [1] have also shown that the sum of the elements on the rising diagonal of the Delannoy triangle gives the terms of the Tribonacci sequence $(T_n)_n$, that is

$$T_{n+1} = \sum_k \binom{n-k}{k}_{[2]}. \tag{1.5}$$

Theorem 1.1 *[1] The sequence $\{T_n\}_{n\geqslant 0}$ satisfies the following recurrence relation*

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

with $T_0 = 0$ and $T_1 = T_2 = 1$.

The first terms of the Tribonacci sequence are $(0,1,1,2,4,7,13,24,\ldots)$, see Sloane* as $A000073$.
Several authors have studied the properties associated with the Tribonacci sequence; Kilic [15] has given a

*Sloane N. The Online Encyclopedia of Integer Sequences. Available online at http://oeis.org/

recurrence relation for the sum of the Tribonacci numbers. Ramirez and Sirvent [23] have studied the incomplete Tribonacci sequence.

Moreover, Barry [3] has proved that the coefficients of the Delannoy triangle satisfy the following two identities

$$\binom{n}{k}_{[2]} = \sum_{j=0}^{k} \binom{k}{j}\binom{n-j}{k}, \tag{1.6}$$

and

$$\binom{n}{k}_{[2]} = \sum_{j=0}^{k} \binom{k}{j}\binom{n-k}{j}2^j. \tag{1.7}$$

Shattuck [25] has provided a combinatorial interpretation for the relation (1.6) by using the tiling approach.

On the other hand, the coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}$ are considered a q-analogue of the binomial coefficients, see [5, 12].

Throughout this paper, we use $\begin{bmatrix} n \\ k \end{bmatrix}^*$ to denote the above-mentioned coefficients, namely

$$\begin{bmatrix} n \\ k \end{bmatrix}^* = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} = \frac{[n]_q!}{[k]_q![n-k]_q!} q^{\binom{k}{2}},$$

with $[n]_q! = [1]_q[2]_q \cdots [n]_q$ and $[n]_q = 1+q+q^2+\cdots+q^{n-1}$, we use the convention $\begin{bmatrix} n \\ k \end{bmatrix}^* = 0$ for $k \notin \{0,\ldots,n\}$. The q-analogue of binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}^*$ satisfies the following recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix}^* = \begin{bmatrix} n-1 \\ k \end{bmatrix}^* + q^{n-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^*, \tag{1.8}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}^* = q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}^* + q^{k-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^*, \tag{1.9}$$

and their generating functions are given by

$$\sum_{n\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}^* x^n = \frac{q^{\binom{k}{2}}x^k}{(x;q)_{k+1}}, \tag{1.10}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^* x^k = (-x;q)_n, \tag{1.11}$$

where $(x;q)_n$ is the q-Pochhammer symbol defined by

$$(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x).$$

Other q-analogues of the Delannoy numbers apear in the literature, such as those defined by Sagan [24] and Pan [22], namely the q-analogues of the Delannoy numbers, see the identities (1.12) and (1.13), respectively,

$$D_{n,k}(q) = \sum_{j\geq 0} \binom{k}{j}\binom{n+k-j}{k}q^j, \tag{1.12}$$

$$D_q(n,k)=\sum_{j=0}^{n}\begin{bmatrix}k\\j\end{bmatrix}_q\begin{bmatrix}n+k-j\\k\end{bmatrix}_q q^{\binom{j+1}{2}}, \tag{1.13}$$

where $D_q(n,k)$ satisfy the following recurrence relation

$$D_q(n,k)=q^kD_q(n-1,k)+D_q(n,k-1)+q^kD_q(n-1,k-1).$$

Belbachir and Benmezai [5] have defined a q-analogue for the bi^{s}nomial coefficients. Bazeniar et al. [4] have established a new type of symmetric function to interpret these coefficients.
Let us focus now on the unimodality property. Given a real positive sequence $U=(u_k)_{k\geq 0}$, we say that U is unimodal if there exists an integer l such that U is increasing for $k\in\{0,\ldots,l\}$ and it is decreasing for $k\geq l$. The integer l is called the mode of the sequence U. The sequence $(u_k)_{k\geq 1}$ is log-concave (resp. log-convex) if it satisfies, $u_k^2\geq u_{k-1}u_{k+1}$ (resp. $u_k^2\leq u_{k-1}u_{k+1}$).
The log concavity of $(u_k)_{k\geq 0}$ implies its unimodality, see [10].
Recently, many papers have dealt with the unimodality of sequences laying along the transversals of arithmetic triangles. The first result has been obtained by Tanny and Zuker [28], they have proved that for all n the sequences $\left\{\binom{n-k}{k}\right\}_k$ laying along the rising diagonal of the Pascal triangle are unimodal. Belbachir and Szalay [7] have shown that for any sequence of binomial coefficients located along a finite ray in Pascal's triangle, $\left\{\binom{n-qk}{p+rk}\right\}_k$ is log-concave and thus unimodal. Su and Wang [27] have proved the unimodality of the infinite sequences in the Pascal triangle $\left\{\binom{n+ak}{p+bk}\right\}_k$. Benoumhani [8] has shown that the sequence $\left\{\frac{n}{n-k}\binom{n-k}{k}\right\}_k$ which is associated with Lucas numbers is strictly log-concave and hence unimodal. Sagan [24] has established that the sequence related to the q-analogue of the Delannoy numbers $\{D_{j,n-j}(q)\}_{j=0}^{n}$ is q-unimodal. Liu and Wang [17] have proved that the central q-analogue of the Delannoy numbers $D_{n,k}(q)$ form a q-log-convex sequence. See also [9, 26, 30].
This paper consists of four sections and is organized as follows. In Section 2, we establish the unimodality of sequences located along the infinite transversals of the Delannoy triangle. The third section is devoted to the description of the recurrence relations associated with the sum of diagonal elements laying along a finite transversal crossing the Delannoy triangle. We precise the generating function of the described sum, then we give an identity for the odd and even terms of the Tribonacci sequence. In the last section, we define a q-analogue of the Delannoy numbers and we give their closed-form expression, we also establish the q-deformation of the Tribonacci sequence.

2. Unimodality of sequences located over rays in the Delannoy triangle

In this section, we prove that the sequences located along the infinite transversals of the Delannoy triangle $\left\{\binom{n-\alpha k}{p+rk}_{[2]}\right\}_{k\geq 0}$ are increasing, hence unimodal. For the particular case where $\alpha=-1, r=1$ and $p=0$, we use the result given by Wang and Yeh [29], to establish that the sequences $\left\{\binom{n+k}{k}_{[2]}\right\}_{k\geq 0}$ are log-concave, for all n.

Definition 2.1 *For all fixed n, the sequences laying along the transversals of the Delannoy triangle are obtained*

by this expression $\left\{\binom{n-\alpha k}{p+rk}_{[2]}\right\}_{k\geq 0}$ *, with* $r,p\in\mathbb{N},\alpha\in\mathbb{Z}$ *and* $0\leq k\leq\lfloor(n-p)\,/\,(\alpha+r)\rfloor$*.*
It should be noted that when, $r+\alpha>0$*, this expression deals with the finite transversals case, otherwise, it deals with the infinite case.*

In order to prove the log-concavity of the sequences $\left\{\binom{n+k}{k}_{[2]}\right\}_{k\geq 0}$, one can consider the linear transformation preserving the log-concavity property given by Wang and Yeh [29].

Theorem 2.2 **([29])** *If the sequence* $\{x_n\}_{n\geq 0}$ *is log-concave, then the linear transformation*

$$y_n:=\sum_{k=0}^{n}\binom{n}{k}x_k,\ \ n=0,1,2\ldots$$

preserves the log-concavity property.

Using Theorem 2.2, for $(r=1,\ p=0$ and $\alpha=-1)$ we establish the following result.

Theorem 2.3 *For a fixed* n*, the sequence* $\left\{\binom{n+k}{k}_{[2]}\right\}_{k\geq 0}$ *is log-concave and thus unimodal.*

Proof From relation (1.7), we have for a fixed n, $\binom{n+k}{k}_{[2]}=\sum_j\binom{k}{j}\binom{n}{j}2^j$. Since the sequence $x_j:=\binom{n}{j}2^j$ is trivially log-concave, then, by Theorem 2.2, the sequence $\left\{\binom{n+k}{k}_{[2]}\right\}_{k\geq 0}$ is log-concave and thus unimodal. $\square$

Let us now explore the general case, when $\alpha+r\leq 0$.

Theorem 2.4 *Let* n*,* p*,* r *be nonnegative integers and* $\alpha\in\mathbb{Z}$*, with* $n\geq p$*,* $0\leq p<r$ *and* $\alpha+r\leq 0$*, the sequences* $\left\{\binom{n-\alpha k}{p+rk}_{[2]}\right\}_{k\geq 0}$ *are increasing and hence unimodal.*

Proof In order to prove that the sequences $\left\{\binom{n-\alpha k}{p+rk}_{[2]}\right\}_{k\geq 0}$ are increasing, one has to prove that the following relation (2.1) is satisfied

$$\binom{n-\alpha(k+1)}{p+r(k+1)}_{[2]}=\binom{n-\alpha k}{p+rk}_{[2]}+\sigma, \tag{2.1}$$

with $\sigma\geq 0$.
Let us consider $\binom{n-\alpha(k+1)}{p+r(k+1)}_{[2]}=\binom{n+ak+a}{p+rk+r}_{[2]}$, with $a=-\alpha$ $(a\geq r)$, using the relation (1.2) we have

$$\begin{aligned}\binom{n+ak+a}{p+rk+r}_{[2]}&=\binom{n+ak+a-1}{p+rk+r}_{[2]}+\binom{n+ak+a-1}{p+rk+r-1}_{[2]}+\binom{n+ak+a-2}{p+rk+r-1}_{[2]}\\&=\binom{n+ak+a-2}{p+rk+r}_{[2]}+\binom{n+ak+a-2}{p+rk+r-1}_{[2]}+\binom{n+ak+a-3}{p+rk+r-1}_{[2]}+\binom{n+ak+a-1}{p+rk+r-1}_{[2]}\\&\quad+\binom{n+ak+a-2}{p+rk+r-1}_{[2]}.\end{aligned}$$

If we use the relation (1.2) $(a-r)$ times , we obtain a relation of the following form

$$\binom{n+ak+a}{p+rk+r}_{[2]}=\binom{n+ak+r}{p+rk+r}_{[2]}+\sigma^{'}, \tag{2.2}$$

with $\sigma^{'}>0$.
One can show that the relation (2.2) can be rewritten as follows

$$\begin{aligned}\binom{n+ak+a}{p+rk+r}_{[2]} &= \binom{n+ak+r-1}{p+rk+r}_{[2]}+\binom{n+ak+r-1}{p+rk+r-1}_{[2]}+\binom{n+ak+r-2}{p+rk+r-1}_{[2]}+\sigma^{'}\\ &= \binom{n+ak+r-1}{p+rk+r}_{[2]}+\binom{n+ak+r-2}{p+rk+r-1}_{[2]}+\binom{n+ak+r-2}{p+rk+r-2}_{[2]}+\binom{n+ak+r-3}{p+rk+r-2}_{[2]}\\ &+\binom{n+ak+r-2}{p+rk+r-1}_{[2]}+\sigma^{'}.\end{aligned}$$

Repeating this process r times we get the desired result

$$\binom{n+ak+a}{p+rk+r}_{[2]}=\binom{n+ak}{p+rk}_{[2]}+\sigma, \tag{2.3}$$

with $\sigma\geq 0$. Consequently the sequences $\left\{\binom{n-\alpha k}{p+rk}_{[2]}\right\}_{k\geq 0}$ are increasing. □

3. Recurrence relations

In this section, we establish the recurrence relations associated with the sum of elements crossing the finite transversals of the Delannoy triangle and we give the generating function of the provided sum. This work generalizes the results given by Belbachir and Szalay [6]. New identities for the odd and even terms of the Tribonacci sequence are also given.
For $\alpha\in\mathbb{N}$, consider the sequence

$$T_{n+1}^{(\alpha)}:=\sum_k\binom{n-\alpha k}{k}_{[2]}.$$

Theorem 3.1 *The sequence $(T_n^{(\alpha)})_n$ satisfies the following recurrence relation*

$$T_{n+1}^{(\alpha)}=T_n^{(\alpha)}+T_{n-\alpha}^{(\alpha)}+T_{n-\alpha-1}^{(\alpha)},$$

with $T_1^{(\alpha)}=1, T_{-i}^{(\alpha)}=0$ for $i\in\{0,1,\ldots,\alpha\}$.

Proof Using the relation (1.2), we have

$$T_{n+1}^{(\alpha)}=\sum_k\binom{n-\alpha k-1}{k}_{[2]}+\sum_k\binom{n-\alpha k-1}{k-1}_{[2]}+\sum_k\binom{n-\alpha k-2}{k-1}_{[2]}$$

$$k^{'}\rightarrow k-1$$

$$= \sum_{k} \binom{n-\alpha k-1}{k}_{[2]} + \sum_{k'} \binom{n-\alpha k'-\alpha-1}{k'}_{[2]} + \sum_{k'} \binom{n-\alpha k'-\alpha-2}{k'}_{[2]}$$

$$= T_n^{(\alpha)} + T_{n-\alpha}^{(\alpha)} + T_{n-\alpha-1}^{(\alpha)}.$$

□

Remark 3.2 *For $\alpha = 1$, we get the Tribonacci sequence.*

Example 3.3 *For $\alpha = 2$,*

$$T_n^{(2)} = T_{n-1}^{(2)} + T_{n-3}^{(2)} + T_{n-4}^{(2)},$$

with $T_0^{(2)} = 0, T_1^{(2)} = T_2^{(2)} = T_3^{(2)} = 1$. The first terms are $(0, 1, 1, 1, 2, 4, 6, 9, 15, 25, \dots)$, see Sloane as A006498.
For other values of α, the sequences in Sloane were also found, see for instance A079972 for $\alpha = 3$ and A121832 for $\alpha = 4$.

The previous result can be extended for all finite transversals $(r + \alpha > 0)$, with $r \in \mathbb{N},\ 0 \le p < r$ and $\alpha \in \mathbb{Z}$. In order to do that, consider the following sequence

$$T_{n+1}^{(r,p,\alpha)} = \sum_{k} \binom{n-\alpha k}{p+rk}_{[2]}.$$

Theorem 3.4 *For $n \ge 2r + \alpha$, the sequence $\{T_n^{(r,p,\alpha)}\}_{n \ge 0}$ satisfies the following linear recurrence relation*

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} T_{n-s}^{(r,p,\alpha)} = \sum_{s=0}^{r} \binom{r}{s} T_{n-r-\alpha-s}^{(r,p,\alpha)}. \tag{3.1}$$

To prove Theorem 3.4, we need the following lemma.

Lemma 3.5 ([6]) *Let a, b, and r be nonnegative integers with $r \le a$, then*

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{a-s}{b} = \binom{a-r}{b-r}.$$

Now we are able to proceed with the proof of Theorem 3.4.

Proof From Lemma 3.5, we get

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} T_{n-s}^{(r,p,\alpha)} = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \sum_{k} \sum_{j} \binom{p+rk}{j} \binom{n-\alpha k-j-s-1}{p+rk}$$

$$= \sum_{k} \sum_{j} \binom{p+rk}{j} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \binom{n-\alpha k-j-s-1}{p+rk}$$

$$= \sum_k \sum_j \binom{p+rk}{j}\binom{n-\alpha k-j-r-1}{p+r(k-1)}$$

$(k^{'} \to k-1)$

$$= \sum_{k'} \sum_j \binom{p+rk^{'}+r}{j}\binom{n-\alpha k^{'}-j-r-\alpha-1}{p+rk'}$$

$$= \sum_{k'} \sum_j \sum_s \binom{r}{s}\binom{p+rk^{'}}{j-s}\binom{n-\alpha k^{'}-j-r-\alpha-1}{p+rk'}$$

$(j^{'} \to j-s)$

$$= \sum_{k'} \sum_{j'} \sum_s \binom{r}{s}\binom{p+rk^{'}}{j'}\binom{n-\alpha k^{'}-j^{'}-s-r-\alpha-1}{p+rk'}$$

$$= \sum_{s=0}^{r} \binom{r}{s} T_{n-r-\alpha-s}^{(r,p,\alpha)}.$$

□

Remark 3.6 *For $r=\alpha=1$ and $p=0$, we obtain the result of Theorem 1.1.*

Example 3.7 *For $r=2, \alpha=1$, and $p=0$, we obtain the sequence*

$$T_{n+1}^{(2,0,1)} = \sum_k \sum_j \binom{2k}{j}\binom{n-j-k}{2k}.$$

This sequence satisfies the following recurrence relation

$$T_n^{(2,0,1)} = 2T_{n-1}^{(2,0,1)} - T_{n-2}^{(2,0,1)} + T_{n-3}^{(2,0,1)} + 2T_{n-4}^{(2,0,1)} + T_{n-5}^{(2,0,1)} \ (n>5),$$

with $T_0^{(2,0,1)}=0, T_1^{(2,0,1)}=T_2^{(2,0,1)}=T_3^{(2,0,1)}=1$ and $T_4^{(2,0,1)}=2$.

If we consider the particular cases, where $r=2, \alpha=-1, p=0$, and $p=1$ of Theorem 3.4 one can get new identities for the odd and even terms of the Tribonacci sequence.
The following Lemma is used in order to obtain the proof of Theorem 3.9.

Lemma 3.8 *The odd and even terms of the Tribonacci sequence $(T_n)_n$ satisfy the two following recurrence relations*

$$T_{2n+1} = 3T_{2n-1} + T_{2n-3} + T_{2n-5}, \ \textit{with } T_1=1, T_3=2, T_5=7, \tag{3.2}$$

and

$$T_{2n} = 3T_{2n-2} + T_{2n-4} + T_{2n-6}, \ \textit{with } T_0=0, T_2=1, T_4=4. \tag{3.3}$$

Proof The relation (3.2) for the odd terms is obtained using the following transformation

$$\begin{aligned}T_{2n+1} &= T_{2n}+T_{2n-1}+T_{2n-2}\\ &= T_{2n-1}+T_{2n-2}+T_{2n-3}+T_{2n-1}+T_{2n-2}\\ &= 2T_{2n-1}+T_{2n-1}-T_{2n-3}-T_{2n-4}+T_{2n-3}+T_{2n-3}+T_{2n-4}+T_{2n-5}\\ &= 3T_{2n-1}+T_{2n-3}+T_{2n-5}.\end{aligned}$$

Using the same idea one can easily verify the relation (3.3). □

Denote by V_n and U_n the respectively odd and even terms of the Tribonacci sequence ($V_n \equiv T_{2n+1}$ and $U_n \equiv T_{2n}$), then

$$V_n = 3V_{n-1}+V_{n-2}+V_{n-3},\ \text{with } V_1=1, V_2=2, V_3=7, \tag{3.4}$$

and

$$U_n = 3U_{n-1}+U_{n-2}+U_{n-3},\ \text{with } U_0=0, U_1=1, U_2=4. \tag{3.5}$$

On the other hand, if we set $r=2, \alpha=-1$ and $p=0$, in the above relation (3.1) we obtain

$$T_n^{(2,0,-1)} = 3T_{n-1}^{(2,0,-1)}+T_{n-2}^{(2,0,-1)}+T_{n-3}^{(2,0,-1)},\ \text{with } T_1^{(2,0,-1)}=1, T_2^{(2,0,-1)}=2, T_3^{(2,0,-1)}=7, \tag{3.6}$$

and if we set $r=2, \alpha=-1$, and $p=1$, in the same relation we obtain

$$T_n^{(2,1,-1)} = 3T_{n-1}^{(2,1,-1)}+T_{n-2}^{(2,1,-1)}+T_{n-3}^{(2,1,-1)},\ \text{with } T_0^{(2,1,-1)}=0, T_1^{(2,1,-1)}=1, T_2^{(2,1,-1)}=4. \tag{3.7}$$

Note that $T_n^{(2,0,-1)}$, and $T_n^{(2,1,-1)}$ are respectively the odd and even terms of the Tribonacci sequence ($T_n^{(2,0,-1)} \equiv V_n$, $T_n^{(2,1,-1)} \equiv U_n$). This allows us to state the following result.

Theorem 3.9 *The odd and even terms of the Tribonacci sequence satisfy the relations*

$$T_{2n+1} = \sum_k \binom{n+k}{2k}_{[2]},$$

$$T_{2n} = \sum_k \binom{n+k}{2k+1}_{[2]}.$$

3.1. Generating function

In this subsection, the generating function of the sequence $T_{n+1}^{(r,p,\alpha)}$ is provided.

Theorem 3.10 *The generating function of the sequence* $(T_n^{(r,p,\alpha)})_n$ *is given by*

$$\sum_{n\geqslant 0} T_{n+1}^{(r,p,\alpha)}x^n = \frac{(1-x)^{r-p-1}(x+x^2)^p}{(1-x)^r - x^{\alpha+r}(1+x)^r}.$$

Proof We have

$$\begin{aligned}\sum_{n\geqslant 0} T_{n+1}^{(r,p,\alpha)} x^n &= \sum_{n\geqslant 0}\sum_{k} \binom{n-\alpha k}{p+rk}_{[2]} x^n \\ &= \sum_{k}\sum_{n\geqslant \alpha k} \binom{n-\alpha k}{p+rk}_{[2]} x^{n-\alpha k}x^{\alpha k} \\ &= \sum_{k} \frac{(x+x^2)^{p+rk}x^{\alpha k}}{(1-x)^{p+rk+1}} \\ &= \frac{(x+x^2)^p}{(1-x)^{p+1}} \sum_{k} \left(\frac{(x+x^2)^r x^\alpha}{(1-x)^r}\right)^k \\ &= \frac{(x+x^2)^p}{(1-x)^{p+1}} \frac{1}{1-(\frac{x^\alpha (x+x^2)^r}{(1-x)^r})} \\ &= \frac{(1-x)^{r-p-1}(x+x^2)^p}{(1-x)^r - x^{\alpha+r}(1+x)^r}.\end{aligned}$$

□

4. q-Delannoy numbers

In this section, we define a q-analogue of the Delannoy numbers which are denoted $\begin{bmatrix} n\\ k\end{bmatrix}_{[2]}$. A closed-form expression and generating function are provided for these numbers, we propose also a q-deformation of the Tribonacci sequence.

Definition 4.1 *We define the q-Delannoy numbers, according to the relation (1.2), as*

$$\begin{bmatrix} n\\ k\end{bmatrix}_{[2]} = \begin{bmatrix} n-1\\ k\end{bmatrix}_{[2]} + q^{n-1}\begin{bmatrix} n-1\\ k-1\end{bmatrix}_{[2]} + q^{n-2}\begin{bmatrix} n-2\\ k-1\end{bmatrix}_{[2]}, \tag{4.1}$$

We use the convention $\begin{bmatrix} 0\\ 0\end{bmatrix}_{[2]} = 1$ and $\begin{bmatrix} n\\ k\end{bmatrix}_{[2]} = 0$ for $k \notin \{0,\dots,n\}$.

The generating function of $\begin{bmatrix} n\\ k\end{bmatrix}_{[2]}$ is given in the following result.

Theorem 4.2 *Let $\mathbb{F}_k(x) := \sum_{n\geq 0} \begin{bmatrix} n\\ k\end{bmatrix}_{[2]} x^n$ be the generating function of the q-Delannoy numbers, then*

$$\mathbb{F}_k(x) = \frac{q^{\binom{k}{2}} x^k (-x;q)_k}{(x;q)_{k+1}}.$$

Proof We have $\mathbb{F}_k(x) = \sum_{n\geq 0} \begin{bmatrix} n\\ k\end{bmatrix}_{[2]} x^n$, using the relation (4.1) one can obtain

$$\begin{aligned}\mathbb{F}_k(x) &= \sum_{n\geq 0} \begin{bmatrix} n-1\\ k\end{bmatrix}_{[2]} x^n + \sum_{n\geq 0} q^{n-1}\begin{bmatrix} n-1\\ k-1\end{bmatrix}_{[2]} x^n + \sum_{n\geq 0} q^{n-2}\begin{bmatrix} n-2\\ k-1\end{bmatrix}_{[2]} x^n \\ &= x\sum_{n\geq 0} \begin{bmatrix} n\\ k\end{bmatrix}_{[2]} x^n + (x+x^2)\sum_{n\geq 0} \begin{bmatrix} n\\ k-1\end{bmatrix}_{[2]} (qx)^n \\ &= x\mathbb{F}_k(x) + (x+x^2)\mathbb{F}_{k-1}(qx),\end{aligned}$$

then

$$\mathbb{F}_k(x) = \frac{x(1+x)}{1-x}\mathbb{F}_{k-1}(qx), \tag{4.2}$$

applying repeatedly this relation we get

$$\mathbb{F}_k(x) = \frac{q^{\binom{k}{2}} x^k (-x;q)_k}{(x;q)_{k+1}}.$$

□

Let us now give another recurrence relation for the q-Delannoy numbers which is equivalent to the relation (4.1).

Proposition 4.3 *The q-Delannoy numbers are also given by the following recurrence relation*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[2]} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{[2]} + q^{k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{[2]} + q^{2(k-1)} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{[2]}. \tag{4.3}$$

Proof We have $\mathbb{F}_k(x) = \sum_{n\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_{[2]} x^n$, using the relation (4.3) one can obtain:

$$\begin{aligned}
\mathbb{F}_k(x) &= q^k \sum_{n\geq 0} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{[2]} x^n + q^{k-1} \sum_{n\geq 0} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{[2]} x^n + q^{2(k-1)} \sum_{n\geq 0} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{[2]} x^n \\
&= q^k x \sum_{n\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_{[2]} x^n + (q^{k-1}x + q^{2(k-1)}x^2) \sum_{n\geq 0} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{[2]} x^n \\
&= q^k x F_k(x) + (q^{k-1}x + q^{2(k-1)}x^2) F_{k-1}(x),
\end{aligned}$$

then

$$\mathbb{F}_k(x) = \frac{q^{k-1}x(1+q^{k-1}x)}{1-q^k x}\mathbb{F}_{k-1}(x), \tag{4.4}$$

applying repeatedly this relation we get

$$\mathbb{F}_k(x) = \frac{q^{\binom{k}{2}} x^k (-x;q)_k}{(x;q)_{k+1}}.$$

Since we have obtained the same generating series, these numbers are the same. □

The following result establishes the explicit formula of the q-Delannoy numbers.

Theorem 4.4 *The closed-form expression of q-Delannoy numbers $\begin{bmatrix} n \\ k \end{bmatrix}_{[2]}$ is*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[2]} = \sum_j \begin{bmatrix} n-j \\ k \end{bmatrix}^* \begin{bmatrix} k \\ j \end{bmatrix}^*. \tag{4.5}$$

Proof We have

$$\sum_{n\geq 0}\sum_j \begin{bmatrix} n-j \\ k \end{bmatrix}^* \begin{bmatrix} k \\ j \end{bmatrix}^* x^n = \sum_j \begin{bmatrix} k \\ j \end{bmatrix}^* x^j \sum_{n\geq 0} \begin{bmatrix} n-j \\ k \end{bmatrix}^* x^{n-j} = \frac{q^{\binom{k}{2}} x^k (-x;q)_k}{(x;q)_{k+1}},$$

the last equality comes from the relations (1.10) and (1.11) respectively. □

A q-analogue of the Fibonacci sequence has already been proposed by Cigler and Carlitz see [11, 13]. Munarini [19, 20] and Munarini and Salvi [21] have considered some q-analogues of the generalized Fibonacci numbers. In what follows, we establish the q-analogue of the Tribonacci sequence.

Theorem 4.5 *Let* $\mathbb{T}_{n+1}^{(2)}(x) := \sum_k \begin{bmatrix} n-k \\ k \end{bmatrix}_{[2]} x^k$ *for* $n \geq 0$ *and* $\mathbb{T}_0^{(2)}(x) = 0$, *then*

$$\mathbb{T}_{n+1}^{(2)}(x) = \mathbb{T}_n^{(2)}(x) + q^{n-2} x \mathbb{T}_{n-1}^{(2)}(x/q) + q^{n-3} x \mathbb{T}_{n-2}^{(2)}(x/q), \tag{4.6}$$

or equivalently

$$\mathbb{T}_{n+1}^{(2)}(x) = \mathbb{T}_n^{(2)}(qx) + x \mathbb{T}_{n-1}^{(2)}(qx) + x \mathbb{T}_{n-2}^{(2)}(q^2 x). \tag{4.7}$$

Proof We have

$$\mathbb{T}_{n+1}^{(2)}(x) = \sum_k \begin{bmatrix} n-k \\ k \end{bmatrix}_{[2]} x^k,$$

by the relation (4.1), we obtain

$$\begin{aligned} \mathbb{T}_{n+1}^{(2)}(x) &= \sum_k \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_{[2]} x^k + \sum_k \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_{[2]} q^{n-k-1} x^k + \sum_k \begin{bmatrix} n-k-2 \\ k-1 \end{bmatrix}_{[2]} q^{n-k-2} x^k \\ &= \sum_k \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_{[2]} x^k + x \sum_{k'} \begin{bmatrix} n-k'-2 \\ k' \end{bmatrix}_{[2]} q^{n-k'-2} x^{k'} + x \sum_{k'} \begin{bmatrix} n-k'-3 \\ k' \end{bmatrix}_{[2]} q^{n-k'-3} x^{k'} \\ &= \mathbb{T}_n^{(2)}(x) + q^{n-2} x \mathbb{T}_{n-1}^{(2)}(x/q) + q^{n-3} x \mathbb{T}_{n-2}^{(2)}(x/q). \end{aligned}$$

□

Similarly, one can prove the relation (4.7) using the relation (4.3).

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