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The 2-adic and 3-adic valuation of the Tripell sequence and an application

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Abstract: Let $(T_n)_{n\geq 0}$ denote the Tripell sequence, defined by the linear recurrence $T_n = 2T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$ with $T_0 = 0$, $T_1 = 1$ and $T_2 = 2$ as initial conditions. In this paper, we study the 2-adic and 3-adic valuation of the Tripell sequence and, as an application, we determine all Tripell numbers which are factorials.

Key words: Tripell sequence, p-adic valuation, factorials

1. Introduction

Linear recurrence sequences are a subject of extensive study in number theory. For instance, the Fibonacci sequence $(F_n)_{n\geq 0}$ is one of the most famous and curious numerical sequences in mathematics and has been widely studied in the literature. Another important example is the Pell sequence $(P_n)_{n\geq 0}$, defined by the recurrence $P_n = 2P_{n-1} + P_{n-2}$ for all $n \geq 2$ with $P_0 = 0$ and $P_1 = 1$ as initial conditions. Pell numbers can be used to deduce important arithmetic properties for a large class of linear recurrence sequences. For the beauty and rich applications of these numbers and their relatives one can see Koshy's book [6].

In number theory, for a given prime number p, the p-adic valuation, or p-adic order, of a nonzero integer n, denoted by $\nu_p(n)$, is the exponent of the highest power of p which divides n. The p-adic order of certain linear recurrence sequences has been studied by many authors. For example, the p-adic order of the Fibonacci numbers was completely characterized by Lengyel in [8]. In 2016, Sanna [13] gave simple formulas for the p-adic order $\nu_p(u_n)$, in terms of $\nu_p(n)$ and the rank of apparition of p in $(u_n)_{n\geq 0}$, where $(u_n)_{n\geq 0}$ is a nondegenerate Lucas sequence. In particular, from the main theorems of Lengyel and Sanna we extract the following results:

Theorem 1.1 *For each positive integer n and each prime number $p \neq 2, 5$, we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

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In addition, $\nu_5(F_n) = \nu_5(n)$ *and*

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{\ell(p)}), & \text{if } n \equiv 0 \pmod{\ell(p)}; \\ 0, & \text{otherwise}; \end{cases}$$

where $\ell(p)$ *is the least positive integer such that* $p \mid F_{\ell(p)}$.

Theorem 1.2 *For each positive integer* n *and each prime number* $p \neq 2$, *we have that* $\nu_2(P_n) = \nu_2(n)$ *and*

$$\nu_p(P_n) = \begin{cases} \nu_p(n) + \nu_p(P_{\ell(p)}), & \text{if } n \equiv 0 \pmod{\ell(p)}; \\ 0, & \text{otherwise}; \end{cases}$$

where $\ell(p)$ *is the least positive integer such that* $p \mid P_{\ell(p)}$.

However, much less is known about the behavior of the p-adic valuation of linear recurrence sequences of higher order. A particular case of linear recurrence sequences of order 3 was studied by Marques and Lengyel in [9]. They catheterized the 2-adic valuation of the Tribonacci sequence. The Tribonacci sequence $(t_n)_{n\geq 0}$ starts with $t_0 = 0$, $t_1 = 1$, $t_2 = 1$ and satisfies the recurrence $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ for all $n \geq 3$. Results on the 2-adic valuation of tetra- and pentanacci numbers can be found in [10]. See also [14, 15] for the behavior of the 2-adic valuation of generalized Fibonacci numbers and some applications to certain Diophantine equations.

The Pell sequence and its generalizations have been studied by some authors (see [3–5]). For example, in [4], Kiliç gave some relations involving Fibonacci and generalized Pell numbers showing that generalized Pell numbers can be expressed as the summation of the Fibonacci numbers.

One of the generalizations of the Pell sequence is what we have called the *Tripell sequence* $(T_n)_{n\geq 0}$. This sequence starts with $T_0 = 0$, $T_1 = 1$, $T_2 = 2$ and each following term is given by the recurrence

$$T_n = 2T_{n-1} + T_{n-2} + T_{n-3}. \tag{1.1}$$

Below we present the first few elements of the Tripell sequence:

$$0, 1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396, \dots$$

In this paper we use the theory of constructing identities given by Zhou in [16] and several congruence results to partially characterize the 2-adic valuation of the Tripell sequence and fully characterize the 3-adic valuation $\nu_3(T_n)$.

We next present our theorems in which we give simple formulas for the 2-adic valuation $\nu_2(T_n)$ (for most of the values of n) and the 3-adic valuation $\nu_3(T_n)$ of the Tripell numbers in terms of $\nu_2(n)$ and $\nu_3(n)$, respectively.

Theorem 1.3 *The* 2*-adic valuation of the nth Tripell number is given by*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \equiv 1, 3, 4, 5 \pmod{7}; \\ 2, & \text{if } n \equiv 9 \pmod{14}; \\ 1, & \text{if } n \equiv 2, 7 \pmod{14}; \\ \nu_2(n) + 1, & \text{if } n \equiv 0 \pmod{14}; \\ \nu_2(n+1) + 1, & \text{if } n \equiv 13 \pmod{14}. \end{cases}$$

If $n \equiv 6 \pmod{14}$, *then* $\nu_2(T_n) = \nu_2(n) + 1$ *except when* $n \equiv 1280 \pmod{1792}$ *or, equivalently, when* n *is of the form*

$$n = 14(2^7 t + 2^6 + 2^4 + 2^3 + 2 + 1) + 6 = 1792t + 1280 \quad \textit{with} \quad t \geq 0.$$

Figure 1 shows the first few values of $\nu_2(T_n)$.

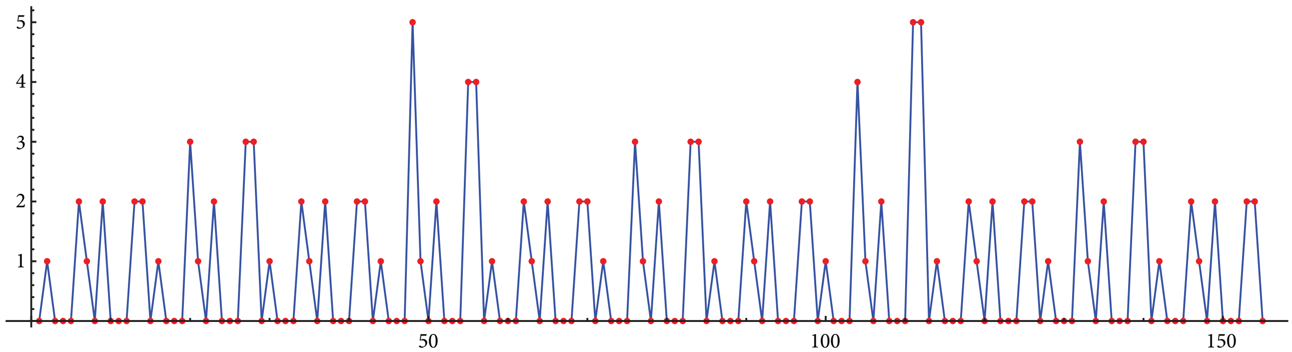

Figure 1. The 2-adic valuation of the Tripell numbers

Theorem 1.4 *The* 3*-adic valuation of the nth Tripell number is given by*

$$\nu_3(T_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2, 3, 4 \pmod{6}; \\ \nu_3(n), & \text{if } n \equiv 0 \pmod{6}; \\ \nu_3(n+1), & \text{if } n \equiv 5 \pmod{6}. \end{cases}$$

As a consequence, we notice that $\nu_3(T_{2n+1}) = \nu_3(T_{2n+2})$ for $n \geq 1$. Figure 2 shows the first few values of $\nu_3(T_n)$.

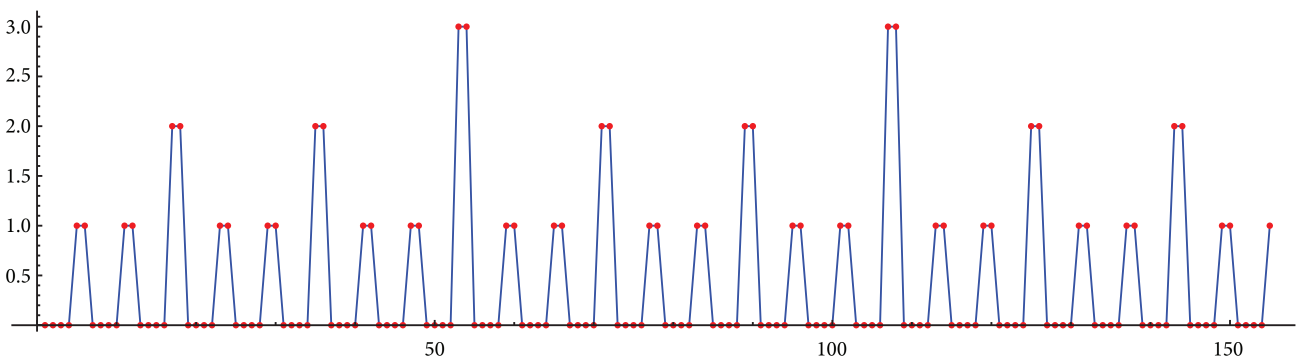

Figure 2. The 3-adic valuation of the Tripell numbers

There are many papers in the literature dealing with Diophantine equations obtained by asking whether members of some fixed binary recurrence sequence are factorials or belong to some other interesting sequence of positive integers. For example, in [2], all Fibonacci numbers which are sums of three factorials were found, while in [11], all factorials which are sums of three Fibonacci numbers were found.

In this paper we also present an application of Theorem 1.4, in which we determine all Tripell numbers which are factorials. We have the following result.

Theorem 1.5 *The only solutions of the Diophantine equation*

$$T_n = m! \tag{1.2}$$

in positive integers n, m *are*

$$(n, m) \in \{(1, 1), (2, 2)\}.$$

We point out that for finding factorials belonging to some binary recurrence sequences, or related problems, the existence of primitive divisors (see [1]) is sometimes used. However, similar divisibility properties for linear recurrences of higher order are not known; therefore, it is necessary to tackle the problem differently. It turns out that one can use the p-adic valuation of the terms of these sequences and use it to establish upper bounds on the solutions of some Diophantine equations.

In this work we prove Theorem 1.5 by a simple method which makes use of the 3-adic valuation of the terms of the Tripell sequence.

2. Auxiliary results

In this section, we present some auxiliary results that are needed in the proofs of the main theorems. To begin with, we give an auxiliary lemma, which is a consequence of Legendre's formula for $\nu_p(m!)$ (see [7]).

Lemma 2.1 *For any integer* $m \geq 1$ *and prime* p, *we have*

$$\frac{m}{p-1} - \left\lfloor \frac{\log m}{\log p} \right\rfloor - 1 \leq \nu_p(m!) \leq \frac{m-1}{p-1},$$

where $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.

A proof of Lemma 2.1 can be found in [12].

We next mention some facts about the Tripell sequence, which will be used later. First, it is easily checked that its characteristic polynomial $f(x) = x^3 - 2x^2 - x - 1$ is irreducible in $\mathbb{Q}[x]$. In addition, $f(x)$ has a real root $\gamma > 1$ and two conjugate complex roots inside the unit circle. In fact,

$$\gamma = \frac{1}{3}\left(2 + \sqrt[3]{\frac{61}{2} - \frac{9\sqrt{29}}{2}} + \sqrt[3]{\frac{61}{2} + \frac{9\sqrt{29}}{2}}\right) = 2.54682\ldots.$$

The following lemma shows the exponential growth of $(T_n)_{n\geq 0}$.

Lemma 2.2 *For all* $n \geq 1$, *we have*

$$\gamma^{n-2} \leq T_n \leq \gamma^{n-1}.$$

Proof We prove Lemma 2.2 by using induction on n. First, note that the result is true for $n = 1, 2, 3$ because

$$\gamma^{-1} \leq T_1 = 1 \leq \gamma^0, \quad \gamma^0 \leq T_2 = 2 \leq \gamma^1, \quad \text{and} \quad \gamma^1 \leq T_3 = 5 \leq \gamma^2.$$

Suppose now that the inequality $\gamma^{m-2} \leq T_m \leq \gamma^{m-1}$ holds for all m such that $1 \leq m \leq n-1$. It then follows from the recurrence relation for $(T_n)_{n\geq 0}$ (1.1) that

$$2\gamma^{n-3} + \gamma^{n-4} + \gamma^{n-5} \leq T_n \leq 2\gamma^{n-2} + \gamma^{n-3} + \gamma^{n-4}.$$

Thus,

$$\gamma^{n-5}(2\gamma^2+\gamma+1) \le T_n \le \gamma^{n-4}(2\gamma^2+\gamma+1),$$

which, combined with the fact that $\gamma^3 = 2\gamma^2+\gamma+1$, gives the desired result. Thus, Lemma 2.2 holds for all positive integers n. □

In [16] Zhou introduced the theory of constructing identities. Basically, it shows how to use certain kinds of polynomial congruences to prove identities for linear recurrence sequences. We apply this technique to obtain the following identity involving Tripell numbers. This result plays a crucial role in the proofs of Theorems 1.3 and 1.4.

Lemma 2.3 *For all m, n, with $m \ge 3$ and $n \ge 0$, we have that*

$$\begin{aligned} T_{m+n} &= T_{m-1}T_{n+2} + (T_{m-2}+T_{m-3})T_{n+1} + T_{m-2}T_n \\ &= T_{m-1}T_{n+2} + (T_m - 2T_{m-1})T_{n+1} + T_{m-2}T_n. \end{aligned}$$

Proof It is easily seen that the lemma holds for $m=3$, so we assume that $m \ge 4$. First, note that $h(x) = x^{m+n} - 2x^{m+n-1} - x^{m+n-2} - x^{m+n-3} \equiv 0 \pmod{f(x)}$, where $f(x)$ is the characteristic polynomial of the sequence $(T_n)_{n\ge 0}$. Thus

$$\begin{aligned} &h(x)(T_1 + T_2x^{-1} + \cdots + T_{m-3}x^{-m+4} + T_{m-2}x^{-m+3}) \\ &= T_1x^{m+n} + T_2x^{m+n-1} + T_3x^{m+n-2} + \cdots + T_{m-3}x^{n+4} + T_{m-2}x^{n+3} \\ &\quad - 2T_1x^{m+n-1} - 2T_2x^{m+n-2} - 2T_3x^{m+n-3} - \cdots - 2T_{m-3}x^{n+3} - 2T_{m-2}x^{n+2} \\ &\quad - T_1x^{m+n-2} - T_2x^{m+n-3} - T_3x^{m+n-4} - \cdots - T_{m-3}x^{n+2} - T_{m-2}x^{n+1} \\ &\quad - T_1x^{m+n-3} - T_2x^{m+n-4} - T_3x^{m+n-5} - \cdots - T_{m-3}x^{n+1} - T_{m-2}x^{n} \\ &= T_1x^{m+n} - (2T_{m-2}+T_{m-3}+T_{m-4})x^{n+2} - (T_{m-2}+T_{m-3})x^{n+1} - T_{m-2}x^n \\ &\equiv 0 \pmod{f(x)}. \end{aligned}$$

By [16, Theorem 2.3] (TCI), we have

$$T_{m+n} = T_{m-1}T_{n+2} + (T_{m-2}+T_{m-3})T_{n+1} + T_{m-2}T_n.$$

□

3. Proof of Theorem 1.3

In order to prove Theorem 1.3, we first prove the following lemma.

Lemma 3.1 *For all $s, t \ge 1$, we have*

$$T_{2^t7s-2} \equiv 1 \pmod{2^{t+2}} \quad \text{and} \quad T_{2^t7s-i} \equiv \begin{cases} 2^{t+1} \pmod{2^{t+2}}, & \text{if } s \equiv 1 \pmod 2; \\ 0 \pmod{2^{t+2}}, & \text{if } s \equiv 0 \pmod 2; \end{cases} \tag{3.1}$$

for $i = 0, 1$.

Proof We first need to prove the congruences

$$T_{14s-2} \equiv 1 \pmod 8 \quad \text{and} \quad T_{14s-i} \equiv \begin{cases} 4 \pmod 8, & \text{if } s \equiv 1 \pmod 2; \\ 0 \pmod 8, & \text{if } s \equiv 0 \pmod 2; \end{cases} \tag{3.2}$$

for $i = 0, 1$. Indeed, suppose s is odd, so $s = 2r + 1$ for some integer $r \geq 0$. Since $(T_n \mod 8)_{n \geq 0}$ is periodic with period 28, we have that

$$\begin{aligned} T_{14(2r+1)-2} &= T_{28r+12} \equiv T_{12} \equiv 22929 \equiv 1 \pmod 8, \\ T_{14(2r+1)-1} &= T_{28r+13} \equiv T_{13} \equiv 58396 \equiv 4 \pmod 8, \\ T_{14(2r+1)} &= T_{28r+14} \equiv T_{14} \equiv 148724 \equiv 4 \pmod 8. \end{aligned}$$

This proves that (3.2) holds when s is odd. A similar argument can be applied in the case where s is even. Thus (3.2) holds for all $s \geq 1$. Now for a fixed s, we use induction on t to prove the congruences given by (3.1). Note that, by (3.2), (3.1) holds for $t = 1$. Suppose now that congruences (3.1) are true for $t-1$. Suppose further that s is odd. The case when s is even can be handled in a similar way. Thus,

$$T_{2^{t-1}7s-2} \equiv 1 \pmod{2^{t+1}} \quad \text{and} \quad T_{2^{t-1}7s-i} \equiv 2^t \pmod{2^{t+1}},$$

for $i = 0, 1$, and so

$$T_{2^{t-1}7s-2} = 1 + 2^{t+1}k_1, \quad T_{2^{t-1}7s-1} = 2^t + 2^{t+1}k_2, \quad \text{and} \quad T_{2^{t-1}7s} = 2^t + 2^{t+1}k_3,$$

for some integers k_1, k_2, and k_3. We derive from all this and Lemma 2.3 that

$$\begin{aligned} T_{2^t7s-2} &= T_{(2^{t-1}7s)+(2^{t-1}7s-2)} \\ &= (2^t + 2^{t+1}k_2)(2^t + 2^{t+1}k_3) + (2^t + 2^{t+1}k_3 - 2(2^t + 2^{t+1}k_2))(2^t + 2^{t+1}k_2) \\ &\quad + (1 + 2^{t+1}k_1)(1 + 2^{t+1}k_1) \\ &\equiv 1 \pmod{2^{t+2}}, \end{aligned}$$

as desired. Similarly, we can prove that

$$\begin{aligned} T_{2^t7s-1} &= T_{(2^{t-1}7s+1)+(2^{t-1}7s-2)} \equiv 2^{t+1} \pmod{2^{t+2}} \quad \text{and} \\ T_{2^t7s} &= T_{(2^{t-1}7s+2)+(2^{t-1}7s-2)} \equiv 2^{t+1} \pmod{2^{t+2}}. \end{aligned}$$

This completes the proof of Lemma 3.1. □

Proof of Theorem 1.3

To prove this theorem, we need to work on each case separately.

(*a*) Let $n \equiv a \pmod 7$ with $a \in \{1, 3, 4, 5\}$. Then it is not difficult to see that $n \equiv 7i + a \pmod{28}$ for some $i \in \{0, 1, 2, 3\}$. Since $(T_n \mod 8)_{n \geq 0}$ is periodic with period 28, it follows that $T_n \equiv T_{7i+a} \pmod 8$. However, one can check by hand that $T_{7i+a} \equiv 1, 3, 5$ or $7 \pmod 8$, and so $\nu_2(T_n) = 0$.

(b) If $n \equiv 9 \pmod{14}$, then $n \equiv 9$ or $23 \pmod{28}$. By using the periodicity of $(T_n \mod 8)_{n\geq 0}$ and taking into account that $T_9 \equiv T_{23} \equiv 4 \pmod 8$, we conclude that $\nu_2(T_n) = 2$.

(c) Suppose now that $n \equiv a \pmod{14}$ with $a \in \{2, 7\}$. Then $n \equiv a$ or $14 + a \pmod{28}$. Here we have that $T_2 \equiv T_{16} \equiv T_{21} \equiv 2 \pmod 8$ and $T_7 \equiv 6 \pmod 8$. Thus, $\nu_2(T_n) = 1$.

(d) If $n \equiv 0 \pmod{14}$, then $n = 2^t 7s$ for some $s, t \geq 1$ with s odd. Hence, $\nu_2(n) = t$. In addition, by Lemma 3.1 we have that $T_n \equiv 2^{t+1} \pmod{2^{t+2}}$. Thus, $\nu_2(T_n) = t + 1 = \nu_2(n) + 1$.

(e) If $n \equiv 13 \pmod{14}$, then $n = 2^t 7s - 1$ for some $s, t \geq 1$ with s odd. From this $\nu_2(n + 1) = t$. Furthermore, by Lemma 3.1 we get $T_n \equiv 2^{t+1} \pmod{2^{t+2}}$. Consequently, $\nu_2(T_n) = t+1 = \nu_2(n+1)+1$.

(f) We finally deal with the special case when $n \equiv 6 \pmod{14}$. Here we have to prove that $\nu_2(T_n) = \nu_2(n)+1$ except for some special case for n that will be fully characterized. In order to do this, we first write n as $n = 14s + 6$ for some $s \geq 1$. We now distinguish two cases.

Case 1. s is even. In this case $n \equiv 6 \pmod{28}$ and so $\nu_2(n) = 1$. In addition, since $(T_n \mod 8)_{n\geq 0}$ is periodic with period 28, we can conclude that $T_n \equiv T_6 \equiv 84 \equiv 4 \pmod 8$; therefore, $\nu_2(T_n) = 2$. Consequently, $\nu_2(T_n) = \nu_2(n) + 1$.

Case 2. s is odd. Here one of the following cases must hold (for some integer $t \geq 0$):

$$\begin{array}{ll} (i)\ s = 2^2 t + 1, & (v)\ s = 2^4 t + 2 + 1, \\ (ii)\ s = 2^3 t + 2^2 + 2 + 1, & (vi)\ s = 2^5 t + 2^3 + 2 + 1, \\ (iii)\ s = 2^7 t + 2^4 + 2^3 + 2 + 1, & (vii)\ s = 2^7 t + 2^6 + 2^4 + 2^3 + 2 + 1. \\ (iv)\ s = 2^6 t + 2^5 + 2^4 + 2^3 + 2 + 1, & \end{array} \tag{3.3}$$

We shall work only with the first case, when $s = 2^2 t + 1$, in order to avoid unnecessary repetitions. The other cases, except the last one, are handled in much the same way. We will prove that

$$\nu_2(T_{14(2^2 t+1)+6}) = \nu_2(14(2^2 t + 1) + 6) + 1 \quad \text{for all} \quad t \geq 0,$$

by using induction on t. First, note that the base case $t = 0$ follows from $\nu_2(20) = 2$ and $\nu_2(T_{20}) = \nu_2(40585304) = 3$. Suppose now that the result holds true for $t - 1$. Then with $m = 14(2^2(t - 1) + 1)$, we have that $\nu_2(T_{m+6}) = 3$ and

$$\begin{aligned} T_{14(2^2 t+1)+6} &= T_{56+14(2^2(t-1)+1)+6} \\ &= T_{55} T_{m+8} + (T_{54} + T_{53}) T_{m+7} + T_{54} T_{m+6} \\ &= 2^4 k_1 T_{m+8} + 2^4 k_2 T_{m+7} + T_{54} 2^3 k_3, \end{aligned}$$

for some odd integers k_1, k_2 and k_3. By using this and taking into account that T_{54} is odd, we conclude that $\nu_2(T_{14(2^2 t+1)+6}) = 3 = \nu_2(14(2^2 t) + 20) + 1$.

Remark 3.2 *As we saw before, we proved Theorem 1.3 in the special case when $n \equiv 6 \pmod{14}$ by using mathematical induction on t and this technique worked for almost all of the forms given in (3.3). However, the induction argument does not work when $s = 2^7 t + 2^6 + 2^4 + 2^3 + 2 + 1$, since the base case $t = 0$ does not hold.*

4. Proof of Theorem 1.4

Here we discuss the 3-adic valuation of the Tripell numbers in a similar way as in the previous section.

Lemma 4.1 *For all* $s, t \geq 1$, $s \not\equiv 0 \pmod 3$, *we have*

$$T_{2\cdot 3^t s-1} \equiv \begin{cases} 2\cdot 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 1 \pmod 3; \\ 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 2 \pmod 3; \end{cases}$$

$$T_{2\cdot 3^t s} \equiv \begin{cases} 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 1 \pmod 3; \\ 2\cdot 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 2 \pmod 3; \end{cases}$$

and

$$T_{2\cdot 3^t s+1} \equiv \begin{cases} 1+2\cdot 3^t \pmod{3^{t+1}}, & \text{if } s \equiv 1 \pmod 3; \\ 1+3^t \pmod{3^{t+1}}, & \text{if } s \equiv 2 \pmod 3. \end{cases}$$

Proof The proof proceeds in a similar way to that of Lemma 3.1. Indeed, using the fact that $(T_n \mod 9)_{n\geq 0}$ is periodic with period 18, one can prove that the congruences are valid for all s and $t = 1$ (details are left to the reader). Thus, we shall consider the general case for any t and fixed s.

Suppose first that $s \equiv 1 \pmod 3$ and the congruences of the lemma are true for $t-1$. Hence, $T_{2\cdot 3^{t-1}s-2} \equiv 1 + 3^{t-1} \pmod{3^t}$ and consequently

$$\begin{aligned} T_{2\cdot 3^{t-1}s-2} &= 1 + 3^{t-1} + 3^t k_3, & T_{2\cdot 3^{t-1}s-1} &= 2\cdot 3^{t-1} + 3^t k_2, \\ T_{2\cdot 3^{t-1}s} &= 3^{t-1} + 3^t k_1, & T_{2\cdot 3^{t-1}s+1} &= 1 + 2\cdot 3^{t-1} + 3^t k_4, \end{aligned} \tag{4.1}$$

for some integers k_1, k_2, k_3, and k_4. We next show that congruences of the lemma are also true for t. To do this, we first need to compute $T_{2(2\cdot 3^{t-1}s)+i}$ for $i \in \{-1, 0, 1\}$. Indeed, by applying the summation identity from Lemma 2.3 and (4.1) we obtain

$$\begin{aligned} T_a := T_{2(2\cdot 3^{t-1}s)-1} &= T_{(2\cdot 3^{t-1}s+1)+(2\cdot 3^{t-1}s-2)} \\ &\equiv 3^{2t-2} + 2\cdot 3^{t-1} + 3^t k_2 + 2\cdot 3^{t-1} + 3^t k_2 + 2\cdot 3^{2t-2} \pmod{3^{t+1}} \\ &\equiv 3^{t-1} + 3^t + 2\cdot 3^t k_2 \pmod{3^{t+1}}, \\ T_b := T_{2(2\cdot 3^{t-1}s)} &= T_{(2\cdot 3^{t-1}s+2)+(2\cdot 3^{t-1}s-2)} \\ &\equiv 3^{t-1} + 3^t k_1 + 2\cdot 3^{2t-2} + 3^{t-1} + 3^{2t-2} + 3^t k_1 \pmod{3^{t+1}} \\ &\equiv 2\cdot 3^{t-1} + 2\cdot 3^t k_1 \pmod{3^{t+1}}, \end{aligned}$$

and

$$\begin{aligned} T_c := T_{2(2\cdot 3^{t-1}s)+1} &= T_{(2\cdot 3^{t-1}s+2)+(2\cdot 3^{t-1}s-1)} \\ &\equiv 1 + 2\cdot 3^{t-1} + 2\cdot 3^{t-1} + 3^t k_4 + 3^t k_4 + 4\cdot 3^{2t-2} + 2\cdot 3^{2t-2} \pmod{3^{t+1}} \\ &\equiv 1 + 3^{t-1} + 3^t + 2\cdot 3^t k_4 \pmod{3^{t+1}}. \end{aligned}$$

We thus get that

$$\begin{aligned}T_{2\cdot 3^t s-1} &= T_{(2\cdot 3^{t-1}s)+(2(2\cdot 3^{t-1}s)-1)}\\ &= T_{2\cdot 3^{t-1}s-1}T_c + (T_{2\cdot 3^{t-1}s} - 2T_{2\cdot 3^{t-1}s-1})T_b + T_{2\cdot 3^{t-1}s-2}T_a\\ &\equiv 2\cdot 3^{t-1} + 3^t k_2 + 2\cdot 3^{2t-2} + 3^{t-1} + 3^{2t-2} + 3^t + 2\cdot 3^t k_2 \pmod{3^{t+1}}\\ &\equiv 2\cdot 3^t \pmod{3^{t+1}},\end{aligned}$$

and

$$\begin{aligned}T_{2\cdot 3^t s} &= T_{(2\cdot 3^{t-1}s+1)+(2(2\cdot 3^{t-1}s)-1)}\\ &= T_{2\cdot 3^{t-1}s}T_c + (T_{2\cdot 3^{t-1}s-1} + T_{2\cdot 3^{t-1}s-2})T_b + T_{2\cdot 3^{t-1}s-1}T_a\\ &\equiv 3^{t-1} + 3^t k_1 + 3^{2t-2} + 2\cdot 3^{t-1} + 2\cdot 3^t k_1 + 2\cdot 3^{2t-2} \pmod{3^{t+1}}\\ &\equiv 3^t \pmod{3^{t+1}}.\end{aligned}$$

A similar argument (which we leave to the reader) shows that

$$T_{2\cdot 3^t s+1} \equiv 1 + 2\cdot 3^t \pmod{3^{t+1}}.$$

We now assume that $s \equiv 2 \pmod 3$. Then $s = 3k+2 = (3k+1)+1$ for some $k \in \mathbb{Z}$. In this case, with $m = 2\cdot 3^t(3k+1)$ and using the previously proved result for the case $s \equiv 1 \pmod 3$, we obtain

$$\begin{aligned}T_{2\cdot 3^t s-1} &= T_{(2\cdot 3^t)+(2\cdot 3^t(3k+1)-1)}\\ &= T_{2\cdot 3^t-1}T_{m+1} + (T_{2\cdot 3^t} - 2T_{2\cdot 3^t-1})T_m + T_{2\cdot 3^t-2}T_{m-1}\\ &\equiv (2\cdot 3^t)(1+2\cdot 3^t) + (3^t - 2(2\cdot 3^t))3^t + (1+3^t)(2\cdot 3^t) \pmod{3^{t+1}}\\ &\equiv 3^t \pmod{3^{t+1}},\end{aligned}$$

and

$$\begin{aligned}T_{2\cdot 3^t s} &= T_{(2\cdot 3^t+1)+(2\cdot 3^t(3k+1)-1)}\\ &= T_{2\cdot 3^t}T_{m+1} + (T_{2\cdot 3^t+1} - 2T_{2\cdot 3^t})T_m + T_{2\cdot 3^t-1}T_{m-1}\\ &\equiv 3^t(1+2\cdot 3^t) + (1+2\cdot 3^t - 2\cdot 3^t)3^t + (2\cdot 3^t)(2\cdot 3^t) \pmod{3^{t+1}}\\ &\equiv 2\cdot 3^t \pmod{3^{t+1}},\end{aligned}$$

as desired. Similarly, we can prove that

$$T_{2\cdot 3^t s+1} \equiv 1 + 3^t \pmod{3^{t+1}}.$$

□

Proof of Theorem 1.4

Suppose first that $n \equiv a \pmod 6$ with $a \in \{-1, 0\}$. Then n can be written as $n = 2\cdot 3^t s + a$ for $s, t \geq 1$ with $s \not\equiv 0 \pmod 3$. Thus, Lemma 4.1 yields $\nu_3(T_{2\cdot 3^t s+a}) = t$, and then

$$\nu_3(T_n) = \nu_3(T_{2\cdot 3^t s+a}) = t = \nu_3(2\cdot 3^t s) = \nu_3(n-a).$$

Suppose now that $n \equiv a \pmod 6$ with $a \in \{1, 2, 3, 4\}$. In this case, by using the fact that $(T_n \mod 3)_{n\geq 0}$ is periodic with period 6, we deduce that $T_n \equiv T_a \pmod 3$. However, one can easily check that $T_a \equiv 1$ or 2 $\pmod 3$ for all $a \in \{1, 2, 3, 4\}$, and so $\nu_3(T_n) = 0$.

5. Proof of Theorem 1.5

In this last section we apply the 3-adic order of the Tripell sequence to completely solve the Diophantine equation (1.2). Indeed, assume first that equation (1.2) holds. If $m \leq 5$, then the only solutions of (1.2) are those shown in Theorem 1.5, so we may assume that $m \geq 6$. Hence, the following inequality holds

$$m! < \left(\frac{m}{2}\right)^m. \tag{5.1}$$

On the other hand, by Theorem 1.4 we get that $\nu_3(T_n) = \nu_3(m!) \leq \nu_3(n) + \nu_3(n+1)$. From this and Lemma 2.1, for $p = 3$, we get

$$\frac{m}{2} - \left\lfloor \frac{\log m}{\log 3} \right\rfloor - 1 \leq \nu_3(m!) \leq 2\max\{\nu_3(n), \nu_3(n+1)\} \leq 2\nu_3(n+\delta),$$

where $\delta = 0, 1$. It then follows that

$$3^{\frac{m}{4} - \frac{\log m}{2\log 3} - \frac{1}{2}} \leq 3^{\nu_3(n+\delta)} \leq n + \delta \leq n + 1,$$

leading to

$$\frac{m}{4} - \frac{\log m}{2\log 3} - \frac{1}{2} \leq \frac{\log(n+1)}{\log 3}. \tag{5.2}$$

Additionally, by Lemma 2.2 and (5.1) we have $2.54^{n-2} < T_n = m! < (m/2)^m$; hence, $n < 2 + 1.1m\log(m/2)$. Inserting this upper bound on n into (5.2), we conclude that

$$\frac{m}{4} - \frac{\log m}{2\log 3} - \frac{1}{2} < \frac{\log(3 + 1.1m\log(m/2))}{\log 3}.$$

This inequality implies that $m < 25$, and therefore $n < 75$. Finally, a computational search with software *Mathematica* revealed that the only solutions to equation (1.2) are those mentioned in Theorem 1.5. Thus, Theorem 1.5 is proved.

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