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On extensions of two results due to Ramanujan

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Abstract: The aim in this note is to provide a generalization of an interesting entry in Ramanujan's notebooks that relate sums involving the derivatives of a function $\varphi(t)$ evaluated at 0 and 1. The generalization obtained is derived with the help of expressions for the sum of terminating ${}_3F_2$ hypergeometric functions of argument equal to 2, recently obtained by Kim et. al. [Two results for the terminating ${}_3F_2(2)$ with applications, Bulletin of the Korean Mathematical Society 2012; 49: 621-633]. Several special cases are given. In addition we generalize a summation formula to include integral parameter differences.

Key words: Hypergeometric series, Ramanujan's sum, sums of Hermite polynomials

1. Introduction

Two of the many interesting results stated by Ramanujan in his notebooks are the following theorems, which appear as Entry 8 [1, p. 51] and Entry 20 [1, p. 36], expressing an infinite sum of derivatives of a function $\varphi(t)$ at the origin to another infinite sum of its derivatives evaluated at $t = 1$.

Entry 8. *Let $\varphi(t)$ be analytic for $|t-1| < R$, where $R > 1$. Suppose that a and $\varphi(t)$ are such that the order of summation in*

$$\sum_{k=0}^{\infty} \frac{2^k (a)_k}{(2a)_k\, k!} \sum_{n=k}^{\infty} \frac{(-1)^n}{n!} (-n)_k \varphi^{(n)}(1)$$

may be inverted. Then

$$\sum_{k=0}^{\infty} \frac{2^k (a)_k \varphi^{(k)}(0)}{(2a)_k\, k!} = \sum_{k=0}^{\infty} \frac{\varphi^{(2k)}(1)}{2^{2k}(a+\frac{1}{2})_k\, k!}. \tag{1.1}$$

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Entry 20. *Let* $\varphi(t) = \sum_{k=0}^{\infty} \varphi^{(k)}(1)(t-1)^k/k!$ *be analytic for* $|t-1| < R$, *where* $R > 1$. *Suppose that* a *and* b *are complex parameters such that the order of summation in*

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k k!} \sum_{n=k}^{\infty} \frac{(-1)^n}{n!}(-n)_k \varphi^{(n)}(1)$$

may be inverted. Then

$$\sum_{k=0}^{\infty} \frac{(a)_k\, \varphi^{(k)}(0)}{(b)_k k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (b-a)_k}{(b)_k k!} \varphi^{(k)}(1). \tag{1.2}$$

Berndt [1] pointed out that Entry 8 can be established with the help of the results

$$ {}_2F_1\left[\begin{matrix} -2n, a \\ 2a \end{matrix}; 2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}, \qquad {}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a \end{matrix}; 2\right] = 0,$$

for nonnegative integer n, where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol.

In [4], each of the above theorems was generalized. The result in (1.1) was extended by replacing the denominatorial parameter $2a$ by $2a+j$, where $j = 0, \pm 1, \ldots, \pm 5$. The second result in (1.2) was extended by the inclusion of an additional pair of numeratorial and denominatorial parameters differing by unity to produce the following theorem.

Theorem 1.1 . *Let* $\varphi(t)$ *be analytic for* $|t-1| < R$, *where* $R > 1$. *Suppose that* a, b, d, *and* $\varphi(t)$ *are such that the order of summation in*

$$\sum_{k=0}^{\infty} \frac{(a)_k (d+1)_k}{(b)_k (d)_k k!} \sum_{n=k}^{\infty} \frac{(-1)^n}{n!}(-n)_k \varphi^{(n)}(1)$$

may be inverted. Then

$$\sum_{k=0}^{\infty} \frac{(a)_k (d+1)_k}{(b)_k (d)_k} \frac{\varphi^{(k)}(0)}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (b-a-1)_k (f+1)_k}{(b)_k (f)_k} \frac{\varphi^{(k)}(1)}{k!}, \tag{1.3}$$

where $f = d(b-a-1)/(d-a)$.

This result was extended to the case where a pair of numeratorial and denominatorial parameters differs by a positive integer m to produce

$$\sum_{k=0}^{\infty} \frac{(a)_k (d+m)_k}{(b)_k (d)_k} \frac{\varphi^{(k)}(0)}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (b-a-m)_k}{(b)_k} \frac{((\xi_m+1))_k}{((\xi_m))_k} \frac{\varphi^{(k)}(1)}{k!},$$

where the $\xi_1, \ldots, \xi_m$ are the zeros of a certain polynomial of degree m.

In this note we shall similarly extend the result in (1.1) (when the parameter $2a$ is replaced by $2a+1$) by the inclusion of a pair of numeratorial and denominatorial parameters differing by unity. For this we shall require the summations of a ${}_3F_2$ hypergeometric function of argument equal to 2 obtained* in [3, Theorem 2]

$$ {}_3F_2\left[\begin{matrix}-2n, a, d+1\\ 2a+1, d\end{matrix}; 2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}, \tag{1.4}$$

and

$$ {}_3F_2\left[\begin{matrix}-2n-1, a, d+1\\ 2a+1, d\end{matrix}; 2\right] = \frac{(1-2a/d)}{2a+1}\,\frac{(\frac{3}{2})_n}{(a+\frac{3}{2})_n}, \tag{1.5}$$

for nonnegative integer n. Several applications are presented in Section 3.

In the final section we generalize the result given in [1, p. 25] as Entry 9:

Entry 9. *If Re $(c-a)>0$, then*

$$\sum_{k=1}^{\infty}\frac{(a)_k}{k(c)_k} = \psi(c)-\psi(c-a), \tag{1.6}$$

where $\psi(x)$ denotes the logarithmic derivative of $\Gamma(x)$.

We extend this summation to include additional numeratorial and denominatorial parameters differing by positive integers. To achieve this we make use of the generalized Karlsson–Minton summation formula for a ${}_{r+2}F_{r+1}$ hypergeometric function of unit argument.

2. Generalization of Ramanujan's result (1.1)

The result to be established in this section is given by the following theorem.

Theorem 2.1 . *Let $\varphi(t)$ be analytic for $|t-1|<R$, where $R>1$. Suppose that a, d, and $\varphi(t)$ are such that the order of summation in*

$$\sum_{k=0}^{\infty}\frac{2^k(a)_k(d+1)_k}{(2a+1)_k(d)_k\,k!}\sum_{n=k}^{\infty}\frac{(-1)^n}{n!}\,(-n)_k\varphi^{(n)}(1)$$

may be inverted. Then

$$\sum_{k=0}^{\infty}\frac{2^k(a)_k(d+1)_k\varphi^{(k)}(0)}{(2a+1)_k(d)_k\,k!} = \sum_{k=0}^{\infty}\frac{\varphi^{(2k)}(1)}{2^{2k}(a+\frac{1}{2})_k\,k!} - \frac{(1-2a/d)}{2a+1}\sum_{k=0}^{\infty}\frac{\varphi^{(2k+1)}(1)}{2^{2k}(a+\frac{3}{2})_k\,k!}. \tag{2.1}$$

Proof. Since $\varphi(t)$ is analytic for $|t-1|<R$, we have

$$\varphi^{(k)}(0) = \sum_{n=k}^{\infty}\frac{(-1)^n(-n)_k}{n!}\,\varphi^{(n)}(1)$$

*It should be noted that the right-hand side of (1.4) is independent of the parameter d.

by suitable differentiation of the associated Taylor series. Then

$$\begin{aligned}S := \sum_{k=0}^{\infty} \frac{2^k (a)_k (d+1)_k}{(2a+1)_k (d)_k} \frac{\varphi^{(k)}(0)}{k!} &= \sum_{k=0}^{\infty} \frac{2^k (a)_k (d+1)_k}{(2a+1)_k (d)_k k!} \sum_{n=k}^{\infty} \frac{(-1)^n (-n)_k}{n!} \varphi^{(n)}(1) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{(n)}(1)}{n!} \sum_{k=0}^{n} \frac{2^k (a)_k (d+1)_k (-n)_k}{(2a+1)_k (d)_k k!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{(n)}(1)}{n!} \, {}_3F_2 \left[\begin{matrix} -n, a, d+1 \\ 2a+1, d \end{matrix} ; 2 \right]\end{aligned}$$

upon inversion of the order of summation by hypothesis.

If we now separate the above sum into terms involving even and odd n, we obtain

$$S = \sum_{n=0}^{\infty} \frac{\varphi^{(2n)}(1)}{(2n)!} \, {}_3F_2 \left[\begin{matrix} -2n, a, d+1 \\ 2a+1, d \end{matrix} ; 2 \right] - \sum_{n=0}^{\infty} \frac{\varphi^{(2n+1)}(1)}{(2n+1)!} \, {}_3F_2 \left[\begin{matrix} -2n-1, a, d+1 \\ 2a+1, d \end{matrix} ; 2 \right].$$

Finally, using the summations in (1.4) and (1.5) and noting that $(2n)! = 2^{2n} (\frac{1}{2})_n n!$, $(2n+1)! = 2^{2n} (\frac{3}{2})_n n!$, we easily arrive at the right-hand side of (2.1). This completes the proof of the theorem. □

When $d = 2a$ it is seen that (2.1) reduces to Ramanujan's result in (1.1).

3. Examples of Theorem 2.1

In this section, we provide some examples of different choices for the function $\varphi(t)$ appearing in (2.1). Throughout this section we let k denote a nonnegative integer.

(a) First we consider the simplest choice with $\varphi(t) = \exp(xt)$, where x is an arbitrary variable (independent of t). Then $\varphi^{(k)}(t) = x^k \varphi(t)$, which satisfies the conditions for the validity of (2.1). Substitution of the derivatives into (2.1) and identification of the resulting series as hypergeometric functions immediately yields

$$e^{-x} {}_2F_2 \left[\begin{matrix} a, d+1 \\ 2a+1, d \end{matrix} ; 2x \right] = {}_0F_1 \left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} ; \tfrac{1}{4}x^2 \right] - \frac{(1-2a/d)x}{2a+1} \, {}_0F_1 \left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ; \tfrac{1}{4}x^2 \right], \tag{3.1}$$

which is a result established by a different method in [9, Theorem 2]. In addition, it is interesting to observe that, since

$${}_0F_1 \left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} ; \tfrac{1}{4}x^2 \right] = \Gamma(a+\frac{1}{2})(\frac{1}{2}x)^{\frac{1}{2}-a} I_{a-\frac{1}{2}}(x),$$

where I_ν is the modified Bessel function of the first kind, the result (3.1) can also be written in terms of I_ν.

(b) If we let $\varphi(t) = \cosh(xt)$, we have $\varphi^{(2k)}(t) = x^{2k} \cosh(xt)$ and $\varphi^{(2k+1)}(t) = x^{2k+1} \sinh(xt)$. Then (2.1), after a little simplification making use of the identity

$$(a)_{2k} = (\frac{1}{2}a)_k (\frac{1}{2}a + \frac{1}{2})_k \, 2^{2k},$$

and letting $d \to 2d$, reduces to

$${}_3F_4 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}, d+1 \\ \frac{1}{2}, a+\frac{1}{2}, a+1, d \end{matrix} ; x^2 \right] = \cosh x \; {}_0F_1 \left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} ; \tfrac{1}{4}x^2 \right] - \frac{(1-a/d)}{2a+1} x \sinh x \; {}_0F_1 \left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ; \tfrac{1}{4}x^2 \right]. \tag{3.2}$$

(c) If $\varphi(t)=(x-t)^{-b}$, where b is an arbitrary parameter and $x>2$, then

$$\varphi^{(k)}(t)=\frac{(b)_k}{(x-t)^{b+k}}.$$

From (2.1), we therefore find

$$\left(\frac{x}{x-1}\right)^{-b}\sum_{k=0}^{\infty}\frac{2^k(a)_k(b)_k(d+1)_k}{(2a+1)_k(d)_k k!}\,x^{-k}=\sum_{k=0}^{\infty}\frac{2^{-2k}(b)_{2k}}{(a+\frac{1}{2})_k k!}\,(x-1)^{-2k}$$

$$-\frac{(1-2a/d)}{2a+1}\sum_{k=0}^{\infty}\frac{2^{-2k}(b)_{2k+1}}{(a+\frac{3}{2})_k k!}\,(x-1)^{-2k-1},$$

which yields

$$\left(\frac{x}{x-1}\right)^{-b}{}_3F_2\left[\begin{matrix}a,b,d+1\\2a+1,d\end{matrix};\frac{2}{x}\right]={}_2F_1\left[\begin{matrix}\frac{1}{2}b,\frac{1}{2}b+\frac{1}{2}\\a+\frac{1}{2}\end{matrix};\frac{1}{(x-1)^2}\right]$$

$$-\frac{(1-2a/d)b}{2a+1}(x-1)^{-1}{}_2F_1\left[\begin{matrix}\frac{1}{2}b+\frac{1}{2},\frac{1}{2}b+1\\a+\frac{3}{2}\end{matrix};\frac{1}{(x-1)^2}\right].$$

If we put $z=x/(1+x)$, this last result becomes

$$(1+z)^{-b}{}_3F_2\left[\begin{matrix}a,b,d+1\\2a+1,d\end{matrix};\frac{2z}{1+z}\right]$$

$$={}_2F_1\left[\begin{matrix}\frac{1}{2}b,\frac{1}{2}b+\frac{1}{2}\\a+\frac{1}{2}\end{matrix};z^2\right]-\frac{(1-2a/d)b}{2a+1}\,z\,{}_2F_1\left[\begin{matrix}\frac{1}{2}b+\frac{1}{2},\frac{1}{2}b+1\\a+\frac{3}{2}\end{matrix};z^2\right], \tag{3.3}$$

which has been obtained by different methods in [3, Theorem 3].

(d) Finally, with $\varphi(t)=\exp(-x^2t^2/4)$, we have [8, p. 442]

$$\varphi^{(k)}(t)=(-1)^k(x/2)^k e^{-x^2t^2/4}\,H_k(xt/2),$$

where H_k is the Hermite polynomial of order k. Since $H_{2k}(0)=(-1)^k(2k)!/k!$ and $H_{2k+1}(0)=0$, it follows from (2.1) that

$$e^{x^2/4}\,{}_3F_3\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2},d+1\\a+\frac{1}{2},a+1,d\end{matrix};-x^2\right]$$

$$=\sum_{k=0}^{\infty}\frac{(x/4)^{2k}}{(a+\frac{1}{2})_k k!}\,H_{2k}(x/2)+\frac{(1-a/d)}{a+\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(x/4)^{2k+1}}{(a+\frac{3}{2})_k k!}\,H_{2k+1}(x/2) \tag{3.4}$$

provided $a\neq-\frac{1}{2},-\frac{3}{2},\ldots$, where we have put $d\to 2d$.

The above series involving the Hermite polynomials can be expressed in terms of ${}_2F_2$ functions since [4, Eqs. (36), (37)]

$$\sum_{k=0}^{\infty}\frac{(x/4)^{2k}}{(a+\frac{1}{2})_k k!}\,H_{2k}(x/2)=e^{x^2/4}\,{}_2F_2\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}\\a,a+\frac{1}{2}\end{matrix};-x^2\right],$$

$$\sum_{k=0}^{\infty}\frac{(x/4)^{2k}}{(a+\frac{3}{2})_k k!}\,H_{2k+1}(x/2)=xe^{x^2/4}\,{}_2F_2\left[\begin{matrix}\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2}\\ a+\frac{3}{2},a+2\end{matrix};-x^2\right],$$

to yield

$${}_3F_3\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2},d+1\\ a+\frac{1}{2},a+1,d\end{matrix};-x^2\right]={}_2F_2\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}\\ a,a+\frac{1}{2}\end{matrix};-x^2\right]+\frac{x^2(1-a/d)}{4a+2}\,{}_2F_2\left[\begin{matrix}\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2}\\ a+\frac{3}{2},a+2\end{matrix};-x^2\right].$$

We remark that this last result can be derived alternatively by writing $(d+1)_k/(d)_k=1+k/d$ in the series expansion of the ${}_3F_3$ function combined with use of the result for contiguous ${}_2F_2$ functions [4]:

$${}_2F_2\left[\begin{matrix}\alpha,\beta\\ \gamma,\delta\end{matrix};z\right]-{}_2F_2\left[\begin{matrix}\alpha,\beta\\ \gamma,\delta+1\end{matrix};z\right]=\frac{\alpha\beta z}{\gamma\delta(\delta+1)}\,{}_2F_2\left[\begin{matrix}\alpha+1,\beta+1\\ \gamma+1,\delta+2\end{matrix};z\right].$$

Finally, the representation (3.4) may be contrasted with the more general result obtained from Theorem 1 with $\varphi(t)=\exp(-x^2t^2/4)$ given in [4, Eq. (40)] (with $x\to 2x$ and $d\to 2d$):

$$e^{x^2}\,{}_3F_3\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2},d+1\\ \frac{1}{2}b,\frac{1}{2}b+\frac{1}{2},d\end{matrix};-x^2\right]=\sum_{k=0}^{\infty}\frac{(b-a-1)_k(f+1)_k}{(b)_k(f)_k}\,x^kH_k(x),$$

where $f=2d(b-a-1)/(2d-a)$.

4. Extension of the summation (1.6)

We employ the usual convention of writing the finite sequence of parameters $(a_1,\ldots,a_p)$ simply by (a_p) and the product of p Pochhammer symbols by $((a_p))_k\equiv(a_1)_k\ldots(a_p)_k$. In order to derive our extension of Ramanujan's sum (1.6) we make use of the generalized Karlsson–Minton summation theorem given below.

Theorem 4.1 *Let (m_r) be a sequence of positive integers and $m:=m_1+\cdots+m_r$. The generalized Karlsson–Minton summation theorem is given by [6, 7]*

$${}_{r+2}F_{r+1}\left[\begin{matrix}a,b,\ (d_r+m_r)\\ c,\quad (d_r)\end{matrix};1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\sum_{k=0}^{m}\frac{(-1)^k(a)_k(b)_kC_{k,r}}{(1+a+b-c)_k}\tag{4.1}$$

provided $\Re(c-a-b)>m$.

The coefficients $C_{k,r}$ appearing in (4.1) are defined for $0\le k\le m$ by

$$C_{k,r}=\frac{1}{\Lambda}\sum_{j=k}^{m}\sigma_j\mathbf{S}_j^{(k)},\qquad \Lambda=(d_1)_{m_1}\ldots(d_r)_{m_r},\tag{4.2}$$

with $C_{0,r}=1$, $C_{m,r}=1/\Lambda$. $\mathbf{S}_j^{(k)}$ denotes the Stirling numbers of the second kind and the σ_j $(0\le j\le m)$ are generated by the relation

$$(d_1+x)_{m_1}\cdots(d_r+x)_{m_r}=\sum_{j=0}^{m}\sigma_jx^j.\tag{4.3}$$

In [5], an alternative representation for the coefficients $C_{k,r}$ is given as the terminating hypergeometric series of unit argument

$$C_{k,r} = \frac{(-1)^k}{k!}\, {}_{r+1}F_r\left[\begin{matrix} -k, & (d_r+m_r) \\ & (d_r) \end{matrix}; 1\right]. \tag{4.4}$$

When $r=1$, with $d_1=d$, $m_1=m$, Vandermonde's summation theorem [10, p. 243] can be used to show that

$$C_{k,1} = \binom{m}{k}\frac{1}{(d)_k}. \tag{4.5}$$

Our extension of Ramanujan's summation in (1.6) is given by the following theorem.

Theorem 4.2 *Let (m_r) be a sequence of positive integers and $m := m_1+\cdots+m_r$. Then, provided Re $(c-a) > m$, we have*

$$\sum_{k=1}^{\infty}\frac{(a)_k\,((d_r+m_r))_k}{(c)_k\,((d_r))_k\,k} = \psi(c)-\psi(c-a)+\sum_{k=1}^{m}\frac{(-1)^k(a)_k\Gamma(k)C_{k,r}}{(1+a-c)_k}, \tag{4.6}$$

where $C_{k,r}$ are the coefficients defined in (4.3) and (4.4).

Proof. We follow the method of proof given in [1, p. 25]. If we differentiate logarithmically the left-hand side of (4.1) with respect to b and then set $b=0$, making use of the simple fact that

$$\frac{d}{dx}(x)_k\bigg|_{x=0} = (k-1)!, \qquad k\geq 1,$$

we immediately obtain

$$\sum_{k=1}^{\infty}\frac{(a)_k\,((d_r+m_r))_k}{(c)_k\,((d_r))_k\,k}$$

when Re $(c-a)>m$. Proceeding in a similar manner for the right-hand side of (4.1), we obtain

$$\psi(c)-\psi(c-a)+\sum_{k=1}^{m}\frac{(-1)^k(a)_k\Gamma(k)C_{k,r}}{(1+a-c)_k}.$$

This completes the proof of the theorem. □

When $r=1$, the coefficients $C_{k,1}$ are given by (4.5) and we obtain the summation

$$\sum_{k=1}^{\infty}\frac{(a)_k(d+m)_k}{(c)_k(d)_k k} = \psi(c)-\psi(c-a)+m!\sum_{k=1}^{m}\frac{(-1)^k(a)_k}{k(m-k)!(1+a-c)_k(d)_k} \tag{4.7}$$

provided Re $(c-a)>m$.

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