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Dissipative canonical type differential operators for first order

Rukiye ÖZTÜRK MERT[1,*], **Zameddin I. ISMAILOV**[2], **Pembe İPEK AL**[2]
[1]Department of Mathematics, Faculty of Art and Sciences, Hitit University, Corum, Turkey
[2]Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon, Turkey

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Abstract: In this paper, using the Calkin-Gorbachuk method, the general form of all maximally dissipative extensions of the minimal operator generated by the first order linear symmetric canonical type quasi-differential expression in the weighted Hilbert space of vector functions has been found. Also, the spectrum set of these extensions has been investigated.

Key words: Dissipative operator, quasi-differential operator, spectrum

1. Introduction

In the development of the studies on the spectral properties of an operator related with a boundary value problem acting on a Hilbert space, operator theory has a big importance. A linear closed operator $\mathcal{T} \subset H \to H$ in a Hilbert space H with a dense domain is called to be dissipative iff

$$Im(\mathcal{T}\psi, \psi)_H \geq 0, \ \psi \in D(T),$$

where $Im(\,\cdot\,,\,\cdot\,)$ and $D(\mathcal{T})$ denote the imaginary part of the inner product and the domain of the operator $\mathcal{T}$, respectively (see [6]). A dissipative operator which does not have any proper dissipative extension is called maximally dissipative [6]. It is well-known that their spectrum lies in the closed upper half-plane. Hence the open lower half-plane does not belong to the spectrum of $\mathcal{T}$. In mathematics and physics, maximally dissipative operators have important roles. In physics, they have many applications in hydrodynamic, laser and nuclear scattering theories.

It is noteworthy to mention that the theory of self-adjoint extensions of symmetric operators in any Hilbert space has been given by Neumann [12].

The further investigations of Vishik and Birman have dealt with the characterization of all nonnegative selfadjoint extensions of a symmetric operator (see [5]). In addition, the general information can be found in [5]. A fundamental technique to study the spectral properties of dissipative operators is the functional model theory which is given by Nagy and Foias [9].

Gorbachuk and Gorbachuk [6] and Rofe-Beketov and Kholkin [11] have researched the maximal dissipative extensions and analyzed the spectral properties of the minimal operator. The minimal operator is generated by formally symmetric differential-operator expression in the Hilbert space of vector-functions defined in one finite or infinite interval case. Also, it has equal deficiency indices.

*Correspondence: rukiyeozturkmert@hitit.edu.tr
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In [1–4, 8], some spectral analysis of a closed extensions based on Weyl function has been researched.

In the present study, we obtain the representation of all maximally dissipative extensions of the minimal operator. This minimal operator is generated by the first order linear symmetric canonical type quasi-differential expression with operator coefficient in the weighted Hilbert spaces of vector-functions defined on right semi-axis. Later on, we investigate the structure of the spectrum of such extensions.

2. Statement of the problem

Let H be a separable Hilbert space and $a \in \mathbb{R}$. In the Hilbert space $L^2_\kappa(H,(a,\infty))$ of vector functions on (a,∞), consider the following linear canonical type quasi-differential operator expression for first order in the form

$$l(\nu) = iJ(\kappa\nu)^{'}(\varsigma) + S\nu(\varsigma),$$

where:
(1) $\kappa : (a,\infty) \to (0,\infty)$;
(2) $\kappa \in C(a,\infty)$;
(3) $\dfrac{1}{\kappa} \in L^1(a,\infty)$;
(4) $S : D(S) \subset H \to H$ is a linear bounded selfadjoint operator with condition $S \geq 0$, $J \in L(H)$, $J^* = J$, $J^2 = I$, $JS = SJ$. Here the operator I will denote the identity operator in corresponding space.

The minimal operator L_0 corresponding to quasi-differential expression $l(\,\cdot\,)$ in $L^2_\kappa(H,(a,\infty))$ can be constructed by using same technique in [7]. The operator $L = (L_0)^*$ is called maximal operator corresponding to $l(\,\cdot\,)$ in $L^2_\kappa(H,(a,\infty))$.

It will be shown that the minimal operator is symmetric and it has nonzero equal deficiency indices in $L^2_\kappa(H,(a,\infty))$.

3. Description of maximally dissipative extensions

In this section, with the use of Calkin-Gorbachuk method, the general form of all maximally dissipative extensions of the operator L_0 in $L^2_\kappa(H,(a,\infty))$ in terms of boundary values has been obtained.

In a Hilbert space, the deficiency indices of any symmetric operator are defined as follows:

Definition 3.1 [10] *Let T be a symmetric operator, λ be an arbitrary nonreal number and H be a Hilbert space. We denote by $\mathcal{R}_{\overline{\lambda}}$ and $\mathcal{R}_\lambda$ the ranges of the operator $\left(T - \overline{\lambda}I\right)$ and $(T - \lambda I)$, respectively, where I is identity operator on H. Clearly, $\mathcal{R}_{\overline{\lambda}}$ and $\mathcal{R}_\lambda$ are subspaces of H, which need not necessarily be closed. We call $\left(H - \mathcal{R}_{\overline{\lambda}}\right)$ and $(H - \mathcal{R}_\lambda)$, which are their orthogonal complements, the deficiency spaces of the operator T and we denote them by $\mathcal{N}_{\overline{\lambda}}$ and $\mathcal{N}_\lambda$ respectively: thus*

$$\mathcal{N}_{\overline{\lambda}} = H - \mathcal{R}_{\overline{\lambda}}, \quad \mathcal{N}_\lambda = H - \mathcal{R}_\lambda.$$

The numbers

$$n_{\overline{\lambda}} = dim\mathcal{N}_{\overline{\lambda}}, \quad n_\lambda = dim\mathcal{N}_\lambda$$

are called deficiency indices of the operator T.

Let us prove the following important result which we will need:

Lemma 3.2 *The deficiency indices of the operator* L_0 *in* $L^2_\kappa(H,(a,\infty))$ *are of the form*

$$(n_+(L_0), n_-(L_0)) = (dimH, dimH).$$

Proof Let us take $S=0$ for the simplicity of calculations. The general solutions of the differential equations

$$iJ(\kappa\nu_\pm)^{'}(\varsigma) \pm i\nu_\pm(\varsigma) = 0, \ \varsigma > a$$

in $L^2_\kappa(H,(a,\infty))$ are of the forms

$$\nu_\pm(\varsigma) = \frac{1}{\kappa(\varsigma)} exp\left(\mp J\int_a^\varsigma \frac{1}{\kappa(\xi)}d\xi\right) f, \ f\in H, \ \varsigma > a.$$

From these representations we have

$$\begin{aligned}
\|\nu_+\|^2_{L^2_\kappa(H,(a,\infty))} &= \int_a^\infty \kappa(\varsigma)\|\nu_+(\varsigma)\|^2_H d\varsigma \\
&= \int_a^\infty \kappa(\varsigma)\|\frac{1}{\kappa(\varsigma)} exp\left(-J\int_a^\varsigma \frac{1}{\kappa(\xi)}d\xi\right) f\|^2_H d\varsigma \\
&\leq \int_a^\infty \frac{1}{\kappa(\varsigma)}\|exp\left(-J\int_a^\varsigma \frac{1}{\kappa(\xi)}d\xi\right)\|^2_H d\varsigma \|f\|^2_H \\
&\leq \int_a^\infty \frac{1}{\kappa(\varsigma)}\left(exp\left(\|J\|\int_a^\varsigma \frac{1}{\kappa(\xi)}d\xi\right)\right)^2 d\varsigma \|f\|^2_H \\
&\leq \int_a^\infty \frac{1}{\kappa(\varsigma)}d\varsigma\left(exp\left(\int_a^\infty \frac{1}{\kappa(\xi)}d\xi\right)\right)^2 \|f\|^2_H < \infty.
\end{aligned}$$

Hence,

$$n_+(L_0) = dimker(L+iI) = dimH.$$

Similarly,

$$n_-(L_0) = dimker(L-iI) = dimH.$$

This completes the proof. □

As a consequence of this result, the minimal operator has a maximally dissipative extension (see [6]). For the description of these extensions, we need to obtain the space of boundary values.

Definition 3.3 [6] *Let* $\mathfrak{H}$ *be any Hilbert space and* $S : D(S) \subset \mathfrak{H} \to \mathfrak{H}$ *be a closed densely defined symmetric operator on the Hilbert space having equal finite or infinite deficiency indices. A triplet* $(\boldsymbol{H}, \beta_1, \beta_2)$*, where* $\boldsymbol{H}$

is a Hilbert space, β_1 *and* β_2 *are linear mappings from* $D(S^*)$ *into* $\boldsymbol{H}$, *is called a space of boundary values for the operator* S, *if for any* $\nu, \vartheta \in D(S^*)$

$$(S^*\nu, \vartheta)_{\mathfrak{H}} - (\nu, S^*\vartheta)_{\mathfrak{H}} = (\beta_1(\nu), \beta_2(\vartheta))_{\boldsymbol{H}} - (\beta_2(\nu), \beta_1(\vartheta))_{\boldsymbol{H}}$$

while for any $\mathcal{G}_1, \mathcal{G}_2 \in \boldsymbol{H}$, *there exists an element* $\nu \in D(S^*)$ *such that* $\beta_1(\nu) = \mathcal{G}_1$ *and* $\beta_2(\nu) = \mathcal{G}_2$.

It is known that for any symmetric operator with equal deficiency indices, we have at least one space of boundary values (see [6]).

Lemma 3.4 *The triplet* (H, β_1, β_2), *where*

$$\begin{aligned} \beta_1 : D(L) \to H,\ \beta_1(\nu) &= \frac{1}{\sqrt{2}} \left(J(\kappa\nu)(\infty) - J(\kappa\nu)(a) \right), \\ \beta_2 : D(L) \to H,\ \beta_2(\nu) &= \frac{1}{i\sqrt{2}} \left((\kappa\nu)(\infty) + (\kappa\nu)(a) \right),\ \nu \in D(L) \end{aligned}$$

is a space of boundary values of the operator L_0 *in* $L^2_\kappa(H, (a, \infty))$.

Proof For any $\nu,\ \vartheta \in D(L)$, one can easily check that

$$\begin{aligned} & (L\nu, \vartheta)_{L^2_\kappa(H,(a,\infty))} - (\nu, L\vartheta)_{L^2_\kappa(H,(a,\infty))} \\ = \ & (iJ(\kappa\nu)' + S\nu, \vartheta)_{L^2_\kappa(H,(a,\infty))} - (\nu, iJ(\kappa\vartheta)' + S\vartheta)_{L^2_\kappa(H,(a,\infty))} \\ = \ & (iJ(\kappa\nu)', \vartheta)_{L^2_\kappa(H,(a,\infty))} - (\nu, iJ(\kappa\vartheta)')_{L^2_\kappa(H,(a,\infty))} \\ = \ & \int_a^\infty (iJ(\kappa\nu)'(\varsigma), \vartheta(\varsigma))_H \, \kappa(\varsigma) d\varsigma - \int_a^\infty (\nu(\varsigma), iJ(\kappa\vartheta)'(\varsigma))_H \, \kappa(\varsigma) d\varsigma \\ = \ & i \left[\int_a^\infty (J(\kappa\nu)'(\varsigma), (\kappa\vartheta)(\varsigma))_H d\varsigma + \int_a^\infty (J(\kappa\nu)(\varsigma), (\kappa\vartheta)'(\varsigma))_H d\varsigma \right] \\ = \ & i \int_a^\infty (J(\kappa\nu)(\varsigma), (\kappa\vartheta)(\varsigma))'_H d\varsigma \\ = \ & i \left[(J(\kappa\nu)(\infty), (\kappa\vartheta)(\infty))_H - (J(\kappa\nu)(a), (\kappa\vartheta)(a))_H \right] \\ = \ & (\beta_1(\nu), \beta_2(\vartheta))_H - (\beta_2(\nu), \beta_1(\vartheta))_H. \end{aligned}$$

Now let $f,\ g \in H$. Then one can find the function $\nu \in D(L)$ such that

$$\beta_1(\nu) = \frac{1}{\sqrt{2}}(J(\kappa\nu)(\infty) - J(\kappa\nu)(a)) = f,$$

$$\beta_2(\nu) = \frac{1}{i\sqrt{2}} \left((\kappa\nu)(\infty) + (\kappa\nu)(a) \right) = g.$$

By the above observations, we have

$$(\kappa\nu)(\infty) = (Jf + ig)/\sqrt{2} \ \text{ and } \ (\kappa\nu)(a) = (-Jf + ig)/\sqrt{2}.$$

If we choose the function $\nu(\,\cdot\,)$ as

$$\nu(\varsigma) = \frac{1}{\kappa(\varsigma)}(1 - e^{a-\varsigma})(Jf + ig)/\sqrt{2} + \frac{1}{\kappa(\varsigma)}e^{a-\varsigma}(-Jf + ig)/\sqrt{2},$$

it is obvious that $\nu \in D(L)$ and $\beta_1(\nu) = f,\ \beta_2(\nu) = g.$ □

By using Calkin-Gorbachuk method [6], one can immediately have the following:

Theorem 3.5 *If $\widetilde{L}$ is a maximally dissipative extension of the operator L_0 in $L^2_\kappa(H,(a,\infty))$, it is generated by the differential operator expression $l(\,\cdot\,)$ and the boundary condition*

$$(\Gamma - I)\left(J(\kappa\nu)(\infty) - J(\kappa\nu)(a)\right) + (\Gamma + I)\left((\kappa\nu)(\infty) + (\kappa\nu)(a)\right) = 0,$$

where $\Gamma : H \to H$ is a contraction operator. Moreover, the contraction operator Γ in H is uniquely determined by the extension $\widetilde{L}$, i.e. $\widetilde{L} = L_\Gamma$, and vice versa.

Proof Each maximally dissipative extension $\widetilde{L}$ of the operator L_0 is described by the differential operator expression $l(\,\cdot\,)$ with the boundary condition

$$(\Gamma - I)\beta_1(\nu) + i(\Gamma + I)\beta_2(\nu) = 0,$$

where $\Gamma : H \to H$ is a contraction operator. Therefore, by Lemma 3.4, we obtain that

$$(\Gamma - I)\left(J(\kappa\nu)(\infty) - J(\kappa\nu)(a)\right) + (\Gamma + I)\left((\kappa\nu)(\infty) + (\kappa\nu)(a)\right) = 0,\ \nu \in D(\widetilde{L}).$$

This completes the proof. □

Corollary 3.6 *In case that*

$$J = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} : H \oplus H \to H \oplus H,$$

$$S = \begin{pmatrix} S_0 & 0 \\ 0 & S_0 \end{pmatrix} : H \oplus H \to H \oplus H,$$

$S_0 \in L(H),\ S_0^* = S_0 \geq 0,$ all maximally dissipative extensions of the operator L_0 in $L^2_\kappa(H \oplus H,(a,\infty))$ are generated by the corresponding differential operator expression $l(\,\cdot\,)$ and the boundary condition

$$(\Gamma - I)\begin{pmatrix} -i(\kappa\nu_2)(\infty) + i(\kappa\nu_2)(a) \\ i(\kappa\nu_1)(\infty) - i(\kappa\nu_1)(a) \end{pmatrix} + (\Gamma + I)\begin{pmatrix} (\kappa\nu_1)(\infty) + (\kappa\nu_1)(a) \\ (\kappa\nu_2)(\infty) + (\kappa\nu_2)(a) \end{pmatrix} = 0,$$

where $\Gamma : H \oplus H \to H \oplus H$ is a contraction operator. Moreover, the contraction operator Γ in $H \oplus H$ is uniquely determined by the extension $\widetilde{L}$, i.e. $\widetilde{L} = L_\Gamma$, and vice versa.

On the other hand, when $\Gamma = \Gamma_1 \oplus \Gamma_2$, where $\Gamma_1 : H_1 \to H_1,\ \Gamma_2 : H_2 \to H_2$ two contraction operators, the boundary condition can be rewritten in the form

$$i(\Gamma_1 - I_1)\left(-(\kappa\nu_2)(\infty) + (\kappa\nu_2)(a)\right) + (\Gamma_1 + I_1)\left((\kappa\nu_1)(\infty) + (\kappa\nu_1)(a)\right) = 0,$$
$$i(\Gamma_2 - I_2)\left((\kappa\nu_1)(\infty) - (\kappa\nu_1)(a)\right) + (\Gamma_2 + I_2)\left((\kappa\nu_2)(\infty) + (\kappa\nu_2)(a)\right) = 0.$$

Hence,

$$i(\Gamma_1 - I_1)(\kappa\nu_2)(\infty) - (\Gamma_1 + I_1)(\kappa\nu_1)(\infty) = i(\Gamma_1 - I_1)(\kappa\nu_2)(a) + (\Gamma_1 + I_1)(\kappa\nu_1)(a),$$
$$i(\Gamma_2 - I_2)(\kappa\nu_1)(\infty) + (\Gamma_2 + I_2)(\kappa\nu_2)(\infty) = i(\Gamma_2 - I_2)(\kappa\nu_1)(a) - (\Gamma_2 + I_2)(\kappa\nu_2)(a),$$

where $I_k : H_k \to H_k,\ k = 1,2$ *are identity operators.*

4. The spectrum of the maximally dissipative extensions

In this section, the structure of the spectrum set of the maximally dissipative extensions L_Γ of the operator L_0 in $L^2_\kappa(H,(a,\infty))$ has been examined.

Theorem 4.1 *In order to* $\lambda \in \sigma(L_\Gamma)$ *the necessary and sufficient condition is*

$$0 \in \sigma(\Delta(\lambda)),$$

where,

$$\Delta(\lambda) = (\Gamma - I)J\left(exp\left(iJ(S-\lambda I)\int_a^\infty \frac{1}{\kappa(\xi)}d\xi\right) - I\right) + (\Gamma + I)\left(exp\left(iJ(S-\lambda I)\int_a^\infty \frac{1}{\kappa(\xi)}d\xi\right) + I\right).$$

Proof Consider the following spectrum problem for the operator L_Γ in $L^2_\kappa(H,(a,\infty))$, i.e.

$$L_\Gamma(\nu) = \lambda\nu + f,\ f \in L^2_\kappa(H,(a,\infty)),\ \lambda \in \mathbb{C},\ \lambda_i = Im\lambda \geq 0.$$

Then, we have

$$iJ(\kappa\nu)'(\varsigma) + S\nu(\varsigma) = \lambda\nu(\varsigma) + f(\varsigma),\ \varsigma > a,$$
$$(\Gamma - I)\left(J(\kappa\nu)(\infty) - J(\kappa\nu)(a)\right) + (\Gamma + I)\left((\kappa\nu)(\infty) + (\kappa\nu)(a)\right) = 0.$$

It is easily to find the general solution of the above differential equation as follows:

$$\nu(\varsigma;\lambda) = \frac{1}{\kappa(\varsigma)}exp\left(iJ(S-\lambda I)\int_a^\varsigma \frac{1}{\kappa(\xi)}d\xi\right)f_\lambda$$
$$+ \frac{i}{\kappa(\varsigma)}J\int_\varsigma^\infty \exp\left(iJ(S-\lambda I)\int_\xi^\varsigma \frac{1}{\kappa(\tau)}d\tau\right)f(\xi)d\xi,\ f_\lambda \in H,\ \varsigma > a.$$

In this case

$$\begin{aligned}
&\left\|\frac{1}{\kappa(\varsigma)}exp\left(iJ(S-\lambda I)\int\limits_a^{\varsigma}\frac{1}{\kappa(\xi)}d\xi\right)f_\lambda\right\|^2_{L^2_\kappa(H,(a,\infty))}\\
=&\int\limits_a^{\infty}\left\|\frac{1}{\kappa(\varsigma)}exp\left(iJ(S-\lambda I)\int\limits_a^{\varsigma}\frac{1}{\kappa(\xi)}d\xi\right)f_\lambda\right\|^2_H\kappa(\varsigma)d\varsigma\\
=&\int\limits_a^{\infty}\frac{1}{\kappa(\varsigma)}exp\left(2\lambda_iJ\int\limits_a^{\varsigma}\frac{1}{\kappa(\xi)}d\xi\right)d\varsigma\|f_\lambda\|^2_H\\
\leq&\ exp\left(2\lambda_iJ\int\limits_a^{\infty}\frac{1}{\kappa(\xi)}d\xi\right)\left(\int\limits_a^{\infty}\frac{1}{\kappa(\varsigma)}d\varsigma\right)\|f_\lambda\|^2_H<\infty
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\frac{i}{\kappa(\varsigma)}J\int\limits_{\varsigma}^{\infty}\exp\left(iJ(S-\lambda I)\int\limits_{\xi}^{\varsigma}\frac{1}{\kappa(\tau)}d\tau\right)f(\xi)d\xi\right\|^2_{L^2_\kappa(H,(a,\infty))}\\
=&\int\limits_a^{\infty}\left\|\frac{1}{\kappa(\varsigma)}J\int\limits_{\varsigma}^{\infty}\exp\left(iJ(S-\lambda I)\int\limits_{\xi}^{\varsigma}\frac{1}{\kappa(\tau)}d\tau\right)f(\xi)d\xi\right\|^2_H\kappa(\varsigma)d\varsigma\\
\leq&\int\limits_a^{\infty}\frac{1}{\kappa(\varsigma)}\left(\int\limits_{\varsigma}^{\infty}\frac{\sqrt{\kappa(\xi)}}{\kappa(\xi)}\exp\left(\lambda_iJ\int\limits_{\xi}^{\varsigma}\frac{1}{\kappa(\tau)}d\tau\right)\|\sqrt{\kappa(\xi)}f(\xi)\|_Hd\xi\right)^2d\varsigma\|J\|\\
\leq&\int\limits_a^{\infty}\frac{1}{\kappa(\varsigma)}\left(\int\limits_{\varsigma}^{\infty}\frac{1}{\kappa(\xi)}\exp\left(2\lambda_iJ\int\limits_{\xi}^{\varsigma}\frac{1}{\kappa(\tau)}d\tau\right)\right)d\xi\left(\int\limits_a^{\infty}\kappa(\xi)\|f(\xi)\|^2_Hd\xi\right)d\varsigma\|J\|\\
=&\int\limits_a^{\infty}\frac{1}{\kappa(\varsigma)}\left(\int\limits_{\varsigma}^{\infty}\frac{1}{\kappa(\xi)}\exp\left(2\lambda_iJ\int\limits_{\xi}^{\varsigma}\frac{1}{\kappa(\tau)}d\tau\right)d\xi\right)d\varsigma\|J\|\|f\|^2_{L^2_\kappa(H,(a,\infty))}\\
=&\ \exp\left(2\lambda_iJ\int\limits_a^{\infty}\frac{1}{\kappa(\tau)}d\tau\right)\left(\int\limits_a^{\infty}\frac{1}{\kappa(\varsigma)}d\varsigma\right)^2\|J\|\|f\|^2_{L^2_\kappa(H,(a,\infty))}<\infty.
\end{aligned}$$

Hence for $\lambda\in\mathbb{C},\ \lambda_i=Im\lambda\geq 0$ we have $\nu(\,\cdot\,,\lambda)\in L^2_\kappa(H,(a,\infty))$.

Using the corresponding boundary condition it is obtained that

$$\left[(\Gamma - I)J\left(exp\left(iJ(S-\lambda I)\int_a^\infty \frac{1}{\kappa(\xi)}d\xi\right)-I\right)+(\Gamma+I)\left(exp\left(iJ(S-\lambda I)\int_a^\infty \frac{1}{\kappa(\xi)}d\xi\right)+I\right)\right]f_\lambda$$

$$= i[(\Gamma - I) - J(\Gamma + I)]\int_a^\infty \exp\left(iJ(A-\lambda I)\int_\xi^a \frac{1}{\kappa(\tau)}d\tau\right)f(\xi)d\xi.$$

Therefore, the proof of the theorem is completed.

□

Now, we give an example as an application of the above theorem.

Example 4.2 *In the Hilbert space* $L^2_{\varsigma^\alpha}(1,\infty) \oplus L^2_{\varsigma^\alpha}(1,\infty)$*, consider the following first order canonical type linear symmetric singular differential expression*

$$l(\nu) = i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} (\varsigma^\alpha \nu_1)'(\varsigma) \\ (\varsigma^\alpha \nu_2)'(\varsigma) \end{pmatrix} + \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} \nu_1(\varsigma) \\ \nu_2(\varsigma) \end{pmatrix},\ \alpha > 1,\ s > 0.$$

Let L_0 *be the minimal operator generated by the differential expression* $l(\cdot)$*. Then, each maximally dissipative extension of the operator* L_0 *is generated by the differential operator expression* $l(\cdot)$ *and the boundary condition*

$$i\begin{pmatrix} r_1 - 1 & r_2 \\ r_3 & r_4 - 1 \end{pmatrix}\begin{pmatrix} (\varsigma^\alpha\nu_2)(\infty) - (\varsigma^\alpha\nu_2)(1) \\ -(\varsigma^\alpha\nu_1)(\infty) + (\varsigma^\alpha\nu_1)(1) \end{pmatrix} + \begin{pmatrix} r_1 + 1 & r_2 \\ r_3 & r_4 + 1 \end{pmatrix}\begin{pmatrix} (\varsigma^\alpha\nu_1)(\infty) + (\varsigma^\alpha\nu_1)(1) \\ (\varsigma^\alpha\nu_2)(\infty) + (\varsigma^\alpha\nu_2)(1) \end{pmatrix} = 0,$$

where $r_1,\ r_2,\ r_3,\ r_4 \in \mathbb{C}$ *and* $\max\limits_{1\le k\le 4}|r_k| \le 1$.
The boundary condition can be written as follows

$$(ir_1 - i + r_2)(\varsigma^\alpha\nu_2)(\infty) + (r_1 + 1 - ir_2)(\varsigma^\alpha\nu_1)(\infty) = (-ir_1 + i - r_2)(\varsigma^\alpha\nu_2)(1) - (r_1 + 1 + ir_2)(\varsigma^\alpha\nu_1)(1),$$
$$(ir_3 + r_4 + 1)(\varsigma^\alpha\nu_2)(\infty) + (r_3 - (r_4 - 1))(\varsigma^\alpha\nu_1)(\infty) = (r_4 + 1 - ir_3)(\varsigma^\alpha\nu_2)(1) - (r_3 + r_4 - 1)(\varsigma^\alpha\nu_1)(1).$$

In order to $\lambda \in \sigma(L_K)$ *the necessary and sufficient condition is*

$$0 \in \sigma(\Delta(\lambda)),$$

where $K = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}$ *and*

$$\begin{aligned}\Delta(\lambda) &= i\begin{pmatrix} -r_2 & r_1 - 1 \\ 1 - r_4 & r_3 \end{pmatrix}\left[exp\left(\begin{pmatrix} 0 & \lambda - s \\ s - \lambda & 0 \end{pmatrix}\frac{1}{\alpha - 1}\right) - 1\right] \\ &\quad + \begin{pmatrix} r_1 + 1 & r_2 \\ r_3 & r_4 + 1 \end{pmatrix}\left[exp\left(\begin{pmatrix} 0 & \lambda - s \\ s - \lambda & 0 \end{pmatrix}\frac{1}{\alpha - 1}\right) + 1\right]\end{aligned}$$

$$
\begin{aligned}
= \quad & i\begin{pmatrix} -r_2 & r_1-1 \\ 1-r_4 & r_3 \end{pmatrix}\left[exp\left(\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}\right) exp\left(\begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}\frac{1}{\alpha-1}\right)-1\right] \\
& +\begin{pmatrix} r_1+1 & r_2 \\ r_3 & r_4+1 \end{pmatrix}\left[exp\left(\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}\right) exp\left(\begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}\frac{1}{\alpha-1}\right)+1\right] \\
= \quad & i\begin{pmatrix} -r_2 & r_1-1 \\ 1-r_4 & r_3 \end{pmatrix}\left[\begin{pmatrix} 1-\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}-\frac{\lambda^6}{6!}+\frac{\lambda^8}{8!}+..., & \lambda-\frac{\lambda^3}{3!}+\frac{\lambda^5}{5!}-\frac{\lambda^7}{7!}+\frac{\lambda^9}{9!}+... \\ -\lambda+\frac{\lambda^3}{3!}-\frac{\lambda^5}{5!}+\frac{\lambda^7}{7!}-\frac{\lambda^9}{9!}+..., & 1-\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}-\frac{\lambda^6}{6!}+\frac{\lambda^8}{8!}+... \end{pmatrix}\right. \\
& \left.\times\begin{pmatrix} 1-\frac{s^2}{2!}+\frac{s^4}{4!}-\frac{s^6}{6!}+\frac{s^8}{8!}+..., & -s+\frac{s^3}{3!}-\frac{s^5}{5!}+\frac{s^7}{7!}-\frac{s^9}{9!}+... \\ s-\frac{s^3}{3!}+\frac{s^5}{5!}-\frac{s^7}{7!}+\frac{s^9}{9!}+..., & 1-\frac{s^2}{2!}+\frac{s^4}{4!}-\frac{s^6}{6!}+\frac{s^8}{8!}+... \end{pmatrix}\frac{1}{\alpha-1}-1\right] \\
& +\begin{pmatrix} r_1+1 & r_2 \\ r_3 & r_4+1 \end{pmatrix}\left[\begin{pmatrix} 1-\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}-\frac{\lambda^6}{6!}+\frac{\lambda^8}{8!}+..., & \lambda-\frac{\lambda^3}{3!}+\frac{\lambda^5}{5!}-\frac{\lambda^7}{7!}+\frac{\lambda^9}{9!}+... \\ -\lambda+\frac{\lambda^3}{3!}-\frac{\lambda^5}{5!}+\frac{\lambda^7}{7!}-\frac{\lambda^9}{9!}+..., & 1-\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}-\frac{\lambda^6}{6!}+\frac{\lambda^8}{8!}+... \end{pmatrix}\right. \\
& \left.\times\begin{pmatrix} 1-\frac{s^2}{2!}+\frac{s^4}{4!}-\frac{s^6}{6!}+\frac{s^8}{8!}+..., & -s+\frac{s^3}{3!}-\frac{s^5}{5!}+\frac{s^7}{7!}-\frac{s^9}{9!}+... \\ s-\frac{s^3}{3!}+\frac{s^5}{5!}-\frac{s^7}{7!}+\frac{s^9}{9!}+..., & 1-\frac{s^2}{2!}+\frac{s^4}{4!}-\frac{s^6}{6!}+\frac{s^8}{8!}+... \end{pmatrix}\frac{1}{\alpha-1}+1\right] \\
= \quad & i\begin{pmatrix} -r_2 & r_1-1 \\ 1-r_4 & r_3 \end{pmatrix}\left[\begin{pmatrix} \frac{1}{2}\left(e^{i\lambda}+e^{-i\lambda}\right) & \frac{1}{2i}\left(e^{i\lambda}-e^{-i\lambda}\right) \\ \frac{1}{2i}\left(-e^{i\lambda}+e^{-i\lambda}\right) & \frac{1}{2}\left(e^{i\lambda}+e^{-i\lambda}\right) \end{pmatrix}\right. \\
& \left.\times\begin{pmatrix} \frac{1}{2}\left(e^{is}+e^{-is}\right) & \frac{1}{2i}\left(-e^{is}+e^{-is}\right) \\ \frac{1}{2i}\left(e^{is}-e^{-is}\right) & \frac{1}{2}\left(e^{is}+e^{-is}\right) \end{pmatrix}\frac{1}{\alpha-1}-1\right] \\
& +\begin{pmatrix} r_1+1 & r_2 \\ r_3 & r_4+1 \end{pmatrix}\left[\begin{pmatrix} \frac{1}{2}\left(e^{i\lambda}+e^{-i\lambda}\right) & \frac{1}{2i}\left(e^{i\lambda}-e^{-i\lambda}\right) \\ \frac{1}{2i}\left(-e^{i\lambda}+e^{-i\lambda}\right) & \frac{1}{2}\left(e^{i\lambda}+e^{-i\lambda}\right) \end{pmatrix}\right. \\
& \left.\times\begin{pmatrix} \frac{1}{2}\left(e^{is}+e^{-is}\right) & \frac{1}{2i}\left(-e^{is}+e^{-is}\right) \\ \frac{1}{2i}\left(e^{is}-e^{is}\right) & \frac{1}{2}\left(e^{is}+e^{-is}\right) \end{pmatrix}\frac{1}{\alpha-1}+1\right] \\
= \quad & i\begin{pmatrix} -r_2 & r_1-1 \\ 1-r_4 & r_3 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\left(e^{i\lambda-is}+e^{-i\lambda+is}\right)\frac{1}{\alpha-1}-1 & \frac{1}{2i}\left(e^{i\lambda-is}-e^{-i\lambda+is}\right)\frac{1}{\alpha-1} \\ \frac{1}{2i}\left(-e^{i\lambda-is}+e^{-i\lambda+is}\right)\frac{1}{\alpha-1} & \frac{1}{2}\left(e^{i\lambda-is}+e^{-i\lambda+is}\right)\frac{1}{\alpha-1}-1 \end{pmatrix}
\end{aligned}
$$

$$+\begin{pmatrix} r_1+1 & r_2 \\ r_3 & r_4+1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\left(e^{i\lambda-is}+e^{-i\lambda+is}\right)\frac{1}{\alpha-1}+1 & \frac{1}{2i}\left(e^{i\lambda-is}-e^{-i\lambda+is}\right)\frac{1}{\alpha-1} \\ \frac{1}{2i}\left(-e^{i\lambda-is}+e^{-i\lambda+is}\right)\frac{1}{\alpha-1} & \frac{1}{2}\left(e^{i\lambda-is}+e^{-i\lambda+is}\right)\frac{1}{\alpha-1}+1 \end{pmatrix}.$$

Furthermore, using Mathematica and by the condition $det\Delta(\lambda)=0$*, we have*

$$\begin{aligned} \lambda_1 &= -iln\Bigg[\frac{1}{4(1-\alpha)}\Big(e^{is}((2+(\alpha-2)\alpha)r_1+2r_4+(\alpha-2)\alpha(i(r_2-r_3)+r_4)) \\ &\quad + \sqrt{e^{2is}(16(\alpha-1)^2(r_2r_3-r_1r_4)+((2+(\alpha-2)\alpha)r_1+2r_4+(\alpha-2)\alpha(i(r_2-r_3)+r_4)^2)}\Big)\Bigg], \\ \lambda_2 &= -iln\Bigg[\frac{1}{4(\alpha-1)}\Big(-e^{is}((2+(\alpha-2)\alpha)r_1+2r_4+(\alpha-2)\alpha(i(r_2-r_3)+r_4)) \\ &\quad + \sqrt{e^{2is}(16(\alpha-1)^2(r_2r_3-r_1r_4)+((2+(\alpha-2)\alpha)r_1+2r_4+(\alpha-2)\alpha(i(r_2-r_3)+r_4)^2)}\Big)\Bigg]. \end{aligned}$$

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