Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 491 – 523

doi:10.3906/mat-1912-67

Conceptions on topological transitivity in products and symmetric products

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Received: 17.12.2019 • **Accepted/Published Online:** 30.01.2020 • **Final Version:** 17.03.2020

Abstract: Having a finite number of topological spaces X_i and functions $f_i : X_i \to X_i$, and considering one of the following classes of functions: exact, transitive, strongly transitive, totally transitive, orbit-transitive, strictly orbit-transitive, ω-transitive, mixing, weakly mixing, mild mixing, chaotic, exactly Devaney chaotic, minimal, backward minimal, totally minimal, TT_{++}, scattering, Touhey or an F-system, in this paper, we study dynamical behaviors of the systems (X_i, f_i), $(\prod X_i, \prod f_i)$, $(\mathcal{F}_n(\prod X_i), \mathcal{F}_n(\prod f_i))$, and $(\mathcal{F}_n(X_i), \mathcal{F}_n(f_i))$.

Key words: Topological transitivity, symmetric products, dynamical systems

1. Introduction

Given a topological space X and a positive integer n, we consider the n-fold symmetric product of X, $\mathcal{F}_n(X)$, consisting of all nonempty subsets of X with at most n points [7]. A function $f : X \to X$ induces a map on $\mathcal{F}_n(X)$ denoted by $\mathcal{F}_n(f) : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ and defined by $\mathcal{F}_n(f)(A) = f(A)$, for each $A \in \mathcal{F}_n(X)$ [3]. Thereby, the discrete dynamical system (X, f) induces the discrete dynamical system $(\mathcal{F}_n(X), \mathcal{F}_n(f))$.

Let $X_1, \ldots, X_m$ be topological spaces, with $m \geq 2$ and for each $i \in \{1, \ldots, m\}$ let $f_i : X_i \to X_i$ be a function. We define the function $\prod_{i=1}^m f_i : \prod_{i=1}^m X_i \to \prod_{i=1}^m X_i$ by $\prod_{i=1}^m f_i((x_1, \ldots, x_m)) = (f_1(x_1), \ldots, f_m(x_m))$, for each $(x_1, \ldots, x_m) \in \prod_{i=1}^m X_i$. This function is called product function. In this way, we can analyze the relationships between the dynamical of the systems (1) $(\mathcal{F}_n(\prod_{i=1}^m X_i), \mathcal{F}_n(\prod_{i=1}^m f_i))$; (2) $(\mathcal{F}_n(X_i), \mathcal{F}_n(f_i))$, for each $i \in \{1, \ldots, m\}$; (3) $(\prod_{i=1}^m X_i, \prod_{i=1}^m f_i)$ and (4) (X_i, f_i), for each $i \in \{1, \ldots, , m\}$. Hou et al. [11] considered two compact metric spaces without isolated points X and Y, and two continuous functions $f : X \to X$ and $g : Y \to Y$, and they showed the following result: if f and g are sensitive functions, then the function $2^{f\times g} : 2^{X\times Y} \to 2^{X\times Y}$ is sensitive. Later, Degirmenci and Kocak [8] considered two metric spaces, X and Y, and two functions $f : X \to X$ and $g : Y \to Y$ (not necessarily continuous) and they analyzed the relationship between f, g and $f \times g$ when any of them is a chaotic function. In particular, they proved the following result: if f is continuous and chaotic, and g is chaotic and mixing (not necessarily continuous), then $f \times g$ is chaotic. Later, Wu and Zhu [21] proved that for each integer $m \geq 2$, if $\prod_{i=1}^m f_i$ is chaotic in the sense of Devaney, then for each $i \in \{1, \ldots, m\}$, f_i is also chaotic in the sense of Devaney. Moreover, they proved that if $\prod_{i=1}^m f_i$ is transitive, then, for each $i \in \{1, \ldots, m\}$, f_i is transitive. The converse problem is not true in general. In [21], Wu and Zhu considered metric spaces without isolated points and continuous functions. Moreover, Li and Zhou

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2010 *AMS Mathematics Subject Classification:* 54B20, 54H20, 37B45, 54F15

[13] analyzed the relationships between f, g and $f \times g$ when any of these are: topologically transitive, topologically weakly mixing, syndetically transitive, cofinitely sensitive, multisensitive and ergodically sensitive, always considering metric spaces and functions not necessarily continuous. Wu et al. [20] studied the $\mathcal{F}$-sensitivity and the multisensitivity of the dynamical system $(2^{X\times Y}, 2^{f\times g})$, when X and Y are both compact metric spaces. Recently, Mangang [6] studied the Li-Yorke chaos of the product dynamical system $(\prod_{i=1}^{m} X_i, \prod_{i=1}^{m} f_i)$ when each dynamical system (X_i, f_i) has the property. In particular, he proved that (X, f) and (Y, g) are two exact dynamical systems if and only if the product dynamical system $(X\times Y, f\times g)$ is exact. In this last paper, X and Y are compact metric spaces and f and g are continuous functions. In order to make a contribution to this line of investigation, let $\mathcal{M}$ be one of the following classes of functions: exact, mixing, transitive, weakly mixing, totally transitive, strongly transitive, chaotic, minimal, orbit-transitive, strictly orbit-transitive, ω-transitive, TT_{++}, mild mixing, exactly Devaney chaotic, backward minimal, totally minimal, scattering, Touhey or an F-system, in this paper we study the relationships between the following four statements:

1. For each $i \in \{1, \dots, m\}$, $f_i \in \mathcal{M}$.
2. $\prod_{i=1}^{m} f_i \in \mathcal{M}$.
3. $\mathcal{F}_n(\prod_{i=1}^{m} f_i) \in \mathcal{M}$.
4. For each $i \in \{1, \dots, m\}$, $\mathcal{F}_n(f_i) \in \mathcal{M}$.

It is important to emphasize that in the aforementioned articles, the authors work with compact metric spaces or without isolated points metric spaces and continuous function. In this paper, we are going to answer similar questions that we can find in [6, 8, 11, 13, 20, 21], considering topological spaces and functions not necessarily continuous.

2. Definitions and notations

Throughout this paper, m is an integer greater than one. A set is said to be nondegenerate if it has more than one point. A (discrete) dynamical system is a pair (X, f), where X is a nondegenerate topological space and $f\colon X \to X$ is a function, X is called the phase space. Let X be a topological space and let A be a subset of X, $\mathrm{cl}_X(A)$ denotes the closure of the set A in X. The symbols $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ denote the set of integers, the set of nonnegative integers and the set of positive integers, respectively. Given a finite collection of topological spaces $X_1, \dots, X_m$, the Cartesian product of these topological spaces is denoted by $\prod_{i=1}^{m} X_i$. This space is considered with the product topology [16, p. 86]. On the other hand, given a finite collection of functions, $f_1 : X_1 \to X_1, \dots, f_m : X_m \to X_m$ (not necessarily continuous), we define the product function $\prod_{i=1}^{m} f_i : \prod_{i=1}^{m} X_i \to \prod_{i=1}^{m} X_i$ by $\prod_{i=1}^{m} f_i((x_1, \dots, x_m)) = (f_1(x_1), \dots, f_m(x_m))$, for each $(x_1, \dots, x_m) \in \prod_{i=1}^{m} X_i$. Particularly, if X is a topological space and $f : X \to X$ is a function, the Cartesian product of X with itself m times is denoted by X^m and the Cartesian product of f with itself m times is denoted by $f^{\times m}$.

Given a dynamical system (X, f), for each $k \in \mathbb{N}$, the kth iteration of f is defined as repeated composition of f with itself k times and is denoted by f^k. This is, $f^k = f \circ f^{k-1}$, where $f^1 = f$ and $f^0 = id_X$, the identity function on X. For a subset A of X and $k \in \mathbb{Z}$, we denote by $f^k(A)$ the image of A under f^k, when $k \geq 0$, and the preimage under $f^{|k|}$ when $k < 0$. If $z \in X$, $f^{-k}(z)$ denotes the set $f^{-k}(\{z\})$, for each $k > 0$.

Let (X, f) be a dynamical system and let $x \in X$. The orbit of x under f is the set $\mathcal{O}(x, f) = \{f^k(x) \mid k \in \mathbb{Z}_+\}$. A point x of X is a transitive point of the function f if the set $\mathcal{O}(x, f)$ is dense in X. The set of transitive points of f is denoted by $trans(f)$. The point x is a fixed point of f if $f(x) = x$. The point x is a periodic point of f if there exists $k \in \mathbb{N}$ such that $f^k(x) = x$. The set of periodic points of f is denoted by $Per(f)$. If $k = \min\{l \in \mathbb{N} \mid f^l(x) = x\}$, we say that k is the period of x under f. A point y in X is an ω-limit point of x under f if for any $k \in \mathbb{N}$ and for any open subset U of X such that $y \in U$, there exists a positive integer $l \geq k$ such that $f^l(x) \in U$. The set of ω-limit points of x under f, is denoted by $\omega(x, f)$ and is called ω-limit set of x. Given a subset A of X, we say that A is $+$invariant under f if $f(A) \subseteq A$, A is $-$invariant under f if $f^{-1}(A) \subseteq A$ and A is invariant under f if $f(A) = A$. A topological space X is $+$invariant over open subsets under f, if for each open subset U of X, U is $+$invariant under f. For subsets A and B of X, it is defined the following subset of $\mathbb{Z}$, $n_f(A, B) = \{k \in \mathbb{Z}_+ \mid A \cap f^{-k}(B) \neq \emptyset\}$. A topological space X is pseudoregular if for any nonempty open subset U of X, there exists a nonempty open subset V of X such that $\mathrm{cl}_X(V) \subseteq U$ [15]. Let X be a topological space, let B be a subset of X and let $b \in B$. We say that b is an isolated point of B if there exists an open subset U of X such that $U \cap B = \{b\}$. Denote by $IP(B)$ the set of isolated points in B. A point x of X is a quasiisolated point of X if there exists a dense subset B of X such that $x \in B$ and x is an isolated point of B [15]. A topological space is perfect if it does not have isolated points. The following definitions can be found in [1, 8, 15].

Let X be a topological space and let $f\colon X \to X$ be a function. Then f is:

- Exact, if for each nonempty open subset U of X, there exists $k \in \mathbb{N}$ such that $f^k(U) = X$.
- Mixing, if for each pair of nonempty open subsets U and V of X, there exists $N \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$, for all $k \geq N$.
- Transitive, if for every pair of nonempty open subsets U and V of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$ (equivalently, for each pair of nonempty open subsets U and V of X, $n_f(U, V) \setminus \{0\} \neq \emptyset$).
- Weakly mixing, if $f^{\times 2}$ is transitive.
- Totally transitive, if f^s is transitive, for all $s \in \mathbb{N}$.
- Strongly transitive, if for each nonempty open subset U of X, there exists $s \in \mathbb{N}$ such that $X = \bigcup_{k=0}^{s} f^k(U)$.
- Chaotic, if it is transitive and $Per(f)$ is dense in X.
- Minimal, if for each nonempty closed subset A of X which is $+$invariant under f, we have $A = X$.
- Orbit-transitive, if there exists $x \in X$ such that $\mathrm{cl}_X(\mathcal{O}(x, f)) = X$.
- Strictly orbit-transitive, if there exists a point x in X such that $\mathrm{cl}_X(\mathcal{O}(f(x), f)) = X$.
- ω-transitive, if there exists $x \in X$ such that $\omega(x, f) = X$.
- TT_{++}, if for any pair of nonempty open subsets U and V of X, the set $n_f(U, V)$ is infinite.
- Mild mixing, if for any transitive function, $f_1 : X_1 \to X_1$, the function $f \times f_1$ is transitive.

- Exactly Devaney chaotic, if f is exact and $Per(f)$ is dense in X.
- Backward minimal, if the subset $\{y \in X : f^n(y) = x, \text{ for some } n \in \mathbb{N}\}$ is dense in X, for every $x \in X$.
- Totally minimal, if f^s is minimal for all $s \in \mathbb{N}$.
- Scattering, if for any minimal function, $f_1 : X_1 \to X_1$, the function $f \times f_1$ is transitive.
- Touhey, if for every pair of nonempty open subsets U and V of X, there exist a periodic point $x \in U$ and $k \in \mathbb{Z}_+$ such that $f^k(x) \in V$.
- An F-system, if f is totally transitive and $Per(f)$ is dense in X.

In the diagram of Figure, we put the inclusions between some of these classes of functions for the general case, that is to say, when X is a topological space and $f\colon X \to X$ is a function. For the proofs of these inclusions see, for instance, [1, 5, 15].

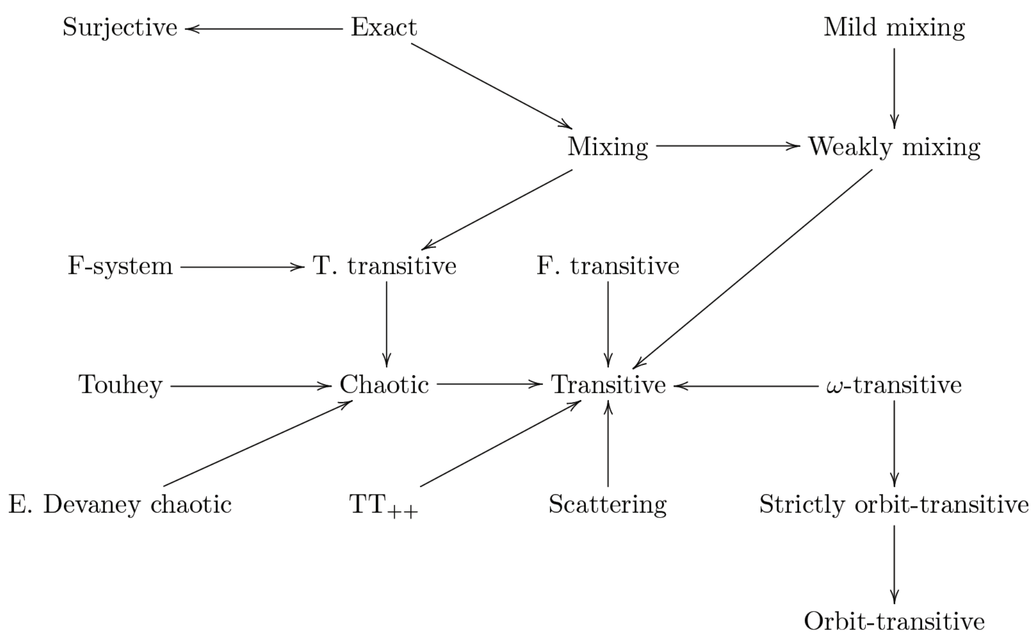

Figure : Inclusions between some classes of functions.

When we add properties to the phase space or to the function in Figure, we obtain other relationships, namely: Let X be a topological space and let $f\colon X \to X$ be a function. If X is a Hausdorff and compact topological space, and f is a surjective continuous function, we have that if f is scattering, then f is totally transitive [2, Theorem 2.9]. Moreover, if f is a continuous function, it follows that if f is chaotic, then f is Touhey [18, Proposition 2.6].

Hyperspace theory started in early 1900, with the work of Hausdorff [9] and Vietoris [19]. Nowadays hyperspaces are widely studied, mainly in continuum theory, see [12, 14, 17].

Given a topological space (X, τ) and a positive integer n, we define the n-fold symmetric product of X by:

$$\mathcal{F}_n(X) = \{A \subseteq X \mid A \neq \emptyset \quad \text{and} \quad A \quad \text{has at most } n \text{ elements}\}.$$

This set, equipped with the Vietoris topology [17], is called a hyperspace. Next we describe this topology.

Let (X, τ) be a topological space. Given a finite collection of nonempty subsets $U_1, \ldots, U_k$ of X, we denote by $\langle U_1, \ldots, U_k \rangle$ the subset of $\mathcal{F}_n(X)$:

$$\left\{ A \in \mathcal{F}_n(X) \mid A \subseteq \bigcup_{i=1}^{k} U_i \quad \text{and} \quad A \cap U_i \neq \emptyset, \quad \text{for each} \quad i \in \{1, \ldots, k\} \right\}.$$

The family:

$$\mathcal{B} = \{\langle U_1, \ldots, U_k \rangle \mid U_i \in \tau, \quad \text{for each} \quad i \in \{1, \ldots, k\} \text{ and } k \in \mathbb{N}\}$$

forms a basis for a topology on $\mathcal{F}_n(X)$ which is denoted by τ_V and called the Vietoris topology.

3. Preliminary results

Let $X_1, \ldots, X_m$ be topological spaces and for each $i \in \{1, \ldots, m\}$ let $f_i : X_i \to X_i$ be a function. In this section, we present some topological and dynamical properties of the space $\prod_{i=1}^m X_i$. Moreover, we review the basic results that we need to know about the function $\prod_{i=1}^m f_i$.

Remark 3.1 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let U_i, V_i be nonempty subsets of X_i, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function and let $k \in \mathbb{N}$. Then the following hold:*

1. $(\prod_{i=1}^m f_i)^k = \prod_{i=1}^m f_i^k$.
2. $[\mathcal{F}_n(\prod_{i=1}^m f_i)]^k = \mathcal{F}_n(\prod_{i=1}^m f_i^k)$.
3. *If* $(\prod_{i=1}^m f_i)^k(\prod_{i=1}^m U_i) = \prod_{i=1}^m V_i$, *then, for each* $i \in \{1, \ldots, m\}$, $f_i^k(U_i) = V_i$.

Lemma 3.2 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let U_i be a nonempty subset of X_i, let $x_i \in X_i$ and let $f_i : X_i \to X_i$ be a function. If for each $i \in \{1, \ldots, m\}$, X_i is $+$invariant over open subsets under f_i and, for each $i \in \{1, \ldots, m\}$, there exists $k_i \in \mathbb{N}$ such that $f_i^{k_i}(x_i) \in U_i$, then, for $k = \max\{k_1, \ldots, k_m\}$, it follows that, for each $i \in \{1, \ldots, m\}$, $f_i^k(x_i) \in U_i$.*

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, X_i is $+$invariant over open subsets under f_i and that there exists $k_i \in \mathbb{N}$ such that $f_i^{k_i}(x_i) \in U_i$. Let $k = \max\{k_1, \ldots, k_m\}$. It follows that, for each $i \in \{1, \ldots, m\}$, there exists $l_i \in \mathbb{Z}_+$ such that $k = k_i + l_i$. Thus, for each $i \in \{1, \ldots, m\}$, $f_i^k(x_i) = f_i^{k_i + l_i}(x_i) = f_i^{l_i}(f_i^{k_i}(x_i))$. Consequently, for each $i \in \{1, \ldots, m\}$, $f_i^k(x_i) \in f_i^{l_i}(U_i)$. By hypothesis, since, for each $i \in \{1, \ldots, m\}$, U_i is $+$invariant under f_i, we have that, for each $i \in \{1, \ldots, m\}$, $f_i^k(x_i) \in U_i$. □

Theorem 3.3 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, and let $(x_1, \ldots, x_m) \in \prod_{i=1}^m X_i$. Then the following hold:*

1. *If $(x_1, \ldots, x_m)$ is a transitive point of $\prod_{i=1}^m f_i$, then, for each $i \in \{1, \ldots, m\}$, x_i is a transitive point of f_i.*
2. *If $\omega((x_1, \ldots, x_m), \prod_{i=1}^m f_i) = \prod_{i=1}^m X_i$, then, for each $i \in \{1, \ldots, m\}$, $\omega(x_i, f_i) = X_i$.*

3. *For each* $i \in \{1, \dots, m\}$, x_i *is an isolated point in* X_i *if and only if* $(x_1, \dots, x_m)$ *is an isolated point in* $\prod_{i=1}^m X_i$.

4. *For each* $i \in \{1, \dots, m\}$, x_i *is a periodic point of* f_i *if and only if* $(x_1, \dots, x_m)$ *is a periodic point of* $\prod_{i=1}^m f_i$.

Proof Suppose that $\mathrm{cl}_{\prod_{i=1}^m X_i}(\mathcal{O}((x_1, \dots, x_m), \prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Let $i_0 \in \{1, \dots, m\}$ and let U_{i_0} be a nonempty open subset of X_{i_0}. For each $i \in \{1, \dots, m\} \setminus \{i_0\}$, let $V_i = X_i$ and $V_{i_0} = U_{i_0}$. It follows that $\prod_{i=1}^m V_i$ is a nonempty open subset of $\prod_{i=1}^m X_i$. By hypothesis, $\mathcal{O}((x_1, \dots, x_m), \prod_{i=1}^m f_i) \cap (\prod_{i=1}^m V_i) \neq \emptyset$. Then, there exists $k \in \mathbb{N}$ such that $(\prod_{i=1}^m f_i)^k((x_1, \dots, x_m)) \in \prod_{i=1}^m V_i$. By Remark 3.1, part (1), $(\prod_{i=1}^m f_i)^k((x_1, \dots, x_m)) = (f_1^k(x_1), \dots, f_m^k(x_m))$, $f_{i_0}^k(x_{i_0}) \in U_{i_0}$. Therefore, $U_{i_0} \cap \mathcal{O}(x_{i_0}, f_{i_0}) \neq \emptyset$ and $\mathrm{cl}_{X_{i_0}}(\mathcal{O}(x_{i_0}, f_{i_0})) = X_{i_0}$.

Suppose that $\omega((x_1, \dots, x_m), \prod_{i=1}^m f_i) = \prod_{i=1}^m X_i$. Let $i_0 \in \{1, \dots, m\}$, let $y_{i_0} \in X_{i_0}$, let $k \in \mathbb{N}$, let U_{i_0} be an open subset of X_{i_0} such that $y_{i_0} \in U_{i_0}$ and for each $j \in \{1, \dots, m\} \setminus \{i_0\}$, let $y_j \in X_j$. Moreover, for each $i \in \{1, \dots, m\} \setminus \{i_0\}$, we put $V_i = X_i$ and $V_{i_0} = U_{i_0}$. It follows that $\prod_{i=1}^m V_i$ is a nonempty open subset of $\prod_{i=1}^m X_i$ such that $(y_1, \dots, y_m) \in \prod_{i=1}^m V_i$. Thus, by hypothesis, there exists $l \in \mathbb{N}$ with $l \geq k$ such that $(\prod_{i=1}^m f_i)^l((x_1, \dots, x_m)) \in \prod_{i=1}^m V_i$. By Remark 3.1, part (1), we have that $f_{i_0}^l(x_{i_0}) \in U_{i_0}$. Therefore, $y_{i_0} \in \omega(x_{i_0}, f_{i_0})$. Consequently, $X_{i_0} = \omega(x_{i_0}, f_{i_0})$.

Suppose that $(x_1, \dots, x_m)$ is an isolated point in $\prod_{i=1}^m X_i$. Then there exists an open subset $\mathcal{U}$ of $\prod_{i=1}^m X_i$ such that $(\prod_{i=1}^m X_i) \cap \mathcal{U} = \{(x_1, \dots, x_m)\}$. Even more, for each $i \in \{1, \dots, m\}$, there exists a nonempty open subset $U_i \subseteq X_i$ such that $(\prod_{i=1}^m U_i) \cap (\prod_{i=1}^m X_i) = \{(x_1, \dots, x_m)\}$. Observe that, for each $i \in \{1, \dots, m\}$, $U_i \cap X_i = \{x_i\}$. Consequently, for each $i \in \{1, \dots, m\}$, x_i is an isolated point in X_i.

Now suppose that, for each $i \in \{1, \dots, m\}$, x_i is an isolated point in X_i. Then, for each $i \in \{1, \dots, m\}$, there exists an open subset $U_i \subseteq X_i$ such that $U_i \cap X_i = \{x_i\}$. Note that, $(x_1, \dots, x_n) \in \prod_{i=1}^m U_i$ and $(\prod_{i=1}^m U_i) \cap (\prod_{i=1}^m X_i) = \{(x_1, \dots, x_m)\}$. Thus, $(x_1, \dots, x_m)$ is an isolated point in $\prod_{i=1}^m X_i$.

Suppose that, for each $i \in \{1, \dots, m\}$, x_i is a periodic point of f_i. Thus, for each $i \in \{1, \dots, m\}$, there exists $k_i \in \mathbb{N}$ such that $f_i^{k_i}(x_i) = x_i$. Let $k = k_1 \cdots k_m$. It follows that, for each $i \in \{1, \dots, m\}$, $f_i^k(x_i) = x_i$. Hence, $(f_1^k(x_1), \dots, f_m^k(x_m)) = (x_1, \dots, x_m)$. By Remark 3.1, part (1), $(\prod_{i=1}^m f_i)^k((x_1, \dots, x_m)) = (x_1, \dots, x_m)$. Therefore, $(x_1, \dots, x_m)$ is a periodic point of $\prod_{i=1}^m f_i$.

Now, suppose that $(x_1, \dots, x_m)$ is a periodic point of $\prod_{i=1}^m f_i$. Then, there exists $k \in \mathbb{N}$ such that $(\prod_{i=1}^m f_i)^k((x_1, \dots, x_m)) = (x_1, \dots, x_m)$. Thus, by Remark 3.1, part (1), for each $i \in \{1, \dots, m\}$, $f_i^k(x_i) = x_i$. Therefore, for each $i \in \{1, \dots, m\}$, x_i is a periodic point of f_i. □

As a consequence of Theorem 3.3, we have the following:

Remark 3.4 *Let* $X_1, \dots, X_m$ *be topological spaces and, for each* $i \in \{1, \dots, m\}$, *let* $f_i : X_i \to X_i$ *be a function. Then the following hold:*

1. $trans(\prod_{i=1}^m f_i) \subseteq \prod_{i=1}^m trans(f_i)$.

2. $\omega((x_1, \dots, x_m), \prod_{i=1}^m f_i) \subseteq \prod_{i=1}^m \omega(x_i, f_i)$.

3. $IP(\prod_{i=1}^{m} X_i) = \prod_{i=1}^{m} IP(X_i)$.

4. $Per(\prod_{i=1}^{m} f_i) = \prod_{i=1}^{m} Per(f_i)$.

Corollary 3.5 *Let $X_1, \dots, X_m$ be topological spaces and, for each $i \in \{1, \dots, m\}$, let $f_i : X_i \to X_i$ be a function. Then the following hold:*

1. $\mathrm{cl}_{\prod_{i=1}^{m} X_i}(trans(\prod_{i=1}^{m} f_i)) \subseteq \prod_{i=1}^{m} \mathrm{cl}_{X_i}(trans(f_i))$.

2. $\mathrm{cl}_{\prod_{i=1}^{m} X_i}(\prod_{i=1}^{m} Per(f_i)) = \prod_{i=1}^{m} \mathrm{cl}_{X_i}(Per(f_i))$.

In Example 3.6 we show that the converse of Theorem 3.3, parts (1) and (2), are not true in general.

Example 3.6 *Let $X = \{1, 2\}$ topologized with $\tau = \{\emptyset, X, \{1\}\}$ and let $f : X \to X$ be a function given by $f(1) = 2$ and $f(2) = 1$. Note that*

1. $\mathrm{cl}_X(\mathcal{O}(1, f)) = X$ *and* $\mathrm{cl}_X(\mathcal{O}(2, f)) = X$. *However,* $\mathcal{O}((1, 2), f \times f) \cap (\{1\} \times \{1\}) = \emptyset$. *Consequently,* $\mathrm{cl}_{X \times X}(\mathcal{O}((1, 2), f \times f)) \neq X \times X$.

2. $\omega(1, f) = X$ *and* $\omega(2, f) = X$. *However,* $\omega((1, 2), f \times f) \neq X \times X$.

There exist conditions that make the converse of Theorem 3.3, parts (1) and (2) true. One of these conditions is given in Theorem 3.7.

Theorem 3.7 *Let $X_1, \dots, X_m$ be topological spaces, for each $i \in \{1, \dots, m\}$, let $x_i \in X_i$, and let $f_i : X_i \to X_i$ be a function. Then the following hold:*

1. *If, for each $i \in \{1, \dots, m\}$, $\omega(x_i, f_i) = X_i$ and X_i is $+$invariant over open subsets under f_i, then* $\omega((x_1, \dots, x_m), \prod_{i=1}^{m} f_i) = \prod_{i=1}^{m} X_i$.

2. *If, for each $i \in \{1, \dots, m\}$, $\mathrm{cl}_{X_i}(\mathcal{O}(x_i, f_i)) = X_i$ and X_i is $+$invariant over open subsets under f_i, then:*

$$\mathrm{cl}_{\prod_{i=1}^{m} X_i}\left(\mathcal{O}\left((x_1, \dots, x_m), \prod_{i=1}^{m} f_i\right)\right) = \prod_{i=1}^{m} X_i.$$

Proof Suppose that, for each $i \in \{1, \dots, m\}$, $\omega(x_i, f_i) = X_i$ and that X_i is $+$invariant over open subsets under f_i. Let $(y_1, \dots, y_m) \in \prod_{i=1}^{m} X_i$, let $k \in \mathbb{N}$ and let $\mathcal{U}$ be an open subset of $\prod_{i=1}^{m} X_i$ such that $(y_1, \dots, y_m) \in \mathcal{U}$. Then, for each $i \in \{1, \dots, m\}$, there exists a nonempty open subset U_i of X_i, such that $(y_1, \dots, y_m) \in \prod_{i=1}^{m} U_i \subseteq \mathcal{U}$. By hypothesis, for each $i \in \{1, \dots, m\}$, there exists $l_i \in \mathbb{N}$ such that $l_i \geq k$ and $f_i^{l_i}(x_i) \in U_i$. For each $i \in \{1, \dots, m\}$, let $l = \max\{l_1, \dots, l_m\}$. By Lemma 3.2, for each $i \in \{1, \dots, m\}$, we have that $f_i^{l}(x_i) \in U_i$. Thus, $(\prod_{i=1}^{m} f_i)^{l}((x_1, \dots, x_m)) \in \mathcal{U}$. Also note that $l \geq k$. Therefore, $(x_1, \dots, x_m) \in \omega((x_1, \dots, x_m), \prod_{i=1}^{m} f_i)$ and $\omega((x_1, \dots, x_m), \prod_{i=1}^{m} f_i) = \prod_{i=1}^{m} X_i$.

Now suppose that, for each $i \in \{1, \dots, m\}$, $\mathrm{cl}_{X_i}(\mathcal{O}(x_i, f_i)) = X_i$ and that X_i is $+$invariant over open subsets under f_i. Let $\mathcal{U}$ be a nonempty open subset of $\prod_{i=1}^{m} X_i$. Then, for each $i \in \{1, \dots, m\}$, there exists a nonempty open subset U_i of X_i such that $\prod_{i=1}^{m} U_i \subseteq \mathcal{U}$. By hypothesis, for each $i \in \{1,\ldots,m\}$, $\mathcal{O}(x_i, f_i) \cap U_i \neq \emptyset$. It follows that, for each $i \in \{1,\ldots,m\}$, there exists $k_i \in \mathbb{N}$ such that $f_i^{k_i}(x_i) \in U_i$. Let $k = \max\{k_1,\ldots,k_m\}$. By Lemma 3.2, for each $i \in \{1,\ldots,m\}$, we have that $f_i^k(x_i) \in U_i$. Consequently, $(\prod_{i=1}^{m} f_i)^k((x_1,\ldots,x_m)) = (f_1^k(x_1),\ldots,f_m^k(x_m)) \in \prod_{i=1}^{m} U_i$. Hence, $\mathcal{O}((x_1,\ldots,x_m), \prod_{i=1}^{m} f_i) \cap \mathcal{U} \neq \emptyset$. Therefore, $\mathrm{cl}_{\prod_{i=1}^{m} X_i}(\mathcal{O}((x_1,\ldots,x_m), \prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i$. □

As a consequence of Theorem 3.3, part (3), we have:

Corollary 3.8 *Let $X_1,\ldots,X_m$ be topological spaces and, for each $i \in \{1,\ldots,m\}$, let $f_i : X_i \to X_i$ be a function. Then $\prod_{i=1}^{m} X_i$ is perfect if and only if, for each $i \in \{1,\ldots,m\}$, X_i is perfect.*

Theorem 3.9 *Let $X_1,\ldots,X_m$ be topological spaces. Then $\prod_{i=1}^{m} X_i$ is pseudoregular if and only if, for each $i \in \{1,\ldots,m\}$, X_i is pseudoregular.*

Proof Suppose that $\prod_{i=1}^{m} X_i$ is pseudoregular. Let $i_0 \in \{1,\ldots,m\}$ and let U_{i_0} be a nonempty open subset of X_{i_0}. For each $i \in \{1,\ldots,m\}\setminus\{i_0\}$, let $V_i = X_i$ and let $V_{i_0} = U_{i_0}$. Thus, $\prod_{i=1}^{m} V_i$ is a nonempty open subset of $\prod_{i=1}^{m} X_i$. Since $\prod_{i=1}^{m} X_i$ is pseudoregular, there exists a nonempty open subset $\mathcal{V}$ of $\prod_{i=1}^{m} X_i$ such that $\mathrm{cl}_{\prod_{i=1}^{m} X_i}(\mathcal{V}) \subseteq \prod_{i=1}^{m} V_i$. Moreover, for each $i \in \{1,\ldots,m\}$, there exists a nonempty open subset $V_i^{'} \subseteq X_i$ such that $\prod_{i=1}^{m} V_i^{'} \subseteq \mathcal{V}$. Consequently, $\mathrm{cl}_{\prod_{i=1}^{m} X_i}(\prod_{i=1}^{m} V_i^{'}) \subseteq \prod_{i=1}^{m} V_i$. Then $\mathrm{cl}_{X_{i_0}}(V_{i_0}') \subseteq U_{i_0}$. Therefore, X_{i_0} is pseudoregular. Because $i_0 \in \{1,\ldots,m\}$ is arbitrary, we have that, for each $i \in \{1,\ldots,m\}$, X_i is pseudoregular.

Now suppose that, for each $i \in \{1,\ldots,m\}$, X_i is pseudoregular. Let $\mathcal{U}$ be a nonempty open subset of $\prod_{i=1}^{m} X_i$. Then, for each $i \in \{1,\ldots,m\}$, there exists a nonempty open subset U_i of X_i such that $\prod_{i=1}^{m} U_i \subseteq \mathcal{U}$. Since, for each $i \in \{1,\ldots,m\}$, X_i is pseudoregular, we have that, for each $i \in \{1,\ldots,m\}$, there exists a nonempty open subset V_i of X_i such that $\mathrm{cl}_{X_i}(V_i) \subseteq U_i$. Hence, $\prod_{i=1}^{m} \mathrm{cl}_{X_i}(V_i) \subseteq \prod_{i=1}^{m} U_i$. On the other hand, since $\mathrm{cl}_{\prod_{i=1}^{m} X_i}(\prod_{i=1}^{m} V_i) \subseteq \prod_{i=1}^{m} \mathrm{cl}_{X_i}(V_i)$, we have that $\mathrm{cl}_{\prod_{i=1}^{m} X_i}(\prod_{i=1}^{m} V_i) \subseteq \mathcal{U}$. Therefore, $\prod_{i=1}^{m} X_i$ is pseudoregular. □

Proposition 3.10 *Let $X_1,\ldots,X_m$ be topological spaces, for each $i \in \{1,\ldots,m\}$, let U_i be an open subset of X_i, and let $f_i : X_i \to X_i$ be a function. Then, for each $i \in \{1,\ldots,m\}$, U_i is $+$invariant under f_i if and only if $\prod_{i=1}^{m} U_i$ is $+$invariant under $\prod_{i=1}^{m} f_i$.*

Proof Suppose that, for each $i \in \{1,\ldots,m\}$, U_i is $+$invariant under f_i. Let $(a_1,\ldots,a_m) \in \prod_{i=1}^{m} f_i(\prod_{i=1}^{m} U_i)$. Then there exists $(x_1,\ldots,x_m) \in \prod_{i=1}^{m} U_i$ such that $\prod_{i=1}^{m} f_i((x_1,\ldots,x_m)) = (a_1,\ldots,a_m)$. It follows that, for each $i \in \{1,\ldots,m\}$, $f_i(x_i) = a_i$. Then, for each $i \in \{1,\ldots,m\}$, $a_i \in f_i(U_i)$. Since, for each $i \in \{1,\ldots,m\}$, U_i is $+$invariant under f_i, we have that, for each $i \in \{1,\ldots,m\}$, $a_i \in U_i$. Therefore, $(a_1,\ldots,a_m) \in \prod_{i=1}^{m} U_i$. Consequently, $\prod_{i=1}^{m} U_i$ is $+$invariant under $\prod_{i=1}^{m} f_i$.

Now suppose that $\prod_{i=1}^{m} U_i$ is $+$invariant under $\prod_{i=1}^{m} f_i$. Let $i_0 \in \{1,\ldots,m\}$ and let $x_{i_0} \in f_{i_0}(U_{i_0})$. Then there exists $u_{i_0} \in U_{i_0}$ such that $f_{i_0}(u_{i_0}) = x_{i_0}$. For each $j \in \{1,\ldots,m\}\setminus\{i_0\}$, let $u_j \in U_j$. Next, $(u_1,\ldots,u_m) \in \prod_{i=1}^{m} U_i$. Since $\prod_{i=1}^{m} U_i$ is $+$invariant under $\prod_{i=1}^{m} f_i$, we have that $\prod_{i=1}^{m} f_i((u_1,\ldots,u_m)) = (f_1(u_1),\ldots,f_m(u_m)) \in \prod_{i=1}^{m} U_i$. Thus, $x_{i_0} = f_{i_0}(u_{i_0}) \in U_{i_0}$. Therefore, $f_{i_0}(U_{i_0}) \subseteq U_{i_0}$. □

Proposition 3.11 *Let $X_1,\dots,X_m$ be topological spaces and, for each $i\in\{1,\dots,m\}$, let $f_i:X_i\to X_i$ be a function. If, for each $i\in\{1,\dots,m\}$, $U_i\subseteq X_i$ is $-$invariant under f_i, then $\prod_{i=1}^m U_i$ is $-$invariant under $\prod_{i=1}^m f_i$.*

Proof Suppose that, for each $i\in\{1,\dots,m\}$, U_i is $-$invariant under f_i. We show that $(\prod_{i=1}^m f_i)^{-1}(\prod_{i=1}^m U_i)\subseteq \prod_{i=1}^m U_i$. Let $(a_1,\dots,a_m)\in(\prod_{i=1}^m f_i)^{-1}(\prod_{i=1}^m U_i)$. We have that $\prod_{i=1}^m f_i((a_1,\dots,a_m))\in\prod_{i=1}^m U_i$. It follows that, for each $i\in\{1,\dots,m\}$, $f_i(a_i)\in U_i$. Thus, for each $i\in\{1,\dots,m\}$, $a_i\in f_i^{-1}(U_i)$. Since, for each $i\in\{1,\dots,m\}$, U_i is $-$invariant under f_i, we obtain that, for each $i\in\{1,\dots,m\}$, $a_i\in U_i$. Consequently, $(a_1,\dots,a_m)\in\prod_{i=1}^m U_i$. Therefore, $\prod_{i=1}^m U_i$ is $-$invariant under $\prod_{i=1}^m f_i$. □

The converse of Proposition 3.11 is not true in general.

Example 3.12 *Let $X=\{1,2,3,4\}$ be a set topologized with $\{X,\emptyset,\{1,2\}\}$, and let $f:X\to X$ be a function given by $f(x)=1$, for each $x\in X$. Let $A=\{1\}\times\{2,3,4\}$. Note that $(f\times f)^{-1}(A)=\emptyset$. Thus, $(f\times f)^{-1}(A)\subseteq A$. Then A is $-$invariant under $f\times f$. On the other hand, $f^{-1}(\{1\})=X$. It follows that, $f^{-1}(\{1\})\nsubseteq\{1\}$. Consequently, $\{1\}$ it is not $-$invariant under f.*

Theorem 3.13 *Let $X_1,\dots,X_m$ be topological spaces, for each $i\in\{1,\dots,m\}$, let U_i be an open subset of X_i, and let $f_i:X_i\to X_i$ be a surjective function. Then $\prod_{i=1}^m U_i$ is $-$invariant under $\prod_{i=1}^m f_i$ if and only if, for each $i\in\{1,\dots,m\}$, U_i is $-$invariant under f_i.*

Proof Suppose that $\prod_{i=1}^m U_i$ is $-$invariant under $\prod_{i=1}^m f_i$. Let $i_0\in\{1,\dots,m\}$ and let $a_{i_0}\in f_{i_0}^{-1}(U_{i_0})$. Thus, $f_{i_0}(a_{i_0})\in U_{i_0}$. On the other hand, since, for each $j\in\{1,\dots,m\}$, f_j is surjective, we have that, for each $j\in\{1,\dots,m\}$, $f_j^{-1}(U_j)\neq\emptyset$. Then, for each $j\in\{1,\dots,m\}\setminus\{i_0\}$, we can take $a_j\in f_j^{-1}(U_j)$. Hence, for each $j\in\{1,\dots,m\}$, $f_j(a_j)\in U_j$. It follows that $(f_1(a_1),\dots,f_m(a_m))\in\prod_{i=1}^m U_i$. Thus, $\prod_{i=1}^m f_i((a_1,\dots,a_m))\in\prod_{i=1}^m U_i$. Then $(a_1,\dots,a_m)\in(\prod_{i=1}^m f_i)^{-1}(\prod_{i=1}^m U_i)$. By hypothesis, since $\prod_{i=1}^m U_i$ is $-$invariant, $(a_1,\dots,a_m)\in\prod_{i=1}^m U_i$. Hence, $a_{i_0}\in U_{i_0}$. Therefore, U_{i_0} is $-$invariant.

The converse implication follows from Proposition 3.11. □

Theorem 3.14 *Let $X_1,\dots,X_m$ be topological spaces and, for each $i\in\{1,\dots,m\}$, let $f_i:X_i\to X_i$ be a function. Then, for each $i\in\{1,\dots,m\}$, X_i is $+$invariant over open subsets under f_i if and only if $\prod_{i=1}^m X_i$ is $+$invariant over open subsets under $\prod_{i=1}^m f_i$.*

Proof Suppose that, for each $i\in\{1,\dots,m\}$, X_i is $+$invariant over open subsets under f_i. Let $\mathcal{U}$ be a nonempty open subset of $\prod_{i=1}^m X_i$ and let $(x_1,\dots,x_m)\in\prod_{i=1}^m f_i(\mathcal{U})$. Then there exists $(a_1,\dots,a_m)\in\mathcal{U}$ such that $\prod_{i=1}^m f_i((a_1,\dots,a_m))=(x_1,\dots,x_m)$. It follows that, for each $i\in\{1,\dots,m\}$, there exists a nonempty open subset U_i of X_i such that $(a_1,\dots,a_m)\in\prod_{i=1}^m U_i\subseteq\mathcal{U}$. By hypothesis and Proposition 3.10, $\prod_{i=1}^m f_i(\prod_{i=1}^m U_i)\subseteq\prod_{i=1}^m U_i$. Thus, $(x_1,\dots,x_m)\in\prod_{i=1}^m U_i\subseteq\mathcal{U}$. Therefore, $\mathcal{U}$ is $+$invariant under $\prod_{i=1}^m f_i$. Because $\mathcal{U}$ is arbitrary, we have that $\prod_{i=1}^m X_i$ is $+$invariant over open subsets under $\prod_{i=1}^m f_i$.

Now, suppose that $\prod_{i=1}^m X_i$ is $+$invariant over open subsets under $\prod_{i=1}^m f_i$. Let $i_0\in\{1,\dots,m\}$, let U_{i_0} be an open subset of X_{i_0} and, for each $i\in\{1,\dots,m\}\setminus\{i_0\}$, let $V_i=X_i$ and $V_{i_0}=U_{i_0}$. Then $\prod_{i=1}^m V_i$ is a nonempty open subset of $\prod_{i=1}^m X_i$. Since $\prod_{i=1}^m X_i$ is $+$invariant over open subsets under $\prod_{i=1}^m f_i$, we have that $\prod_{i=1}^m V_i$ is $+$invariant under $\prod_{i=1}^m f_i$. Then, by Proposition 3.10, U_{i_0} is $+$invariant under f_{i_0}. □

Theorem 3.15 *Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. Then $Per(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$ if and only if, for each $i \in \{1, \ldots, m\}$, $Per(f_i)$ is dense in X_i.*

Proof Suppose that $Per(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$. Thus, $\mathrm{cl}_{\prod_{i=1}^m X_i}(Per(\prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Then, by Corollary 3.5, part (2), $\prod_{i=1}^m \mathrm{cl}_{X_i}(Per(f_i)) = \prod_{i=1}^m X_i$. Consequently, for each $i \in \{1, \ldots, m\}$, $\mathrm{cl}_{X_i}(Per(f_i)) = X_i$. Therefore, for each $i \in \{1, \ldots, m\}$, $Per(f_i)$ is dense in X_i.

Now suppose that, for each $i \in \{1, \ldots, m\}$, $Per(f_i)$ is dense in X_i. In consequence, we have that, $\prod_{i=1}^m \mathrm{cl}_{X_i}(Per(f_i)) = \prod_{i=1}^m X_i$. On the other hand, by Remark 3.4 and Corollary 3.5, part (2), we have that $\prod_{i=1}^m \mathrm{cl}_{X_i}(Per(f_i)) = \mathrm{cl}_{\prod_{i=1}^m X_i}(\prod_{i=1}^m Per(f_i)) = \mathrm{cl}_{\prod_{i=1}^m X_i}(Per(\prod_{i=1}^m f_i))$. It follows that $\mathrm{cl}_{\prod_{i=1}^m X_i}(Per(\prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Therefore, $Per(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$. □

Proposition 3.16 *Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. If $trans(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$, then, for each $i \in \{1, \ldots, m\}$, $trans(f_i)$ is dense in X_i.*

Proof Suppose that $trans(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$. Hence, $\mathrm{cl}_{\prod_{i=1}^m X_i}(trans(\prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Thus, by Corollary 3.5, part (1), $\prod_{i=1}^m X_i \subseteq \prod_{i=1}^m \mathrm{cl}_{X_i}(trans(f_i))$. Consequently, for each $i \in \{1, \ldots, m\}$, $X_i \subseteq \mathrm{cl}_{X_i}(trans(f_i))$. Therefore, for each $i \in \{1, \ldots, m\}$, $trans(f_i)$ is dense in X_i. □

The converse of Proposition 3.16 is not true in general.

Example 3.17 *Let $X = \{1, 2\}$ be a set topologized with $\tau = \{\emptyset, X, \{1\}, \{2\}\}$, and let $f : X \to X$ be a function given by $f(1) = 2$ and $f(2) = 1$. Note that*

1. *$\mathcal{O}(1, f) = \{1, 2\}$ is dense in X and $\mathcal{O}(2, f) = \{2, 1\}$ is dense in X. Thus, $trans(f)$ is dense in X.*

2. *$trans(f \times f) = \emptyset$.*

Theorem 3.18 *Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. If, for each $i \in \{1, \ldots, m\}$, $trans(f_i)$ is dense in X_i and X_i is $+$invariant over open subsets under f_i, then $trans(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$.*

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, $trans(f_i)$ is dense in X_i and that X_i is $+$invariant over open subsets under f_i. Let $\mathcal{U}$ be a nonempty open subset of $\prod_{i=1}^m X_i$. Then, for each $i \in \{1, \ldots, m\}$, there exists a nonempty open subset U_i of X_i such that $\prod_{i=1}^m U_i \subseteq \mathcal{U}$. By hypothesis, for each $i \in \{1, \ldots, m\}$, $U_i \cap trans(f_i) \neq \emptyset$. Consequently, for each $i \in \{1, \ldots, m\}$, there exists $x_i \in U_i$ such that x_i is a transitive point of f_i. Since, for each $i \in \{1, \ldots, m\}$, X_i is $+$invariant over open subsets under f_i, by Theorem 3.7, part (2), we have that $(x_1, \ldots, x_m)$ is a transitive point of $\prod_{i=1}^m f_i$. Even more, $(x_1, \ldots, x_m) \in \mathcal{U}$. Therefore, $trans(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$. □

Lemma 3.19 *Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. If $j \in \{1, \ldots, m\}$, let U_j, V_j be two nonempty open subsets of X_j and, for each $i \in \{1, \ldots, m\} \setminus \{j\}$, we put $U_i = X_i$ and $V_i = X_i$, then $n_{\prod_{i=1}^m f_i}(\prod_{i=1}^m U_i, \prod_{i=1}^m V_i) \subseteq n_{f_j}(U_j, V_j)$.*

Proof Let $k \in n_{\prod_{i=1}^m f_i}(\prod_{i=1}^m U_i, \prod_{i=1}^m V_i)$. Then $(\prod_{i=1}^m U_i) \cap (\prod_{i=1}^m f_i)^{-k} (\prod_{i=1}^m V_i) \neq \emptyset$. Let $(y_1, \ldots, y_m) \in (\prod_{i=1}^m U_i) \cap (\prod_{i=1}^m f_i)^{-k}(\prod_{i=1}^m V_i)$. It follows that $(\prod_{i=1}^m f_i)^k((y_1, \ldots, y_m)) \in \prod_{i=1}^m V_i$. Then, by Remark 3.1, part (1), we have that $(f_1^k(y_1), \ldots, f_m^k(y_m)) \in \prod_{i=1}^m V_i$. Consequently, $y_j \in U_j \cap f_j^{-k}(V_j)$. Then, $k \in n_{f_j}(U_j, V_j)$. Thus, $n_{\prod_{i=1}^m f_i}(\prod_{i=1}^m U_i, \prod_{i=1}^m V_i) \subseteq n_{f_j}(U_j, V_j)$. □

4. Dynamic properties of product functions

Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. In this section, we present the relationships that exist between the functions $\prod_{i=1}^m f_i$ and f_i, for each $i \in \{1, \ldots, m\}$, when any of them is exact, mixing, transitive, weakly mixing, totally transitive, strongly transitive, chaotic, minimal, orbit-transitive, strictly orbit-transitive, ω-transitive, TT_{++}, mild mixing, exactly Devaney chaotic, backward minimal, totally minimal, scattering, Touhey or an F-system.

Theorem 4.1 *Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. Let $\mathcal{M}$ be one of the following classes of functions: transitive, weakly mixing, totally transitive, strongly transitive, chaotic, orbit-transitive, strictly orbit-transitive, ω-transitive, TT_{++}, backward minimal, Touhey, an F-system, scattering or mild mixing. If $\prod_{i=1}^m f_i \in \mathcal{M}$, then, for each $i \in \{1, \ldots, m\}$, $f_i \in \mathcal{M}$.*

Proof Suppose that $\prod_{i=1}^m f_i$ is transitive. Let $i_0 \in \{1, \ldots, m\}$ and let U_{i_0}, V_{i_0} be nonempty open subsets of X_{i_0}. For every $i \in \{1, \ldots, m\} \setminus \{i_0\}$, let $U_i = X_i$ and let $V_i = X_i$. Then $\prod_{i=1}^m U_i$ and $\prod_{i=1}^m V_i$ are nonempty open subsets of $\prod_{i=1}^m X_i$. Since, $\prod_{i=1}^m f_i$ is transitive, there exists $k \in \mathbb{N}$ such that $(\prod_{i=1}^m f_i)^k(\prod_{i=1}^m U_i) \cap (\prod_{i=1}^m V_i) \neq \emptyset$. Let $(u_1, \ldots, u_m) \in \prod_{i=1}^m U_i$ such that $(\prod_{i=1}^m f_i)^k((u_1, \ldots, u_m)) \in \prod_{i=1}^m V_i$. Thus, by Remark 3.1, part (1), we have that $f_{i_0}^k(u_{i_0}) \in V_{i_0}$. Therefore, $f_{i_0}^k(u_{i_0}) \in f_{i_0}^k(U_{i_0}) \cap V_{i_0}$, $f_{i_0}^k(U_{i_0}) \cap V_{i_0} \neq \emptyset$ and f_{i_0} is transitive.

Suppose that $\prod_{i=1}^m f_i$ is weakly mixing. Let $i_0 \in \{1, \ldots, m\}$ and let $\mathcal{U}, \mathcal{V}$ be nonempty open subsets of $X_{i_0} \times X_{i_0}$. Then there exist nonempty open subsets $U_{i_0}^1, U_{i_0}^2, V_{i_0}^1$ and $V_{i_0}^2$ of X_{i_0} such that $U_{i_0}^1 \times U_{i_0}^2 \subseteq \mathcal{U}$ and $V_{i_0}^1 \times V_{i_0}^2 \subseteq \mathcal{V}$. For each $i \in \{1, \ldots, m\} \setminus \{i_0\}$, let $U_i^1 = U_i^2 = V_i^1 = V_i^2 = X_i$. Hence, $(\prod_{i=1}^m U_i^1) \times (\prod_{i=1}^m U_i^2)$ and $(\prod_{i=1}^m V_i^1) \times (\prod_{i=1}^m V_i^2)$ are nonempty open subsets of $(\prod_{i=1}^m X_i) \times (\prod_{i=1}^m X_i)$. By hypothesis, there exists $((a_1, \ldots, a_m), (b_1, \ldots, b_m)) \in (\prod_{i=1}^m U_i^1) \times (\prod_{i=1}^m U_i^2)$ and $k \in \mathbb{N}$ such that $((\prod_{i=1}^m f_i) \times (\prod_{i=1}^m f_i))^k((a_1, \ldots, a_m), (b_1, \ldots, b_m)) \in (\prod_{i=1}^m V_i^1) \times (\prod_{i=1}^m V_i^2)$. Then by Remark 3.1, part (1), $(f_{i_0} \times f_{i_0})^k((a_{i_0}, b_{i_0})) \in V_{i_0}^1 \times V_{i_0}^2$. Even more, $(a_{i_0}, b_{i_0}) \in U_{i_0}^1 \times U_{i_0}^2$. Therefore, $(f_{i_0} \times f_{i_0})^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ and hence $f_{i_0}^{\times 2}$ is transitive. Finally, f_{i_0} is weakly mixing.

Suppose that $\prod_{i=1}^m f_i$ is totally transitive. Let $i_0 \in \{1, \ldots, m\}$ and let $s \in \mathbb{N}$. By hypothesis, $(\prod_{i=1}^m f_i)^s$ is transitive. By Remark 3.1, part (1), $\prod_{i=1}^m f_i^s$ is transitive. Thus, by the first paragraph of the proof of this theorem, we have that $f_{i_0}^s$ is transitive. Therefore, f_{i_0} is totally transitive.

Suppose that $\prod_{i=1}^m f_i$ is strongly transitive. Let $i_0 \in \{1, \ldots, m\}$ and let U_{i_0} be a nonempty open subset of X_{i_0}. For every $i \in \{1,\dots,m\}\backslash\{i_0\}$, let $U_i = X_i$. Then $\prod_{i=1}^m U_i$ is a nonempty open subset of $\prod_{i=1}^m X_i$. By hypothesis, there exists $s \in \mathbb{N}$ such that $\prod_{i=1}^m X_i = \bigcup_{k=0}^s (\prod_{i=1}^m f_i)^k(\prod_{i=1}^m U_i)$. Let $x_{i_0} \in X_{i_0}$ and, for each $i \in \{1,\dots,m\}\backslash\{i_0\}$, let $x_i \in X_i$. Then there exists $k_1 \in \{0,\dots,s\}$ such that $(x_1,\dots,x_m) \in (\prod_{i=1}^m f_i)^{k_1}(\prod_{i=1}^m U_i)$. Thus, by Remark 3.1, part (1), we have that $x_{i_0} \in f_{i_0}^{k_1}(U_{i_0})$. Therefore, $X_{i_0} = \bigcup_{k=0}^s f_{i_0}^k(U_{i_0})$ and hence f_{i_0} is strongly transitive.

Suppose that $\prod_{i=1}^m f_i$ is chaotic. By the first paragraph of the proof of this theorem, for all $i \in \{1,\dots,m\}$, f_i is transitive. Moreover, by Theorem 3.15, for every $i \in \{1,\dots,m\}$, $Per(f_i)$ is dense in X_i. Thus, for each $i \in \{1,\dots,m\}$, f_i is chaotic.

Suppose that $\prod_{i=1}^m f_i$ is orbit-transitive. Consequently, there exists $(x_1,\dots,x_m) \in \prod_{i=1}^m X_i$ such that $\mathrm{cl}_{\prod_{i=1}^m X_i}(\mathcal{O}((x_1,\dots,x_m),\prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Thus, by Theorem 3.3, part (1), for every $i \in \{1,\dots,m\}$, we have that $\mathrm{cl}_{X_i}(\mathcal{O}(x_i,f_i)) = X_i$. Thus, for all $i \in \{1,\dots,m\}$, f_i is orbit-transitive.

Suppose that $\prod_{i=1}^m f_i$ is strictly orbit-transitive. Consequently, there exists $(x_1,\dots,x_m) \in \prod_{i=1}^m X_i$ such that $\mathrm{cl}_{\prod_{i=1}^m X_i}(\mathcal{O}(\prod_{i=1}^m f_i((x_1,\dots,x_m)),\prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Therefore, by Theorem 3.3, part (1), for every $i \in \{1,\dots,m\}$, $\mathrm{cl}_{X_i}(\mathcal{O}(f_i(x_i),f_i)) = X_i$ and hence, for all $i \in \{1,\dots,m\}$, f_i is strictly orbit-transitive.

Suppose that $\prod_{i=1}^m f_i$ is ω-transitive. Consequently, there exists $(x_1,\dots,x_m) \in \prod_{i=1}^m X_i$ such that $\omega((x_1,\dots,x_m),\prod_{i=1}^m f_i) = \prod_{i=1}^m X_i$. Thus, by Theorem 3.3, part (2), for each $i \in \{1,\dots,m\}$, $\omega(x_i,f_i) = X_i$. Therefore, for each $i \in \{1,\dots,m\}$, f_i is ω-transitive.

Suppose that $\prod_{i=1}^m f_i$ is TT_{++}. Let $i_0 \in \{1,\dots,m\}$ and let U_{i_0}, V_{i_0} be nonempty open subsets of X_{i_0}. For every $i \in \{1,\dots,m\}\backslash\{i_0\}$, let $U_i = X_i$ and $V_i = X_i$. Then by Lemma 3.19, $n_{\prod_{i=1}^m f_i}(\prod_{i=1}^m U_i, \prod_{i=1}^m V_i) \subseteq n_{f_{i_0}}(U_{i_0},V_{i_0})$. Moreover, by hypothesis, $n_{\prod_{i=1}^m f_i}(\prod_{i=1}^m U_i, \prod_{i=1}^m V_i)$ is infinite. Therefore, $n_{f_{i_0}}(U_{i_0},V_{i_0})$ is infinite and hence f_{i_0} is TT_{++}.

Suppose that $\prod_{i=1}^m f_i$ is backward minimal. Let $i_0 \in \{1,\dots,m\}$, let $x_{i_0} \in X_{i_0}$ and let U_{i_0} be a nonempty open subset of X_{i_0}. For each $i \in \{1,\dots,m\}\backslash\{i_0\}$, let $U_i = X_i$ and let $x_i \in X_i$. Then $\prod_{i=1}^m U_i$ is a nonempty open subset of $\prod_{i=1}^m X_i$. By hypothesis, we deduce that $\{A \in \prod_{i=1}^m X_i : (\prod_{i=1}^m f_i)^l(A) = (x_1,\dots,x_m),\ \text{for some } l \in \mathbb{N}\} \cap \prod_{i=1}^m U_i \neq \emptyset$. Let $(u_1,\dots,u_m) \in \prod_{i=1}^m U_i$ and let $l \in \mathbb{N}$ such that $(\prod_{i=1}^m f_i)^l((u_1,\dots,u_m)) = (x_1,\dots,x_m)$. It follows that, $u_{i_0} \in \{y \in X_{i_0} : f_{i_0}^l(y) = x_{i_0},\ \text{for some } l \in \mathbb{N}\} \cap U_{i_0} \neq \emptyset$. Thus, the set $\{y \in X_{i_0} : f_{i_0}^l(y) = x_{i_0},\ \text{for some } l \in \mathbb{N}\}$ is dense in X_{i_0}. Since $x_{i_0} \in X_{i_0}$ is arbitrary, we have that f_{i_0} is backward minimal.

Suppose that $\prod_{i=1}^m f_i$ is Touhey. Let $i_0 \in \{1,\dots,m\}$ and let U_{i_0}, V_{i_0} be nonempty open subsets of X_{i_0}. For each $i \in \{1,\dots,m\}\backslash\{i_0\}$, let $U_i = X_i$ and $V_i = X_i$. Then, $\prod_{i=1}^m U_i$ and $\prod_{i=1}^m V_i$ are nonempty open subsets of $\prod_{i=1}^m X_i$. By hypothesis, there exist a periodic point $(x_1,\dots,x_m) \in \prod_{i=1}^m U_i$ and $k \in \mathbb{Z}_+$ such that $(\prod_{i=1}^m f_i)^k((x_1,\dots,x_m)) \in \prod_{i=1}^m V_i$. By Theorem 3.3, part (4), x_{i_0} is a periodic point of f_{i_0} such that $x_{i_0} \in U_{i_0}$ and by Remark 3.1, part (1), $f_{i_0}^k(x_{i_0}) \in V_{i_0}$. Therefore, f_{i_0} is Touhey.

Suppose that $\prod_{i=1}^m f_i$ is an F-system. Thus, $\prod_{i=1}^m f_i$ is totally transitive and $Per(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$. By the third paragraph of this proof, we have that, for each $i \in \{1,\dots,m\}$, f_i is totally transitive. Moreover, by Theorem 3.15, for each $i \in \{1,\dots,m\}$, $Per(f_i)$ is dense in X_i. Therefore, for each $i \in \{1,\dots,m\}$, f_i is an F-system.

Suppose that $\prod_{i=1}^m f_i$ is scattering. Let $i_0 \in \{1,\dots,m\}$, let Y be a topological space and let $g : Y \to Y$ be a minimal function. Let $\mathcal{U}$ and $\mathcal{V}$ be nonempty open subsets of $X_{i_0} \times Y$. Then, there exist nonempty open subsets $U_{i_0}^1, U_{i_0}^2$ of X_{i_0} and nonempty open subsets V_1, V_2 of Y such that $U_{i_0}^1 \times V_1 \subseteq \mathcal{U}$ and $U_{i_0}^2 \times V_2 \subseteq \mathcal{V}$. For each $i \in \{1,\dots,m\}\backslash\{i_0\}$, let $U_i^1 = U_i^2 = X_i$. Thus, $\prod_{i=1}^m U_i^1$ and $\prod_{i=1}^m U_i^2$ are nonempty open subsets of $\prod_{i=1}^m X_i$. By hypothesis, there exist $((u_1,\dots,u_m),v_1) \in (\prod_{i=1}^m U_i^1) \times V_1$ and $k \in \mathbb{N}$ such that $((\prod_{i=1}^m f_i) \times g)^k((u_1,\dots,u_m),v_1) \in (\prod_{i=1}^m U_i^2) \times V_2$. It follows that $(u_{i_0}, v_1) \in U_{i_0}^1 \times V_1$ and by Remark 3.1, part (1), $(f_{i_0} \times g)^k((u_{i_0}, v_1)) \in U_{i_0}^2 \times V_2$. Therefore, $(f_{i_0} \times g)^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ and hence f_{i_0} is scattering.

The proof for mild mixing is similar to that given for scattering. □

The converse of Theorem 4.1 is not true in general. Let us see a partial example of this in the following:

Example 4.2 *Let $f : [0,2] \to [0,2]$ be a function given by:*

$$f(x) = \begin{cases} 2x+1, & 0 \le x \le \frac{1}{2}, \\ -2x+3, & \frac{1}{2} \le x \le 1, \\ -x+2, & 1 \le x \le 2. \end{cases}$$

In [8, Example 1], it is proved that f is a chaotic function. Moreover, it is proved that $f \times f : [0,2]\times[0,2] \to [0,2] \times [0,2]$ is not transitive and, therefore, it is not chaotic. Furthermore, in [1, 15], it is proved that for continua and continuous functions, the notions: transitive, orbit-transitive, strictly orbit-transitive, ω-transitive and TT_{++} are equivalent. Therefore, the converse of Theorem 4.1, for all these classes of functions are not true in general.

Theorem 4.3 *Let $X_1,\dots,X_m$ be topological spaces and, for each $i \in \{1,\dots,m\}$, let $f_i : X_i \to X_i$ be a function. Then, for each $i \in \{1,\dots,m\}$, f_i is exact if and only if $\prod_{i=1}^m f_i$ is exact.*

Proof Suppose that $\prod_{i=1}^m f_i$ is exact. Let $i_0 \in \{1,\dots,m\}$ and let U_{i_0} be a nonempty open subset of X_{i_0}. For each $i \in \{1,\dots,m\}\backslash\{i_0\}$, let $U_i = X_i$. Then $\prod_{i=1}^m U_i$ is an open subset of $\prod_{i=1}^m X_i$. By hypothesis, there exists $k \in \mathbb{N}$ such that $(\prod_{i=1}^m f_i)^k(\prod_{i=1}^m U_i) = \prod_{i=1}^m X_i$. By Remark 3.1, part (3), $f_{i_0}^k(U_{i_0}) = X_{i_0}$. Thus, f_{i_0} is exact.

Now, suppose that, for each $i \in \{1,\dots,m\}$, f_i is exact. Let $\mathcal{U}$ be a nonempty open subset of $\prod_{i=1}^m X_i$. Then, for each $i \in \{1,\dots,m\}$, there exists a nonempty open subset U_i of X_i such that $\prod_{i=1}^m U_i \subseteq \mathcal{U}$. By hypothesis, for each $i \in \{1,\dots,m\}$, there exists $k_i \in \mathbb{N}$ such that $f_i^{k_i}(U_i) = X_i$. On the other hand, by the diagram on Figure, we have that, for each $i \in \{1,\dots,m\}$, f_i is surjective. Then, for each $i \in \{1,\dots,m\}$ and for each $l \in \mathbb{N}$, $f_i^l(X_i) = X_i$. Let $k = \max\{k_1,\dots,k_m\}$. It follows that, for each $i \in \{1,\dots,m\}$, there exists $l_i \in \mathbb{Z}_+$ such that $k = k_i + l_i$. Thus, for each $i \in \{1,\dots,m\}$, $f_i^k(U_i) = f_i^{l_i+k_i}(U_i) = f_i^{l_i}(f_i^{k_i}(U_i)) = f_i^{l_i}(X_i) = X_i$. Consequently, by Remark 3.1, part (1), $(\prod_{i=1}^m f_i)^k(\prod_{i=1}^m U_i) = \prod_{i=1}^m f_i^k(U_i) = \prod_{i=1}^m X_i$. Therefore, $(\prod_{i=1}^m f_i)^k(\mathcal{U}) = \prod_{i=1}^m X_i$ and $\prod_{i=1}^m f_i$ is exact. □

Theorem 4.4 *Let $X_1,\dots,X_m$ be topological spaces and, for each $i \in \{1,\dots,m\}$, let $f_i : X_i \to X_i$ be a function. Then $\prod_{i=1}^m f_i$ is mixing if and only if, for each $i \in \{1,\dots,m\}$, f_i is mixing.*

Proof Suppose that $\prod_{i=1}^{m} f_i$ is mixing. Let $i_0 \in \{1,\ldots,m\}$ and let U_{i_0}, V_{i_0} be two nonempty open subsets of X_{i_0}. For each $i \in \{1,\ldots,m\}\setminus\{i_0\}$, we put $U_i = X_i$ and $V_i = X_i$. It follows that $\prod_{i=1}^{m} U_i$ and $\prod_{i=1}^{m} V_i$ are nonempty open subsets of $\prod_{i=1}^{m} X_i$. Since $\prod_{i=1}^{m} f_i$ is mixing, there exists $N \in \mathbb{N}$ such that $(\prod_{i=1}^{m} f_i)^k(\prod_{i=1}^{m} U_i) \cap (\prod_{i=1}^{m} V_i) \neq \emptyset$, for each $k \geq N$. Let $k \geq N$ and let $(a_1,\ldots,v_{i_0},\ldots,a_m) \in (\prod_{i=1}^{m} f_i)^k(\prod_{i=1}^{m} U_i) \cap (\prod_{i=1}^{m} V_i)$. Then there exists $(x_1,\ldots,u_{i_0},\ldots,x_m) \in \prod_{i=1}^{m} U_i$ such that

$$\left(\prod_{i=1}^{m} f_i\right)^k ((x_1,\ldots,u_{i_0},\ldots,x_m)) = (a_1,\ldots,v_{i_0},\ldots,a_m).$$

Thus, $f_{i_0}^k(u_{i_0}) = v_{i_0}$. Thereby, $v_{i_0} \in f_{i_0}^k(U_{i_0}) \cap V_{i_0}$. Consequently, $f_{i_0}^k(U_{i_0}) \cap V_{i_0} \neq \emptyset$, for each $k \geq N$. Therefore, f_{i_0} is mixing.

Now, suppose that, for each $i \in \{1,\ldots,m\}$, f_i is mixing. Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty open subsets of $\prod_{i=1}^{m} X_i$. Then, for each $i \in \{1,\ldots,m\}$, there exist nonempty open subsets U_i and V_i of X_i, such that $\prod_{i=1}^{m} U_i \subseteq \mathcal{U}$ and $\prod_{i=1}^{m} V_i \subseteq \mathcal{V}$. Since f_i is mixing, for each $i \in \{1,\ldots,m\}$, there exists $N_i \in \mathbb{N}$ such that $f_i^k(U_i) \cap V_i \neq \emptyset$, for each $k \geq N_i$. Let $N = \max\{N_1,\ldots,N_m\}$ and let $l \geq N$. Thus, by hypothesis $f_i^l(U_i) \cap V_i \neq \emptyset$. For each $i \in \{1,\ldots,m\}$, let $a_i \in U_i$ be such that $f_i^l(a_i) \in V_i$. Then $(a_1,\ldots,a_m) \in \prod_{i=1}^{m} U_i$ and $(\prod_{i=1}^{m} f_i)^l(a_1,\ldots,a_m) \in \prod_{i=1}^{m} V_i$. Hence, $(\prod_{i=1}^{m} f_i)^l(a_1,\ldots,a_m) \in [(\prod_{i=1}^{m} f_i)^l(\prod_{i=1}^{m} U_i)] \cap (\prod_{i=1}^{m} V_i)$. Hence, for each $l \geq N$, $[(\prod_{i=1}^{m} f_i)^l(\prod_{i=1}^{m} U_i)] \cap (\prod_{i=1}^{m} V_i) \neq \emptyset$. Therefore, $\prod_{i=1}^{m} f_i$ is mixing. □

By Theorems 3.15 and 4.3, we have the following result.

Proposition 4.5 *Let $X_1,\ldots,X_m$ be topological spaces and, for each $i \in \{1,\ldots,m\}$, let $f_i : X_i \to X_i$ be a function. Then, for each $i \in \{1,\ldots,m\}$, f_i is exactly Devaney chaotic if and only if $\prod_{i=1}^{m} f_i$ is exactly Devaney chaotic.*

Theorem 4.6 *Let $X_1,\ldots,X_m$ be topological spaces and, for each $i \in \{1,\ldots,m\}$, let $f_i : X_i \to X_i$ be a continuous function. If $\prod_{i=1}^{m} f_i$ is minimal, then, for each $i \in \{1,\ldots,m\}$, f_i is minimal.*

Proof Let $i_0 \in \{1,\ldots,m\}$. Since f_{i_0} is continuous, it is enough to show that, for each $x \in X_{i_0}$, $\mathrm{cl}_{X_{i_0}}(\mathcal{O}(x, f_{i_0})) = X_{i_0}$. Let $x \in X_{i_0}$, for each $i \in \{1,\ldots,m\}\setminus\{i_0\}$, let $x_i \in X_i$ and let $x_{i_0} = x$. Then, $(x_1,\ldots,x_m) \in \prod_{i=1}^{m} X_i$. Since, for each $i \in \{1,\ldots,m\}$, f_i is continuos, we have that, $\prod_{i=1}^{m} f_i$ is a minimal and continuous function. Thus, we have that $\mathrm{cl}_{\prod_{i=1}^{m} X_i}(\mathcal{O}((x_1,\ldots,x_m), \prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i$. Later, by Theorem 3.3, part (1), for each $i \in \{1,\ldots,m\}$, $\mathrm{cl}_{X_i}(\mathcal{O}(x_i, f_i)) = X_i$. In particular, $\mathrm{cl}_{X_{i_0}}(\mathcal{O}(x, f_{i_0})) = X_{i_0}$. Considering that $x \in X_{i_0}$ is arbitrary, by [15, Proposition 6.2], f_{i_0} is minimal. □

Corollary 4.7 *Let $X_1,\ldots,X_m$ be topological spaces and, for each $i \in \{1,\ldots,m\}$, let $f_i : X_i \to X_i$ be a continuous function. If $\prod_{i=1}^{m} f_i$ is totally minimal, then, for each $i \in \{1,\ldots,m\}$, f_i is totally minimal.*

Proof Let $s \in \mathbb{N}$. By hypothesis, $(\prod_{i=1}^{m} f_i)^s$ is minimal. Then, by Remark 3.1, part (1), $\prod_{i=1}^{m} f_i^s$ is minimal. Thus, by Theorem 4.6, for each $i \in \{1,\ldots,m\}$, f_i^s is minimal. □

Lemma 4.8 *Let $X_1,\dots,X_{m+1}$ be topological spaces and, for each $i\in\{1,\dots,m+1\}$, let $f_i:X_i\to X_i$ be a function. If, for each $i\in\{1,\dots,m\}$, X_i is $+$invariant over open subsets under f_i and $f_i\times f_{m+1}$ is transitive, then $(\prod_{i=1}^m f_i)\times f_{m+1}$ is transitive.*

Proof Suppose that, for each $i\in\{1,\dots,m\}$, X_i is $+$invariant over open subsets under f_i and $f_i\times f_{m+1}$ is transitive. Let $\mathcal{U}$, $\mathcal{V}$ be two nonempty open subsets of $(\prod_{i=1}^m X_i)\times X_{m+1}$. It follows that, there exist nonempty open subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\prod_{i=1}^m X_i$ and there exist nonempty open subsets V_1 and V_2 of X_{m+1} such that, $\mathcal{U}_1\times V_1\subseteq\mathcal{U}$ and $\mathcal{U}_2\times V_2\subseteq\mathcal{V}$. Hence, for each $i\in\{1,\dots,m\}$, there exist nonempty open subsets U_i^1,U_i^2 of X_i such that $\prod_{i=1}^m U_i^1\subseteq\mathcal{U}_1$ and $\prod_{i=1}^m U_i^2\subseteq\mathcal{U}_2$. By hypothesis, there exists $k_i\in\mathbb{N}$ such that $(f_i\times f_{m+1})^{k_i}(U_i^1\times V_1)\cap(U_i^2\times V_2)\neq\emptyset$. Then, for each $i\in\{1,\dots,m\}$, there exists $(u_i,v_i)\in U_i^1\times V_1$ such that $(f_i\times f_{m+1})^{k_i}((u_i,v_i))\in U_i^2\times V_2$. Consequently, for every $i\in\{1,\dots,m\}$, $f_i^{k_i}(u_i)\in U_i^2$. Let $k=\max\{k_1,\dots,k_m\}$. Then, by Lemma 3.2, we have that, for all $i\in\{1,\dots,m\}$, $f_i^k(u_i)\in U_i^2$. Let $i_0\in\{1,\dots,m\}$ be such that $k=k_{i_0}$, and let $v=v_{i_0}$. Thus, $f_{m+1}^k(v)\in V_2$. Hence, $((u_1,\dots,u_m),v))\in(\prod_{i=1}^m U_i^1)\times V_1$ and $((\prod_{i=1}^m f_i)\times f_{m+1})^k(((u_1,\dots,u_m),v))\in(\prod_{i=1}^m U_i^2)\times V_2$. Consequently, $[(\prod_{i=1}^m f_i)\times f_{m+1}]^k(\mathcal{U}_1\times V_1)\cap(\mathcal{U}_2\times V_2)\neq\emptyset$. Therefore, $(\prod_{i=1}^m f_i)\times f_{m+1}$ is transitive. □

Remark 4.9 *Let X be a topological space and let $f:X\to X$ be a function. Observe that if X is $+$invariant over open subsets under f, then f cannot be strongly transitive unless X has the trivial topology.*

Theorem 4.10 *Let $X_1,\dots,X_m$ be topological spaces and, for each $i\in\{1,\dots,m\}$, let $f_i:X_i\to X_i$ be a function. Let $\mathcal{M}$ be one of the following classes of functions: transitive, weakly mixing, totally transitive, chaotic, orbit-transitive, strictly orbit-transitive, ω-transitive, TT_{++}, Touhey, scattering, an F-system or mild mixing. If, for each $i\in\{1,\dots,m\}$, $f_i\in\mathcal{M}$ and X_i is $+$invariant over open subsets under f_i, then $\prod_{i=1}^m f_i\in\mathcal{M}$.*

Proof Suppose that, for each $i\in\{1,\dots,m\}$, f_i is transitive. Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty open subsets of $\prod_{i=1}^m X_i$. Then, for each $i\in\{1,\dots,m\}$, there exist nonempty open subsets U_i and V_i of X_i such that $\prod_{i=1}^m U_i\subseteq\mathcal{U}$ and $\prod_{i=1}^m V_i\subseteq\mathcal{V}$. By hypothesis, for each $i\in\{1,\dots,m\}$, there exists $k_i\in\mathbb{N}$ such that $f_i^{k_i}(U_i)\cap V_i\neq\emptyset$. For each $i\in\{1,\dots,m\}$, let $u_i\in U_i$ be such that $f_i^{k_i}(u_i)\in V_i$ and let $k=\max\{k_1,\dots,k_m\}$. By Lemma 3.2, we have that, for each $i\in\{1,\dots,m\}$, $f_i^k(u_i)\in V_i$. Hence, $(u_1,\dots,u_m)\in\prod_{i=1}^m U_i$ and $(f_1^k(u_1),\dots,f_m^k(u_m))\in\prod_{i=1}^m V_i$. Consequently, $(\prod_{i=1}^m f_i)^k((u_1,\dots,u_m))\in\prod_{i=1}^m V_i$. It follows that $(\prod_{i=1}^m f_i)^m(\prod_{i=1}^m U_i)\cap(\prod_{i=1}^m V_i)\neq\emptyset$. Therefore, $(\prod_{i=1}^m f_i)^k(\mathcal{U})\cap\mathcal{V}\neq\emptyset$ and $\prod_{i=1}^m f_i$ is transitive.

Suppose that, for each $i\in\{1,\dots,m\}$, f_i is weakly mixing. Let $\mathcal{U}_1,\mathcal{U}_2,\mathcal{V}_1$, and $\mathcal{V}_2$ be four nonempty open subsets of $\prod_{i=1}^m X_i$. Then, for each $i\in\{1,\dots,m\}$, there exist nonempty open subsets U_i^1,U_i^2,V_i^1 and V_i^2 of X_i, such that $\prod_{i=1}^m U_i^1\subseteq\mathcal{U}_1$, $\prod_{i=1}^m U_i^2\subseteq\mathcal{U}_2$, $\prod_{i=1}^m V_i^1\subseteq\mathcal{V}_1$ and $\prod_{i=1}^m V_i^2\subseteq\mathcal{V}_2$. Since, f_i is weakly mixing, for every $i\in\{1,\dots,m\}$, there exists $k_i\in\mathbb{N}$ such that $f_i^{k_i}(U_i^j)\cap V_i^j\neq\emptyset$, for each $j\in\{1,2\}$. For each $i\in\{1,\dots,m\}$, let $a_i\in U_i^1$ be such that $f_i^{k_i}(a_i)\in V_i^1$ and let $a_i^{'}\in U_i^2$ be such that $f_i^{k_i}(a_i^{'})\in V_i^2$. Let $k=\max\{k_1,\dots,k_m\}$. Hence, by Lemma 3.2, for each $i\in\{1,\dots,m\}$, $f_i^k(a_i)\in V_i^1$ and $f_i^k(a_i^{'})\in V_i^2$. It follows that $(\prod_{i=1}^m f_i)^k((a_1,\dots,a_m))\in\prod_{i=1}^m V_i^1$ and $(\prod_{i=1}^m f_i)^k((a_1^{'},\dots,a_m^{'}))\in\prod_{i=1}^m V_i^2$. Consequently, $(\prod_{i=1}^m f_i)^k(\mathcal{U}_1)\cap\mathcal{V}_1\neq\emptyset$ and $(\prod_{i=1}^m f_i)^k(\mathcal{U}_2)\cap\mathcal{V}_2\neq\emptyset$. Therefore, $\prod_{i=1}^m f_i$ is weakly mixing.

Suppose that, for each $i \in \{1, \ldots, m\}$, f_i is totally transitive. Let $s \in \mathbb{N}$ and let $\mathcal{U}$, $\mathcal{V}$ be two nonempty open subsets of $\prod_{i=1}^m X_i$. Then, for each $i \in \{1, \ldots, m\}$, there exist nonempty open subsets U_i, V_i of X_i such that $\prod_{i=1}^m U_i \subseteq \mathcal{U}$ and $\prod_{i=1}^m V_i \subseteq \mathcal{V}$. Since, for each $i \in \{1, \ldots, m\}$, f_i is totally transitive, for each $i \in \{1, \ldots, m\}$, there exists $k_i \in \mathbb{N}$ such that $(f_i^s)^{k_i}(U_i) \cap V_i \neq \emptyset$. Hence, for all $i \in \{1, \ldots, m\}$, $f_i^{sk_i}(U_i) \cap V_i \neq \emptyset$. For every $i \in \{1, \ldots, m\}$, let $u_i \in U_i$ be such that $f_i^{sk_i}(u_i) \in V_i$. Let $k = \max\{k_1, \ldots, k_m\}$. By Lemma 3.2, for each $i \in \{1, \ldots, m\}$, $f_i^{sk}(u_i) \in V_i$. Thus, $(f_1^{sk}(u_1), \ldots, f_m^{sk}(u_m)) \in \prod_{i=1}^m f_i^{sk}(U_i)$ and $(f_1^{sk}(u_1), \ldots, f_m^{sk}(u_m)) \in \prod_{i=1}^m V_i$. By Remark 3.1, part (1), we have that, $(\prod_{i=1}^m f_i)^{sk}((u_1, \ldots, u_m)) \in (\prod_{i=1}^m f_i)^{sk}(\prod_{i=1}^m U_i)$ and $(\prod_{i=1}^m f_i)^{sk}((u_1, \ldots, u_m)) \in \prod_{i=1}^m V_i$. Consequently:

$$\left(\prod_{i=1}^m f_i\right)^{sk}((u_1, \ldots, u_m)) \in \left(\left(\prod_{i=1}^m f_i\right)^{sk}\left(\prod_{i=1}^m U_i\right)\right) \cap \prod_{i=1}^m V_i.$$

Hence, $(\prod_{i=1}^m f_i)^s$ is transitive. Since $s \in \mathbb{N}$ is arbitrary, we have that $\prod_{i=1}^m f_i$ is totally transitive.

Suppose that, for each $i \in \{1, \ldots, m\}$, f_i is a chaotic function. Then, for each $i \in \{1, \ldots, m\}$, f_i is transitive and $Per(f_i)$ is dense in X_i. By the first part of the proof of this theorem, we have that, $\prod_{i=1}^m f_i$ is transitive and by Theorem 3.15, $Per(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$. Therefore, $\prod_{i=1} f_i$ is chaotic.

Suppose that, for each $i \in \{1, \ldots, m\}$, f_i is orbit-transitive. Thus, for all $i \in \{1, \ldots, m\}$, there exists $x_i \in X_i$ such that $\mathrm{cl}_{X_i}(\mathcal{O}(x_i, f_i)) = X_i$. Then, by Theorem 3.7, part (2), $\mathrm{cl}_{\prod_{i=1}^m X_i}(\mathcal{O}((x_1, \ldots, x_m), \prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Thence, $\prod_{i=1}^m f_i$ is orbit-transitive.

Suppose that, for each $i \in \{1, \ldots, m\}$, f_i is strictly orbit-transitive. Then, for every $i \in \{1, \ldots, m\}$, there exists $x_i \in X_i$ such that $\mathrm{cl}_{X_i}(\mathcal{O}(f_i(x_i), f_i)) = X_i$. By Theorem 3.7, part (2):

$$\mathrm{cl}_{\prod_{i=1}^m X_i}(\mathcal{O}((f_1(x_1), \ldots, f_n(x_m)), \prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i.$$

Consequently $\mathrm{cl}_{\prod_{i=1}^m X_i}(\mathcal{O}((\prod_{i=1}^m f_i)((x_1, \ldots, x_m)), \prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Therefore, $\prod_{i=1}^m f_i$ is strictly orbit-transitive.

Suppose that, for each $i \in \{1, \ldots, m\}$, f_i is ω-transitive. Then, for every $i \in \{1, \ldots, m\}$, there exists $x_i \in X_i$ such that $\omega(x_i, f_i) = X_i$. By Theorem 3.7, part (1), $\omega((x_1, \ldots, x_m), \prod_{i=1}^m f_i) = \prod_{i=1}^m X_i$. Therefore, $\prod_{i=1}^m f_i$ is ω-transitive.

Suppose that, for each $i \in \{1, \ldots, m\}$, f_i is TT_{++}. Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty open subsets of $\prod_{i=1}^m X_i$. Then, for every $i \in \{1, \ldots, m\}$, there exist nonempty open subsets U_i, V_i of X_i such that $\prod_{i=1}^m U_i \subseteq \mathcal{U}$ and $\prod_{i=1}^m V_i \subseteq \mathcal{V}$. Since, for all $i \in \{1, \ldots, m\}$, f_i is TT_{++}, we have that, for each $i \in \{1, \ldots, m\}$, $n_{f_i}(U_i, V_i)$ is infinite. For every $i \in \{1, \ldots, m\}$, let $k_i \in n_{f_i}(U_i, V_i)$. Then, for each $i \in \{1, \ldots, m\}$, $f_i^{k_i}(U_i) \cap V_i \neq \emptyset$. It follows that, for all $i \in \{1, \ldots, m\}$, there exists $u_i \in U_i$ such that $f_i^{k_i}(u_i) \in V_i$. Let $k = \max\{k_1, \ldots, k_m\}$. By Lemma 3.2, for every $i \in \{1, \ldots, m\}$, $f_i^k(u_i) \in V_i$. Then $[\prod_{i=1}^m f_i]^k((u_1, \ldots, u_m)) \in [\prod_{i=1}^m f_i]^k(\prod_{i=1}^m U_i) \cap \prod_{i=1}^m V_i$. Consequently, $[\prod_{i=1}^m f_i]^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. Therefore, $k \in n_{\prod_{i=1}^m f_i}(\mathcal{U}, \mathcal{V})$. Now, since, for each $i \in \{1, \ldots, m\}$, $n_{f_i}(U_i, V_i)$ is infinite, for every $i \in \{1, \ldots, m\}$, we can take $k_i^{'} \in n_{f_i}(U_i, V_i)$ such that $k_i^{'} > k$. Let $k_1 = \max\{k_1^{'}, \ldots, k_m^{'}\}$. By Lemma 3.2, for every $i \in \{1, \ldots, m\}$, $f_i^{k_1}(u_i) \in V_i$. It follows that,

$(\prod_{i=1}^m f_i)^{k_1}(\prod_{i=1}^m U_i) \cap \prod_{i=1}^m V_i \neq \emptyset$. Consequently, $(\prod_{i=1}^m f_i)^{k_1}(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. Therefore, $k_1 \in n_{\prod_{i=1}^m f_i}(\mathcal{U}, \mathcal{V})$ and $k_1 > k$. Continuing with this process, we have that $n_{\prod_{i=1}^m f_i}(\mathcal{U}, \mathcal{V})$ is an infinite set. Since $\mathcal{U}$ and $\mathcal{V}$ are arbitrary, we have that the function $\prod_{i=1}^m f_i$ is TT_{++}.

Suppose that, for each $i \in \{1, \dots, m\}$, f_i is Touhey. Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty open subsets of $\prod_{i=1}^m X_i$. Then, for every $i \in \{1, \dots, m\}$, there exist two nonempty open subsets U_i and V_i of X_i such that $\prod_{i=1}^m U_i \subseteq \mathcal{U}$ and $\prod_{i=1}^m V_i \subseteq \mathcal{V}$. Since, for all $i \in \{1, \dots, m\}$, f_i is Touhey, for each pair of nonempty open subsets U_i and V_i, there exist a periodic point $x_i \in U_i$ and $k_i \in \mathbb{Z}_+$ such that $f_i^{k_i}(x_i) \in V_i$. Let $k = \max\{k_1, \dots, k_m\}$. Then, by Lemma 3.2, we have that for each $i \in \{1, \dots, m\}$, $f_i^k(x_i) \in V_i$. By Theorem 3.3, part (4), we obtain that $(x_1, \dots, x_m)$ is a periodic point of $\prod_{i=1}^m f_i$ such that $(x_1, \dots, x_m) \in \prod_{i=1}^m U_i \subseteq \mathcal{U}$ and $(\prod_{i=1}^m f_i)^k((x_1, \dots, x_m)) \in \prod_{i=1}^m V_i \subseteq \mathcal{V}$. Therefore, $\prod_{i=1}^m f_i$ is Touhey.

Suppose that, for each $i \in \{1, \dots, m\}$, f_i is an F-system. Then, for every $i \in \{1, \dots, m\}$, f_i is totally transitive and $Per(f_i)$ is dense in X_i. By the third paragraph of the proof of this theorem, we have that $\prod_{i=1}^m f_i$ is totally transitive. Moreover, by Theorem 3.15, we know that $Per(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$. Therefore, $\prod_{i=1}^m f_i$ is an F-system.

Suppose that, for each $i \in \{1, \dots, m\}$, f_i is mild mixing. Let Y be a topological space, let $g : Y \to Y$ be a transitive function. By hypothesis, for each $i \in \{1, \dots, m\}$, $f_i \times g$ is transitive. Since, for each $i \in \{1, \dots, m\}$, X_i is +invariant over open subsets under f_i, by Lemma 4.8, $(\prod_{i=1}^m f_i) \times g$ is transitive. Therefore, $\prod_{i=1}^m f_i$ is mild mixing.

Suppose that, for each $i \in \{1, \dots, m\}$, f_i is scattering. Let Y be a topological space and let $g : Y \to Y$ be a minimal function. By hypothesis, for each $i \in \{1, \dots, m\}$, $f_i \times g$ is transitive. Since, for each $i \in \{1, \dots, m\}$, X_i is +invariant over open subsets under f_i, by Lemma 4.8, $(\prod_{i=1}^m f_i) \times g$ is transitive. Therefore, $\prod_{i=1}^m f_i$ is scattering. □

Proposition 4.11 *Let $X_1, \dots, X_m$ be topological spaces and, for each $i \in \{1, \dots, m\}$, let $f_i : X_i \to X_i$ be a continuous function. If for every $i \in \{1, \dots, m\}$, f_i is minimal and X_i is +invariant over open subsets under f_i, then $\prod_{i=1}^m f_i$ is minimal.*

Proof Suppose that, for each $i \in \{1, \dots, m\}$, f_i is minimal and X_i is +invariant over open subsets under f_i. By hypothesis, we have that $\prod_{i=1}^m f_i$ is a continuous function. Thus, it is sufficient to show that, for all $(x_1, \dots, x_m) \in \prod_{i=1}^m X_i$, $\mathrm{cl}_{\prod_{i=1}^m X_i}(\mathcal{O}((x_1, \dots, x_m), \prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Let $(x_1, \dots, x_m) \in \prod_{i=1}^m X_i$. Since, for each $i \in \{1, \dots, m\}$, f_i is minimal, we have that $\mathrm{cl}_{X_i}(\mathcal{O}(x_i, f_i)) = X_i$. Since, for every $i \in \{1, \dots, m\}$, X_i is +invariant over open subsets under f_i, by Theorem 3.7, part (2), we have that $\mathrm{cl}_{\prod_{i=1}^m X_i}(\mathcal{O}((x_1, \dots, x_m), \prod_{i=1}^m f_i)) = \prod_{i=1}^m X_i$. Thus, since $\prod_{i=1}^m f_i$ is continuous, we have that $\prod_{i=1}^m f_i$ is minimal. □

Corollary 4.12 *Let $X_1, \dots, X_m$ be topological spaces and for each $i \in \{1, \dots, m\}$, let $f_i : X_i \to X_i$ be a continuous function. If for each $i \in \{1, \dots, m\}$, f_i is totally minimal and X_i is +invariant over open subsets under f_i, then $\prod_{i=1}^m f_i$ is totally minimal.*

Proof Let $s \in \mathbb{N}$. By hypothesis, for every $i \in \{1, \dots, m\}$, f_i^s is minimal and continuous. Thus, by

Proposition 4.11, $\prod_{i=1}^{m} f_i^s$ is minimal. Then, by Remark 3.1, part (1), $(\prod_{i=1}^{m} f_i)^s$ is minimal. Finally, since $s \in \mathbb{N}$ is arbitrary, we have that $\prod_{i=1}^{m} f_i$ is totally minimal. □

5. Dynamic properties of n-fold symmetric product of a product space.

Let $X_1, \ldots, X_m$ be topological spaces. In this section we analyze some topological and dynamical properties of the hyperspace $\mathcal{F}_n(\prod_{i=1}^{m} X_i)$ and their relationships with the spaces $\mathcal{F}_n(X_i)$ and X_i, for each $i \in \{1, \ldots, m\}$.

Lemma 5.1 *Let $X_1, \ldots, X_m$ be topological spaces, let $i_0 \in \{1, \ldots, m\}$, let $n \in \mathbb{N}$, let $\{a_1, \ldots, a_r\} \in \mathcal{F}_n(X_{i_0})$ with $r \leq n$, and let $U_1, \ldots, U_n$ be nonempty open subsets of X_{i_0} such that $\{a_1, \ldots, a_r\} \in \langle U_1, \ldots, U_n \rangle$. For each $i \in \{1, \ldots, m\} \setminus \{i_0\}$ and for every $l \in \{1, \ldots, r\}$, let $a_i^l \in X_i$ and let $a_{i_0}^l = a_l$.*

1. *If, for each $l \in \{1, \ldots, r\}$, $b_l = (a_1^l, \ldots, a_{i_0}^l, \ldots, a_m^l)$, then $\{b_1, \ldots, b_r\} \in \mathcal{F}_n(\prod_{i=1}^{m} X_i)$.*
2. *If, for every $i \in \{1, \ldots, m\} \setminus \{i_0\}$ and for each $j \in \{1, \ldots, n\}$, $V_i^j = X_i$ and $V_{i_0}^j = U_j$, then $\{b_1, \ldots, b_r\} \in \langle U_1', \ldots, U_n' \rangle$, where, for all $j \in \{1, \ldots, n\}$, $U_j' = \prod_{i=1}^{m} V_i^j$.*

Proof It is not difficult to see that (1) is satisfied. We show that (2) is true. Let $p \in \{1, \ldots, r\}$. Since $\{a_1, \ldots, a_r\} \in \langle U_1, \ldots, U_n \rangle$, there exists $j_0 \in \{1, \ldots, n\}$ such that $a_p = a_{j_0}^p \in U_{j_0}$. Thus, $b_p = (a_1^p, \ldots, a_{i_0}^p, \ldots, a_m^p) \in \prod_{i=1}^{m} V_i^{j_0} = U_{j_0}'$. Therefore, $b_p \in \bigcup_{j=1}^{n} U_j'$. Consequently, $\{b_1, \ldots, b_r\} \subseteq \bigcup_{j=1}^{n} U_j'$. Now, we will prove that, for each $j \in \{1, \ldots, n\}$, $\{b_1, \ldots, b_r\} \cap U_j' \neq \emptyset$. Let $k \in \{1, \ldots, n\}$. Then, $U_k' = \prod_{i=1}^{m} V_i^k$. Since $\{a_1, \ldots, a_r\} \cap U_k \neq \emptyset$, there exists $l_0 \in \{1, \ldots, r\}$ such that $a_{l_0} \in U_k$. Hence, $(a_1^k, \ldots, a_{l_0}, \ldots, a_m^k) \in U_k'$. Consequently, for each $j \in \{1, \ldots, n\}$, $\{b_1, \ldots, b_r\} \cap U_j' \neq \emptyset$. Therefore, $\{b_1, \ldots, b_r\} \in \langle U_1', \ldots, U_n' \rangle$. □

Lemma 5.2 *Let $X_1, \ldots, X_m$ be topological spaces, let $l, n \in \mathbb{N}$ be such that $l \leq n$, for each $i \in \{1, \ldots, m\}$, let $U_1^i, \ldots, U_n^i$ be nonempty open subsets of X_i, and for every $j \in \{1, \ldots, l\}$, let $(x_1^j, \ldots, x_m^j) \in \prod_{i=1}^{m} X_i$. If $\{(x_1^j, \ldots, x_m^j) : j \in \{1, \ldots, l\}\} \in \langle \prod_{i=1}^{m} U_1^i, \ldots, \prod_{i=1}^{m} U_n^i \rangle$, then, for each $i \in \{1, \ldots, m\}$, $\{x_i^1, \ldots, x_i^l\} \in \langle U_1^i, \ldots, U_n^i \rangle$.*

Proof Let $i_0 \in \{1, \ldots, m\}$. We will show that $\{x_{i_0}^1, \ldots, x_{i_0}^l\} \in \langle U_1^{i_0}, \ldots, U_n^{i_0} \rangle$. First we will prove that $\{x_{i_0}^1, \ldots, x_{i_0}^l\} \subseteq \bigcup_{j=1}^{n} U_j^{i_0}$. Let $k \in \{1, \ldots, l\}$. By hypothesis, there exists $s \in \{1, \ldots, n\}$ such that $(x_1^k, \ldots, x_m^k) \in \prod_{p=1}^{m} U_s^p$. Then $x_{i_0}^k \in U_s^{i_0}$. Thus, $x_{i_0}^k \in \bigcup_{j=1}^{n} U_j^{i_0}$. Therefore, $\{x_{i_0}^1, \ldots, x_{i_0}^l\} \subseteq \bigcup_{j=1}^{n} U_j^{i_0}$.

Now we will see that, for each $j \in \{1, \ldots, n\}$, $\{x_{i_0}^1, \ldots, x_{i_0}^l\} \cap U_j^{i_0} \neq \emptyset$. Let $p \in \{1, \ldots, n\}$. By hypothesis, $\{(x_1^j, \ldots, x_m^j) : j \in \{1, \ldots, l\}\} \cap \prod_{i=1}^{m} U_p^i \neq \emptyset$. Thus, there exists $j \in \{1, \ldots, l\}$ such that $(x_1^j, \ldots, x_m^j) \in \prod_{i=1}^{m} U_p^i$. Then, $x_{i_0}^j \in U_p^{i_0}$. Hence, $\{x_{i_0}^1, \ldots, x_{i_0}^l\} \cap U_p^{i_0} \neq \emptyset$. Because $p \in \{1, \ldots, n\}$ is arbitrary, we have that, for every $p \in \{1, \ldots, n\}$, $\{x_{i_0}^1, \ldots, x_{i_0}^l\} \cap U_p^{i_0} \neq \emptyset$. Therefore, $\{x_{i_0}^1, \ldots, x_{i_0}^l\} \in \langle U_1^{i_0}, \ldots, U_n^{i_0} \rangle$. Finally, since $i_0 \in \{1, \ldots, m\}$ is arbitrary, we have that, for all $i \in \{1, \ldots, m\}$, $\{x_i^1, \ldots, x_i^l\} \in \langle U_1^i, \ldots, U_n^i \rangle$. □

Lemma 5.3 *Let $X_1,\dots,X_m$ be topological spaces, for each $i\in\{1,\dots,m\}$, let $f_i:X_i\to X_i$ be a function, let $n\in\mathbb{N}$ and let $U_1^i,\dots,U_n^i,V_1^i,\dots,V_n^i$ be nonempty open subsets of X_i. Then, for each $i\in\{1,\dots,m\}$:*

$$n_{\mathcal{F}_n(\prod_{i=1}^m f_i)}\left(\left\langle\prod_{i=1}^m U_1^i,\dots,\prod_{i=1}^m U_n^i\right\rangle,\left\langle\prod_{i=1}^m V_1^i,\dots,\prod_{i=1}^m V_n^i\right\rangle\right)\subseteq n_{\mathcal{F}_n(f_i)}(\langle U_1^i,\dots,U_n^i\rangle,\langle V_1^i,\dots,V_n^i\rangle).$$

Proof Let $k\in n_{\mathcal{F}_n(\prod_{i=1}^m f_i)}(\langle\prod_{i=1}^m U_1^i,\dots,\prod_{i=1}^m U_n^i\rangle,\langle\prod_{i=1}^m V_1^i,\dots,\prod_{i=1}^m V_n^i\rangle)$. Then

$$\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k\left(\left\langle\prod_{i=1}^m U_1^i,\dots,\prod_{i=1}^m U_n^i\right\rangle\right)\cap\left\langle\prod_{i=1}^m V_1^i,\dots,\prod_{i=1}^m V_n^i\right\rangle\neq\emptyset.$$

Now, let $l\leq n$ and let $\{(x_1^j,\dots,x_m^j):j\in\{1,\dots,l\}\}\in\langle\prod_{i=1}^m U_1^i,\dots,\prod_{i=1}^m U_n^i\rangle$, such that

$$\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\{x_1^j,\dots,x_m^j):j\in\{1,\dots,l\}\})\in\left\langle\prod_{i=1}^m V_1^i,\dots,\prod_{i=1}^m V_n^i\right\rangle.$$

By Remark 3.1, parts (1) and (2), we have that $\{(f_1^k(x_1^j),\dots,f_m^k(x_m^j)):j\in\{1,\dots,l\}\}\in\langle\prod_{i=1}^m V_1^i,\dots,\prod_{i=1}^m V_n^i\rangle$. Thus, by Lemma 5.2, for every $i\in\{1,\dots,m\}$, $\{x_i^1,\dots,x_i^l\}\in\langle U_1^i,\dots,U_n^i\rangle$ and $\{f_i^k(x_i^1),\dots,f_i^k(x_i^l)\}\in\langle V_1^i,\dots,V_n^i\rangle$. Hence, for all $i\in\{1,\dots,m\}$, $(\mathcal{F}_n(f_i))^k(\{x_i^1,\dots,x_i^l\})\in(\mathcal{F}_n(f_i))^k(\langle U_1^i,\dots,U_n^i\rangle)\cap\langle V_1^i,\dots,V_n^i\rangle$. Therefore, for every $i\in\{1,\dots,m\}$, $k\in n_{\mathcal{F}_n(f_i)}(\langle U_1^i,\dots,U_n^i\rangle,\langle V_1^i,\dots,V_n^i\rangle)$. □

By Corollary 3.8 and by [4, Theorem 3.14], we have the following result.

Proposition 5.4 *Let $X_1,\dots,X_m$ be topological spaces, for each $i\in\{1,\dots,m\}$, let $f_i:X_i\to X_i$ be a function, and let $n\in\mathbb{N}$. Then the following hold:*

1. *For each $i\in\{1,\dots,m\}$, $\mathcal{F}_n(X_i)$ is perfect if and only if $\prod_{i=1}^n X_i$ is perfect.*
2. *For each $i\in\{1,\dots,m\}$, X_i is perfect if and only if $\mathcal{F}_n(\prod_{i=1}^m X_i)$ is perfect.*
3. *For each $i\in\{1,\dots,m\}$, $\mathcal{F}_n(X_i)$ is perfect if and only if $\mathcal{F}_n(\prod_{i=1}^m X_i)$ is perfect.*

By Theorem 3.9 and [4, Theorem 3.8], we have the following result.

Proposition 5.5 *Let $X_1,\dots,X_m$ be topological spaces and let $n\in\mathbb{N}$. Then the following hold:*

1. *For each $i\in\{1,\dots,m\}$, X_i is pseudoregular if and only if $\mathcal{F}_n(\prod_{i=1}^m X_i)$ is pseudoregular.*
2. *For every $i\in\{1,\dots,m\}$, $\mathcal{F}_n(X_i)$ is pseudoregular if and only if $\prod_{i=1}^m X_i$ is pseudoregular.*

Theorem 5.6 *Let $X_1,\dots,X_m$ be topological spaces, for each $i\in\{1,\dots,m\}$, let $f_i:X_i\to X_i$ be a function, and let $l,n\in\mathbb{N}$ be such that $l\leq n$. If $\mathcal{A}=\{(x_1^j,\dots,x_m^j):j\in\{1,\dots,l\}\}\in\mathcal{F}_n(\prod_{i=1}^m X_i)$ is a transitive point of $\mathcal{F}_n(\prod_{i=1}^m f_i)$, then, for every $i\in\{1,\dots,m\}$, $\{x_i^1,\dots,x_i^l\}$ is a transitive point of $\mathcal{F}_n(f_i)$.*

Proof Suppose that $\mathcal{A}$ is a transitive point of $\mathcal{F}_n(\prod_{i=1}^m f_i)$. Let $i_0 \in \{1,\ldots,m\}$ and let $\mathcal{U}$ be a nonempty open subset of $\mathcal{F}_n(X_{i_0})$. Hence, by [10, Lemma 4.2], there exist nonempty open subsets $U_1,\ldots,U_n$ of X_{i_0} such that $\langle U_1,\ldots,U_n\rangle \subseteq \mathcal{U}$. For each $i \in \{1,\ldots,m\}\backslash\{i_0\}$ and for every $j \in \{1,\ldots,n\}$, let $V_i^j = X_i$ and $V_{i_0}^j = U_j$. Then, for all $j \in \{1,\ldots,n\}$, let $U_j^{'} = \prod_{i=1}^n V_i^j$. Thus, $\langle U_1^{'},\ldots,U_n^{'}\rangle$ is a nonempty open subset of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. By hypothesis, $\langle U_1^{'},\ldots,U_n^{'}\rangle \cap \mathcal{O}(\mathcal{A},\mathcal{F}_n(\prod_{i=1}^m f_i)) \neq \emptyset$. In consequence, there exists $k \in \mathbb{N}$ such that $[\mathcal{F}_n(\prod_{i=1}^m f_i)]^k(\mathcal{A}) \in \langle U_1^{'},\ldots,U_n^{'}\rangle$. Then $\{(f_1^k(x_1^j),\ldots,f_n^k(x_m^j)) : j \in \{1,\ldots,l\}\} \in \langle U_1^{'},\ldots,U_n^{'}\rangle$. By Lemma 5.2, we have that $\{f_{i_0}^k(x_{i_0}^1),\ldots,f_{i_0}^k(x_{i_0}^l)\} \in \langle U_1,\ldots,U_n\rangle$. Hence, $[\mathcal{F}_n(f_{i_0})]^k(\{x_{i_0}^1,\ldots,x_{i_0}^l\}) \in \langle U_1,\ldots,U_n\rangle \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{O}(\{x_{i_0}^1,\ldots,x_{i_0}^l\},\mathcal{F}_n(f)) \neq \emptyset$. Therefore, $\{x_{i_0}^1,\ldots,x_{i_0}^l\}$ is a transitive point of $\mathcal{F}_n(f_{i_0})$. Because $i_0 \in \{1,\ldots,m\}$ is arbitrary, we have that, for each $i \in \{1,\ldots,m\}$, $\{x_i^1,\ldots,x_i^l\}$ is a transitive point of $\mathcal{F}_n(f_i)$. □

Theorem 5.7 *Let $X_1,\ldots,X_m$ be topological spaces, for each $i \in \{1,\ldots,m\}$, let $f_i : X_i \to X_i$ be a function, let $l,n \in \mathbb{N}$ be such that $l \leq n$, and let $\mathcal{A} = \{(x_1^j,\ldots,x_m^j) : j \in \{1,\ldots,l\}\} \in \mathcal{F}_n(\prod_{i=1}^m X_i)$. If $\omega(\mathcal{A},\mathcal{F}_n(\prod_{i=1}^m f_i)) = \mathcal{F}_n(\prod_{i=1}^m X_i)$, then, for each $i \in \{1,\ldots,m\}$, $\omega(\{x_i^1,\ldots,x_i^l\},\mathcal{F}_n(f_i)) = \mathcal{F}_n(X_i)$.*

Proof Suppose that $\omega(\mathcal{A},\mathcal{F}_n(\prod_{i=1}^m f_i)) = \mathcal{F}_n(\prod_{i=1}^m X_i)$. Let $i_0 \in \{1,\ldots,m\}$. Now we show that $\omega(\{x_{i_0}^1,\ldots,x_{i_0}^l\},\mathcal{F}_n(f_{i_0})) = \mathcal{F}_n(X_{i_0})$. Let $\{a_1,\ldots,a_r\} \in \mathcal{F}_n(X_{i_0})$ with $r \leq n$, let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X_{i_0})$ such that $\{a_1,\ldots,a_r\} \in \mathcal{U}$ and let $k \in \mathbb{N}$. By [10, Lemma 4.2], there exist nonempty open subsets $U_1,\ldots,U_n$ of X_{i_0} such that $\{a_1,\ldots,a_r\} \in \langle U_1,\ldots,U_n\rangle \subseteq \mathcal{U}$. For each $l \in \{1,\ldots,r\}$ and for every $i \in \{1,\ldots,m\}\backslash\{i_0\}$, let $a_i^l \in X_i$ and let $a_{i_0}^l = a_l$. Then, for all $l \in \{1,\ldots,r\}$, let $a_l^{'} = (a_1^l,\ldots,a_m^l)$. On the other hand, for each $i \in \{1,\ldots,m\}\backslash\{i_0\}$ and for every $j \in \{1,\ldots,n\}$, let $V_i^j = X_i$ and $V_{i_0}^j = U_j$. Finally, for all $j \in \{1,\ldots,n\}$, let $U_j^{'} = \prod_{i=1}^m V_i^j$. By Lemma 5.1, part (1), $\{a_1^{'},\ldots,a_r^{'}\} \in \mathcal{F}_n(\prod_{i=1}^m X_i)$. Hence, by hypothesis, $\{a_1^{'},\ldots,a_r^{'}\} \in \omega(\mathcal{A},\mathcal{F}_n(\prod_{i=1}^m f_i))$. By Lemma 5.1, part (2), $\{a_1^{'},\ldots,a_r^{'}\} \in \langle U_1^{'},\ldots,U_n^{'}\rangle$. Thus, there exists $s \geq k$, such that $[\mathcal{F}_n(\prod_{i=1}^m f_i)]^s(\mathcal{A}) \in \langle U_1^{'},\ldots,U_n^{'}\rangle$. By Remark 3.1, parts (1) and (2), we have that $\{(f_1^s(x_1^p),\ldots,f_{i_0}^s(x_{i_0}^p),\ldots,f_m^s(x_m^p)) : p \in \{1,\ldots,l\}\} \in \langle U_1^{'},\ldots,U_n^{'}\rangle$. By Lemma 5.2, $\{f_{i_0}^s(x_{i_0}^1),\ldots,f_{i_0}^s(x_{i_0}^l)\} \in \langle U_1,\cdots,U_n\rangle$. Thus, $[\mathcal{F}_n(f_{i_0})]^s(\{x_{i_0}^1,\ldots,x_{i_0}^l\}) \in \langle U_1,\ldots,U_n\rangle \subseteq \mathcal{U}$. Then $\{a_1,\ldots,a_r\} \in \omega(\{x_{i_0}^1,\ldots,x_{i_0}^l\},\mathcal{F}_n(f_{i_0}))$. Thus, $\omega(\{x_{i_0}^1,\ldots,x_{i_0}^l\},\mathcal{F}_n(f_{i_0})) = \mathcal{F}_n(X_{i_0})$. □

By Theorem 3.15 and [4, Theorem 3.4], we have the following result.

Theorem 5.8 *Let $X_1,\ldots,X_m$ be topological spaces, for each $i \in \{1,\ldots,m\}$, let $f_i : X_i \to X_i$ be a function, and let $n \in \mathbb{N}$. Then the following hold:*

1. *For every $i \in \{1,\ldots,m\}$, $Per(f_i)$ is dense in X_i if and only if $Per(\mathcal{F}_n(\prod_{i=1}^m f_i))$ is dense in $\mathcal{F}_n(\prod_{i=1}^m X_i)$.*
2. *For each $i \in \{1,\ldots,m\}$, $Per(\mathcal{F}_n(f_i))$ is dense in $\mathcal{F}_n(X_i)$ if and only if $Per(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$.*

By Proposition 3.10 and [4, Theorem 3.3], we have the following result.

Proposition 5.9 *Let $X_1,\ldots,X_m$ be topological spaces, for each $i \in \{1,\ldots,m\}$, let $f_i : X_i \to X_i$ be a function, and let $n \in \mathbb{N}$. Then the following hold:*

1. *For every $i \in \{1,\ldots,m\}$, U_i is $+$invariant under f_i if and only if $\langle \prod_{i=1}^m U_i\rangle$ is $+$invariant under $\mathcal{F}_n(\prod_{i=1}^m f_i)$.*

2. *For each $i \in \{1,\ldots,m\}$, $\langle U_i\rangle$ is $+$invariant under $\mathcal{F}_n(f_i)$ if and only if $\prod_{i=1}^m U_i$ is $+$invariant under $\prod_{i=1}^m f_i$.*

6. Induced functions to n-fold symmetric products of product spaces

Let $X_1,\ldots,X_m$ be topological spaces and for each $i \in \{1,\ldots,m\}$, let $f_i : X_i \to X_i$ be a function. In this section we analyze the relationships between the functions $\mathcal{F}_n(\prod_{i=1}^m f_i)$, $\mathcal{F}_n(f_i)$ and f_i, for every $i \in \{1,\ldots,m\}$, when any of this is exact, mixing, transitive, weakly mixing, totally transitive, strongly transitive, chaotic, minimal, totally minimal, orbit-transitive, strictly orbit-transitive, ω-transitive, TT_{++}, mild mixing, exactly Devaney chaotic, backward minimal, scattering, Touhey or an F-system.

Theorem 6.1 *Let X, Y be topological spaces, let $f : X \to X$, $g : Y \to Y$ be functions and let $n \in \mathbb{N}$. If $\mathcal{F}_n(f) \times g$ is transitive, then $f \times g$ is transitive.*

Proof Suppose that $\mathcal{F}_n(f) \times g$ is transitive. Let $\mathcal{U}, \mathcal{V}$ be two nonempty open subsets of $X \times Y$. Then there exist nonempty open subsets U_1, U_2 of X and V_1, V_2 of Y such that $U_1 \times V_1 \subseteq \mathcal{U}$ and $U_2 \times V_2 \subseteq \mathcal{V}$. Thus, $\langle U_1\rangle$ and $\langle U_2\rangle$ are nonempty open subsets of $\mathcal{F}_n(X)$. By hypothesis, there exists $k \in \mathbb{N}$ such that $(\mathcal{F}_n(f) \times g)^k(\langle U_1\rangle \times V_2) \cap (\langle U_2\rangle \times V_2) \neq \emptyset$. It follows that there exists $(\{x_1,\ldots,x_r\}, v_1) \in \langle U_1\rangle \times V_2$ such that $[\mathcal{F}_n(f) \times g]^k((\{x_1,\ldots,x_r\}, v_1)) \in \langle U_2\rangle \times V_2$. Let $x \in \{x_1,\ldots,x_r\}$. We have that, $x \in U_1$ and $f^k(x) \in U_2$. Consequently, for each $x \in \{x_1,\ldots,x_r\}$, $(x, v_1) \in U_1 \times V_1$ and $(f \times g)^k((x, v_1)) \in U_2 \times V_2$. Thus, $(f \times g)^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ and $f \times g$ is transitive. □

The proof of Proposition 6.2 is followed by [4, Theorems 3.4 and 4.10].

Proposition 6.2 *Let X be a topological space, let $f : X \to X$ be a function, and let $n \in \mathbb{N}$. Then f is exactly Devaney chaotic if and only if $\mathcal{F}_n(f)$ is exactly Devaney chaotic.*

Theorem 6.3 *Let X be a topological space, let $f : X \to X$ be a function and let $n \in \mathbb{N}$. Let $\mathcal{M}$ be one of the following classes of functions: Touhey, an F-system, backward minimal, totally minimal, mild mixing or scattering. If $\mathcal{F}_n(f) \in \mathcal{M}$, then $f \in \mathcal{M}$.*

Proof Suppose that $\mathcal{F}_n(f)$ is Touhey. Let U, V be nonempty open subsets of X. Hence, $\langle U\rangle$ and $\langle V\rangle$ are nonempty open subsets of $\mathcal{F}_n(X)$. Since $\mathcal{F}_n(f)$ is Touhey, there exist a periodic point $\{x_1,\ldots,x_r\} \in \langle U\rangle$ and $k \in \mathbb{Z}_+$ such that $[\mathcal{F}_n(f)]^k(\{x_1,\ldots,x_r\}) \in \langle V\rangle$. Then, by [4, Theorem 3.4], for each $i \in \{1,\ldots,r\}$, x_i is a periodic point of f. Furthermore, for every $i \in \{1,\ldots,r\}$, $x_i \in U$ and $f^k(x_i) \in V$. Therefore, f is Touhey.

Suppose that $\mathcal{F}_n(f)$ is an F-system. Then $\mathcal{F}_n(f)$ is totally transitive and $Per(\mathcal{F}_n(f))$ is dense in $\mathcal{F}_n(X)$. Thus, by [4, Theorem 4.14], f is totally transitive and, by [4, Theorem 3.4], $Per(f)$ is dense in X. Therefore, f is an F-system.

Suppose that $\mathcal{F}_n(f)$ is backward minimal. Let $x \in X$ and let U be a nonempty open subset of X. Then $\langle U\rangle$ is a nonempty open subset of $\mathcal{F}_n(X)$ and $\{x\} \in \mathcal{F}_n(X)$. Since $\mathcal{F}_n(f)$ is backward minimal, the set $\{A \in \mathcal{F}_n(X) : (\mathcal{F}_n(f))^l(A) = \{x\}$, for some $l \in \mathbb{N}\}$, is dense in $\mathcal{F}_n(X)$. Thus, there exist $\{x_1,\ldots,x_r\} \in \langle U\rangle$ and

$l \in \mathbb{N}$ such that $[\mathcal{F}_n(f)]^l(\{x_1, \ldots, x_r\}) = \{x\}$. It follows that, for each $i \in \{1, \ldots, r\}$, $x_i \in U$ and $f^l(x_i) = x$. Thus, $\{y \in X : f^l(y) = x, \text{ for some } l \in \mathbb{N}\} \cap U \neq \emptyset$. Therefore, the set $\{y \in X : f^l(y) = x, \text{ for some } l \in \mathbb{N}\}$ is dense in X. Because $x \in X$ is arbitrary, we have that f is backward minimal.

Suppose that $\mathcal{F}_n(f)$ is totally minimal. Let $s \in \mathbb{N}$. By hypothesis, $(\mathcal{F}_n(f))^s$ is minimal. Then, by Remark 3.1, part (1), $\mathcal{F}_n(f^s)$ is minimal. Hence, by [4, Theorem 4.18], f^s is minimal.

Suppose that $\mathcal{F}_n(f)$ is mild mixing. Let Y be a topological space and let $g : Y \to Y$ be a transitive function. By hypothesis, $\mathcal{F}_n(f) \times g$ is transitive. Thus, by Theorem 6.1, $f \times g$ is transitive. Therefore, f is mild mixing.

Suppose that $\mathcal{F}_n(f)$ is scattering. Let Y be a topological space, let $g : Y \to Y$ be a minimal function. By hypothesis, $\mathcal{F}_n(f) \times g$ is transitive. By Theorem 6.1, $f \times g$ is transitive. Therefore, f is scattering. □

The converse of Theorem 6.3 is not true in general. Let us see a partial example of this in the following:

Example 6.4 *Let $X = [0, 1]$ and let $f : X \to X$ be a function given by:*

$$f(x) = \begin{cases} 2x + \frac{1}{2}, & \text{if} \quad x \in [0, \frac{1}{4}]; \\ \frac{3}{2} - 2x, & \text{if} \quad x \in [\frac{1}{4}, \frac{1}{2}]; \\ 1 - x, & \text{if} \quad x \in [\frac{1}{2}, 1]. \end{cases}$$

In [10, Example 4.10], it is shown that f is a chaotic function; however, the function $\mathcal{F}_n(f)$ is not chaotic. On the other hand, observe that f is a continuous function. Thus, by [18, Proposition 2.6], f is Touhey. If we suppose that $\mathcal{F}_n(f)$ is Touhey, again, by [18, Proposition 2.6], $\mathcal{F}_n(f)$ is a chaotic function, which is a contradiction. Therefore, $\mathcal{F}_n(f)$ is not Touhey.

Theorem 6.5 *Let X, Y be topological spaces, let $f : X \to X$, $g : Y \to Y$ be functions and let $n \in \mathbb{N}$. If X is $+$invariant over open subsets under f and $f \times g$ is transitive, then $\mathcal{F}_n(f) \times g$ is transitive.*

Proof Suppose that X is $+$invariant over open subsets under f and $f \times g$ is transitive. Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X) \times Y$. Then there exist nonempty open subsets $\mathcal{U}_1, \mathcal{U}_2$ of $\mathcal{F}_n(X)$ and V_1, V_2 of Y such that $\mathcal{U}_1 \times V_1 \subseteq \mathcal{U}$ and $\mathcal{U}_2 \times V_2 \subseteq \mathcal{V}$. By [10, Lemma 4.2], there exist nonempty open subsets $U_1^1 \ldots, U_n^1$, $U_1^2, \ldots, U_n^2$ of X such that $\langle U_1^1, \ldots, U_n^1 \rangle \subseteq \mathcal{U}_1$ and $\langle U_1^2, \ldots, U_n^2 \rangle \subseteq \mathcal{U}_2$. Since $f \times g$ is transitive, for each $i \in \{1, \ldots, n\}$, there exists $k_i \in \mathbb{N}$ such that $(f \times g)^{k_i}(U_i^1 \times V_1) \cap (U_i^2 \times V_2) \neq \emptyset$. Hence, for every $i \in \{1, \ldots, n\}$, there exists $(u_i, v_i) \in U_i^1 \times V_1$ such that $(f \times g)^{k_i}(u_i, v_i) \in U_i^2 \times V_2$. It follows that, for all $i \in \{1, \ldots, n\}$, $f^{k_i}(u_i) \in U_i^2$. Let $k = \max\{k_1, \ldots, k_n\}$. By Lemma 3.2, for each $i \in \{1, \ldots, n\}$, $f^k(u_i) \in U_i^2$. Consequently, $\{f^k(u_1), \ldots, f^k(u_n)\} \in \langle U_1^2, \ldots, U_n^2 \rangle$ which means that $[\mathcal{F}_n(f)]^k(\{u_1, \ldots, u_n\}) \in \langle U_1^2, \ldots, U_n^2 \rangle$. Moreover, $\{u_1, \ldots, u_n\} \in \langle U_1^1, \ldots, U_n^1 \rangle$. Suppose that $k = k_{i_0}$, where $i_0 \in \{1, \ldots, n\}$, and let $v = v_{i_0}$. Then $g^k(v) \in V_2$ and $v \in V_1$. Finally, $[\mathcal{F}_n(f) \times g]^k((\{u_1, \ldots, u_n\}, v)) \in \langle U_1^2, \ldots, U_n^2 \rangle \times V_2$ and $(\{u_1, \ldots, u_n\}, v) \in \langle U_1^1, \ldots, U_n^1 \rangle \times V_2$. Therefore, $[\mathcal{F}_n(f) \times g]^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ and $\mathcal{F}_n(f) \times g$ is transitive. □

Theorem 6.6 *Let X be a topological space, let $f : X \to X$ be a function, and let $n \in \mathbb{N}$. Let $\mathcal{M}$ be one of the following classes of function: transitive, totally transitive, chaotic, Touhey, an F-system, mild mixing or scattering. Then, if $f \in \mathcal{M}$ and X is $+$invariant over open subsets under f, then $\mathcal{F}_n(f) \in \mathcal{M}$.*

Proof Suppose that f is transitive. Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X)$. Thence, by [10, Lemma 4.2], there exist nonempty open subsets $U_1, \dots, U_n$, $V_1, \dots, V_n$ of X such that $\langle U_1, \dots, U_n \rangle_n \subseteq \mathcal{U}$ and $\langle V_1, \dots, V_n \rangle_n \subseteq \mathcal{V}$. Since f is transitive, for each $i \in \{1, \dots, n\}$, there exists $k_i \in \mathbb{N}$ such that $f^{k_i}(U_i) \cap V_i \neq \emptyset$. Then, for every $i \in \{1, \dots, n\}$, there exists $u_i \in U_i$ such that $f^{k_i}(u_i) \in V_i$. Let $k = \max\{k_1, \dots, k_n\}$. By Lemma 3.2, for all $i \in \{1, \dots, n\}$, $f^k(u_i) \in V_i$. It follows that, $\{u_1, \dots, u_n\} \in \langle U_1, \dots, U_n \rangle$ and $[\mathcal{F}_n(f)]^k(\{u_1, \dots, u_n\}) \in \langle V_1, \dots, V_n \rangle$. Therefore, $[\mathcal{F}_n(f)]^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ and $\mathcal{F}_n(f)$ is transitive.

Suppose that f is totally transitive. Let $s \in \mathbb{N}$ and let $\mathcal{U}$, $\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X)$. Then by [10, Lemma 4.2], we have that, there exist nonempty open subsets $U_1, \dots, U_n, V_1, \dots, V_n$ of X such that, $\langle U_1, \dots, U_n \rangle \subseteq \mathcal{U}$ and $\langle V_1, \dots, V_n \rangle \subseteq \mathcal{V}$. Since f^s is transitive, for each $i \in \{1, \dots, n\}$, there exists $k_i \in \mathbb{N}$ such that $(f^s)^{k_i}(U_i) \cap V_i \neq \emptyset$. For every $i \in \{1, \dots, n\}$, let $u_i \in U_i$ such that $(f^s)^{k_i}(u_i) \in V_i$. Let $k = \max\{k_1, \dots k_n\}$. Thus, by Lemma 3.2, for all $i \in \{1, \dots, n\}$, $(f^s)^k(u_i) \in V_i$. Thus, $\{u_1, \dots, u_n\} \in \langle U_1, \dots, U_n \rangle$ and $\{(f^s)^k(u_1), \dots, (f^s)^k(u_n)\} \in \langle V_1, \dots, V_n \rangle$. So, $([\mathcal{F}_n(f)]^s)^k(\{u_1, \dots, u_n\}) \in \langle V_1, \dots, V_n \rangle$. It follows that, $([\mathcal{F}_n(f)]^s)^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. Consequently, $[\mathcal{F}_n(f)]^s$ is transitive. Finally, because s is arbitrary, we have that $\mathcal{F}_n(f)$ is totally transitive.

Suppose that f is chaotic. Then f is transitive and $Per(f)$ is dense in X. Thus, by [4, Theorem 3.4], we have that $Per(\mathcal{F}_n(f))$ is dense in $\mathcal{F}_n(X)$. Moreover, by the first part of this proof, if f is transitive then $\mathcal{F}_n(f)$ is transitive. Therefore, $\mathcal{F}_n(f)$ is chaotic.

Suppose that f is Touhey. Let $\mathcal{U}$, $\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X)$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1, \dots, U_n$, $V_1, \dots, V_n$ of X such that $\langle U_1, \dots, U_n \rangle \subseteq \mathcal{U}$ and $\langle V_1, \dots, V_n \rangle \subseteq \mathcal{V}$. Since f is Touhey, for every $i \in \{1, \dots, n\}$, there exist a periodic point $x_i \in U_i$ and $k_i \in \mathbb{Z}_+$ such that $f^{k_i}(x_i) \in V_i$. Let $k = \max\{k_1, \dots, k_n\}$. Then, by Lemma 3.2, for each $i \in \{1, \dots, n\}$, $f^k(x_i) \in V_i$. Consequently, $[\mathcal{F}_n(f)]^k(\{x_1, \dots, x_n\}) \in \langle V_1, \dots, V_n \rangle$. Furthermore, $\{x_1, \dots, x_n\} \in \langle U_1, \dots, U_n \rangle$. On the other hand, since, for all $i \in \{1, \dots, n\}$, x_i is a periodic point of f_i, by [4, Theorem 3.4], $\{x_1, \dots, x_n\}$ is a periodic point of $\mathcal{F}_n(f)$. Therefore, $\mathcal{F}_n(f)$ is Touhey.

Suppose that f is an F-system. Then f is totally transitive and $Per(f)$ is dense in X. Thus, by the second part of this proof, we have that $\mathcal{F}_n(f)$ is totally transitive. Moreover, by [4, Theorem 3.4], $Per(\mathcal{F}_n(f))$ is dense. Therefore, $\mathcal{F}_n(f)$ is an F-system.

Suppose that f is mild mixing. Let Y be a topological space and let $g : Y \to Y$ be a transitive function. By hypothesis, $f \times g$ is transitive. Since X is +invariant over open subsets under f, by Theorem 6.5, $\mathcal{F}_n(f) \times g$ is transitive. Therefore, $\mathcal{F}_n(f)$ is mild mixing.

Suppose that f is scattering. Let Y be a topological space, let $g : Y \to Y$ be a minimal function. By hypothesis, $f \times g$ is transitive. Since, X is +invariant over open subsets under f, by Theorem 6.5, $\mathcal{F}_n(f) \times g$ is transitive. Therefore, $\mathcal{F}_n(f)$ is scattering. □

Theorem 6.7 *Let X be a topological space, let $f : X \to X$ be a continuous function and let $n \in \mathbb{N}$. If f is minimal and X is +invariant over open subsets under f, then $\mathcal{F}_n(f)$ is minimal.*

Proof Suppose that f is minimal and that X is +invariant over open subsets under f. Since f is a continuous function, by [4, Theorem 6.1] $\mathcal{F}_n(f)$ is a continuous function. Thus, to show that $\mathcal{F}_n(f)$ is minimal, by [15, Proposition 6.2], we need to prove that, for each $A \in \mathcal{F}_n(X)$, $\mathrm{cl}_{\mathcal{F}_n(X)}(\mathcal{O}(A, \mathcal{F}_n(f))) = \mathcal{F}_n(X)$. Let

$\{x_1,\dots,x_r\} \in \mathcal{F}_n(X)$. Since f is minimal, for each $i \in \{1,\dots,m\}$, $\mathrm{cl}_X(\mathcal{O}(x_i,f)) = X$. Let $\mathcal{U}$ be a nonempty open subset of $\mathcal{F}_n(X)$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1,\dots,U_n$ of X such that $\langle U_1,\dots,U_n\rangle \subseteq \mathcal{U}$. Consider the following cases:

Case (i): $r = n$. In this case, for each $i \in \{1,\dots,n\}$, there exists $k_i \in \mathbb{N}$ such that $f^{k_i}(x_i) \in U_i$. Let $k = \max\{k_1,\dots,k_n\}$. Then, by Lemma 3.2, we have that, for every $i \in \{1,\dots,n\}$, $f^k(x_i) \in U_i$. Thus, $[\mathcal{F}_n(f)]^k(\{x_1,\dots,x_r\}) \in \langle U_1,\dots,U_n\rangle$. This implies that $\mathcal{O}(\{x_1,\dots,x_r\},\mathcal{F}_n(f)) \cap \mathcal{U} \neq \emptyset$. Therefore, $\mathrm{cl}_{\mathcal{F}_n(X)}(\mathcal{O}(\{x_1,\dots,x_r\},\mathcal{F}_n(f))) = \mathcal{F}_n(X)$. Finally, since $\{x_1,\dots,x_r\} \in \mathcal{F}_n(X)$ is arbitrary, we have that $\mathcal{F}_n(f)$ is minimal.

Case (ii): $r < n$. In this case, for each $i \in \{1,\dots,r\}$, $\mathcal{O}(x_i,f) \cap U_i \neq \emptyset$ and for every $j \in \{r+1,\dots,n\}$, $\mathcal{O}(x_r,f) \cap U_j \neq \emptyset$. Then, for all $i \in \{1,\dots,r\}$, there exists $k_i \in \mathbb{N}$ such that $f^{k_i}(x_i) \in U_i$ and for each $j \in \{r+1,\dots,n\}$, there exists $k_j \in \mathbb{N}$ such that $f^{k_j}(x_r) \in U_j$. Let $k = \max\{k_1,\dots,k_n\}$. Then, by Lemma 3.2, for every $i \in \{1,\dots,r\}$, $f^k(x_i) \in U_i$ and for all $i \in \{1,\dots,n\}$, $f^k(x_r) \in U_i$. It follows that $\{f^k(x_1),\dots,f^k(x_r)\} \in \langle U_1,\dots,U_n\rangle \subseteq \mathcal{U}$. Consequently, $[\mathcal{F}_n(f)]^k(\{x_1,\dots,x_r\}) \in \mathcal{U}$. Thus, $\mathcal{O}(\{x_1,\dots,x_r\},\mathcal{F}_n(f)) \cap \mathcal{U} \neq \emptyset$. Therefore, $\mathrm{cl}_{\mathcal{F}_n(X)}(\mathcal{O}(\{x_1,\dots,x_r\},\mathcal{F}_n(f))) = \mathcal{F}_n(X)$. Because $\{x_1,\dots,x_r\} \in \mathcal{F}_n(X)$ is arbitrary, $\mathcal{F}_n(f)$ is minimal. □

Proposition 6.8 *Let X be a topological space, let $f : X \to X$ be a continuous function, and let $n \in \mathbb{N}$. If f is totally minimal and X is $+$invariant over open subsets under f, then $\mathcal{F}_n(f)$ is totally minimal.*

Proof Let $s \in \mathbb{N}$. By hypothesis, f^s is minimal and continuous. Hence, by Theorem 6.7, $\mathcal{F}_n(f^s)$ is minimal. Then, by Remark 3.1, part (1), $(\mathcal{F}_n(f))^s$ is minimal. Since $s \in \mathbb{N}$ is arbitrary, we have that $\mathcal{F}_n(f)$ is totally minimal. □

Theorem 6.9 *Let $X_1,\dots,X_m$ be topological spaces, for each $i \in \{1,\dots,m\}$, let $f_i : X_i \to X_i$ be a function, and let $n \in \mathbb{N}$. Then the following hold:*

1. *$\mathcal{F}_n(\prod_{i=1}^m f_i)$ is exact if and only if, for each $i \in \{1,\dots,m\}$, $\mathcal{F}_n(f_i)$ is exact.*
2. *$\mathcal{F}_n(\prod_{i=1}^m f_i)$ is exact if and only if, for each $i \in \{1,\dots,m\}$, f_i is exact.*

Proof Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is exact. Let $i_0 \in \{1,\dots,m\}$ and let $\mathcal{U}$ be a nonempty open subset of $\mathcal{F}_n(X_{i_0})$. By [10, Lemma 4.2], there exist nonempty open subsets $U_1,\dots,U_n$ of X_{i_0} such that $\langle U_1,\dots,U_n\rangle \subseteq \mathcal{U}$. For each $i \in \{1,\dots,m\}\setminus\{i_0\}$ and for every $j \in \{1,\dots,n\}$, let $U_i^j = X_i$ and $U_{i_0}^j = U_j$. Moreover, for all $j \in \{1,\dots,n\}$, let $U_j^{'} = \prod_{i=1}^m U_i^j$. Note that $\langle U_1^{'},\dots,U_n^{'}\rangle$ is a nonempty open subset of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. Since $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is exact, there exists $k \in \mathbb{N}$ such that $[\mathcal{F}_n(\prod_{i=1}^m f_i)]^k(\langle U_1^{'},\dots,U_n^{'}\rangle) = \mathcal{F}_n(\prod_{i=1}^m X_i)$. Let $\{x_1,\dots,x_r\} \in \mathcal{F}_n(X_{i_0})$, with $r \leq n$. For each $j \in \{1,\dots,m\}\setminus\{i_0\}$ and for every $l \in \{1,\dots,r\}$ let $a_j^l \in X_j$ and let $a_{i_0}^l = x_l$. Finally, for all $l \in \{1,\dots,r\}$, let $x_l^{'} = (a_1^l,\dots,a_m^l)$. By Lemma 5.1, part (1), $\{x_1^{'},\dots,x_r^{'}\} \in \mathcal{F}_n(\prod_{i=1}^m X_i)$. Then $\{x_1^{'},\dots,x_r^{'}\} \in [\mathcal{F}_n(\prod_{i=1}^m f_i)]^k(\langle U_1^{'},\dots,U_n^{'}\rangle)$. Thus, there exists $\{(b_1^j,\dots,b_m^j) : j \in \{1,\dots,p\}\} \in \langle U_1^{'},\dots,U_n^{'}\rangle$ such that $[\mathcal{F}_n(\prod_{i=1}^m f_i)]^k(\{(b_1^j,\dots,b_m^j) : j \in \{1,\dots,p\}\}) = \{x_1^{'},\dots,x_r^{'}\}$. Hence, $\{f_{i_0}^k(b_{i_0}^1),\dots,f_{i_0}^k(b_{i_0}^p)\} = \{x_1,\dots,x_r\}$. It follows that $[\mathcal{F}_n(f_{i_0})]^k(\{b_{i_0}^1,\dots,b_{i_0}^p\}) = \{x_1,\dots,x_r\}$. On the other hand, by Lemma 5.2, $\{x_1,\dots,x_r\} \in [\mathcal{F}_n(f_{i_0})]^k(\langle U_1\dots,U_n\rangle)$. Therefore, $\mathcal{F}_n(X_{i_0}) = [\mathcal{F}_n(f_{i_0})]^k(\mathcal{U})$ and $\mathcal{F}_n(f_{i_0})$ is exact.

Suppose that, for each $i \in \{1,\ldots,m\}$, $\mathcal{F}_n(f_i)$ is exact. Then, by [4, Theorem 4.10], for every $i \in \{1,\ldots,m\}$, f_i is exact. Thus, by Theorem 4.3, $\prod_{i=1}^m f_i$ is exact. Finally, by [4, Theorem 4.10], $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is exact.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is exact. By [4, Theorem 4.10], $\prod_{i=1}^m f_i$ is exact. Then, by Theorem 4.3, for each $i \in \{1,\ldots,m\}$, f_i is exact.

Finally, suppose that, for every $i \in \{1,\ldots,m\}$, f_i is exact. By Theorem 4.3, $\prod_{i=1}^m f_i$ is exact. Then, by [4, Theorem 4.10], $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is exact. □

By Theorems 6.2 and 4.5, we have the following result.

Theorem 6.10 *Let $X_1,\ldots,X_m$ be topological spaces, for each $i \in \{1,\ldots,m\}$ let $f_i : X_i \to X_i$ be a function, and let $n \in \mathbb{N}$. Then the following are equivalent:*

1. *For each $i \in \{1,\ldots,m\}$, f_i is exactly Devaney chaotic.*
2. *$\mathcal{F}_n(\prod_{i=1}^m f_i)$ is exactly Devaney chaotic.*
3. *For every $i \in \{1,\ldots,m\}$, $\mathcal{F}_n(f_i)$ is exactly Devaney chaotic.*

By [4, Theorem 4.8] and Theorem 4.4, we have the following result.

Theorem 6.11 *Let $X_1,\ldots,X_m$ be topological spaces, for each $i \in \{1,\ldots,m\}$ let $f_i : X_i \to X_i$ be a function and let $n \in \mathbb{N}$. Then the following are equivalent:*

1. *For each $i \in \{1,\ldots,m\}$, f_i is mixing.*
2. *$\mathcal{F}_n(\prod_{i=1}^m f_i)$ is mixing.*
3. *For every $i \in \{1,\ldots,m\}$, $\mathcal{F}_n(f_i)$ is mixing.*

Theorem 6.12 *Let $X_1,\ldots,,X_{m+1}$ be topological spaces, let $n \in \mathbb{N}$ and, for each $i \in \{1,\ldots,m+1\}$, let $f_i : X_i \to X_i$ be a function. If $\mathcal{F}_n(\prod_{i=1}^m f_i) \times f_{m+1}$ is transitive, then, for each $i \in \{1,\ldots,m\}$, $\mathcal{F}_n(f_i) \times f_{m+1}$ is transitive.*

Proof Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i) \times f_{m+1}$ is transitive. Let $i_0 \in \{1,\ldots,m\}$ and let $\mathcal{U}_1$, $\mathcal{U}_2$ be two nonempty open subsets of $\mathcal{F}_n(X_{i_0}) \times X_{m+1}$. Then there exist nonempty open subsets $\mathcal{U}$, $\mathcal{V}$ of $\mathcal{F}_n(X_{i_0})$ and F_1, F_2 of X_{m+1} such that $\mathcal{U} \times F_1 \subseteq \mathcal{U}_1$ and $\mathcal{V} \times F_2 \subseteq \mathcal{U}_2$. Thus, by [10, Lemma 4.2], there exist nonempty open subsets $U_1,\ldots,U_n$, $V_1,\ldots,V_n$ of X_{i_0} such that $\langle U_1,\ldots,U_n\rangle \subseteq \mathcal{U}$ and $\langle V_1,\ldots,V_n\rangle \subseteq \mathcal{V}$. For each $i \in \{1,\ldots,m\}\setminus\{i_0\}$ and, for every $j \in \{1,\ldots,n\}$, let $U_i^j = X_i$, $V_i^j = X_i$, $U_{i_0}^j = U_j$ and $V_{i_0}^j = V_j$. Finally, for all $j \in \{1,\ldots,n\}$, let $U_j^{'} = \prod_{i=1}^m U_i^j$ and let $V_j^{'} = \prod_{i=1}^m V_i^j$. It follows that, $\langle U_1^{'},\ldots,U_n^{'}\rangle$ and $\langle V_1^{'},\ldots,V_n^{'}\rangle$ are nonempty open subsets of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. By hypothesis, we have that, there exists $k \in \mathbb{N}$ such that $[\mathcal{F}_n(\prod_{i=1}^m f_i) \times f_{m+1}]^k(\langle U_1^{'},\ldots,U_n^{'}\rangle \times F_1) \cap (\langle V_1^{'},\ldots,V_n^{'}\rangle \times F_2) \neq \emptyset$. Thus, there exists $(\{(x_1^l,\ldots,x_m^l) : l \leq n\}, v_1) \in \langle U_1^{'},\ldots,U_n^{'}\rangle \times F_1$ such that $[\mathcal{F}_n(\prod_{i=1}^m f_i) \times f_{m+1}]^k((\{(x_1^l,\ldots,x_m^l) : l \leq n\} \times v_1)) \in \langle V_1^{'},\ldots,V_n^{'}\rangle \times F_2$. Then, by Lemma 5.2, $\{x_{i_0}^1,\ldots,x_{i_0}^l\} \in \langle U_1,\ldots,U_n\rangle$ and $\{f_{i_0}^k(x_{i_0}^1),\ldots,f_{i_0}^k(x_{i_0}^l)\} \in \langle V_1,\ldots,V_n\rangle$. Hence, we have that $(\{x_{i_0}^1,\ldots,x_{i_0}^l\}, v_1) \in \langle U_1,\ldots,U_n\rangle \times F_1$ and $[\mathcal{F}_n(f_{i_0}) \times f_{m+1}]^k((\{x_{i_0}^1,\ldots,x_{i_0}^l\}, v_1)) \in \langle V_1,\ldots,V_n\rangle \times F_2$. Therefore, $[\mathcal{F}_n(f_{i_0}) \times f_{m+1}]^k(\mathcal{U}_1) \cap (\mathcal{U}_2) \neq \emptyset$ and hence $\mathcal{F}_n(f_{i_0}) \times f_{m+1}$ is transitive. □

Theorem 6.13 *Let $X_1,\dots,X_m$ be topological spaces, for each $i\in\{1,\dots,m\}$ let $f_i:X_i\to X_i$ be a function, let $n\in\mathbb{N}$, and let $\mathcal{M}$ be one of the following classes of functions: transitive, weakly mixing, totally transitive, strongly transitive, chaotic, orbit-transitive, strictly orbit-transitive, ω-transitive, Touhey, an F-system, backward minimal, mild mixing, scattering or TT_{++}. If $\mathcal{F}_n(\prod_{i=1}^m f_i)\in\mathcal{M}$, then, for every $i\in\{1,\dots,m\}$, $\mathcal{F}_n(f_i)\in\mathcal{M}$.*

Proof Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is transitive. Let $i_0\in\{1,\dots,m\}$ and let $\mathcal{U}$, $\mathcal{V}$ be nonempty open subsets of $\mathcal{F}_n(X_{i_0})$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1,\dots,U_n$, $V_1,\dots,V_n$ of X_{i_0} such that $\langle U_1,\dots,U_n\rangle\subseteq\mathcal{U}$ and $\langle V_1,\dots,V_n\rangle\subseteq\mathcal{V}$. For each $i\in\{1,\dots,m\}\setminus\{i_0\}$ and for every $j\in\{1,\dots,n\}$, let $U_i^j=X_i$, $U_{i_0}^j=U_j$, $V_i^j=X_i$ and $V_{i_0}^j=V_j$. Finally, for all $j\in\{1,\dots,n\}$, let $U_j^{'}=\prod_{i=1}^m U_i^j$ and $V_j^{'}\prod_{i=1}^m V_i^j$. Note that $\langle U_1^{'},\dots,U_n^{'}\rangle$ and $\langle V_1^{'},\dots,V_n^{'}\rangle$ are nonempty open subsets of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. By hypothesis, there exists $k\in\mathbb{N}$ such that $\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\langle U_1^{'},\dots,U_n^{'}\rangle)\cap\langle V_1^{'},\dots,V_n^{'}\rangle\neq\emptyset$. Hence, there exists $\{(x_1^j,\dots,x_m^j):j\in\{1,\dots,r\}\}\in\langle U_1^{'},\dots,U_n^{'}\rangle$, with $r\le n$ such that

$$\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\{(x_1^j,\dots,x_m^j):j\in\{1,\dots,r\}\})\in\langle V_1^{'},\dots,V_n^{'}\rangle.$$

By Remark 3.1, parts (1) and (2), $\{(f_1^k(x_1^j),\dots,f_m^k(x_m^j)):j\in\{1,\dots,r\}\}\in\langle V_1^{'},\dots,V_n^{'}\rangle$. Consequently, by Lemma 5.2, $\{f_{i_0}^k(x_{i_0}^1),\dots,f_{i_0}^k(x_{i_0}^r)\}\in\langle V_1,\dots,V_n\rangle$. Which means that $(\mathcal{F}_n(f_{i_0}))^k(\{x_{i_0}^1,\dots,x_{i_0}^r\})\in\langle V_1,\dots,V_n\rangle$. On the other hand, $\{x_{i_0}^1,\dots,x_{i_0}^r\}\in\langle U_1,\dots,U_n\rangle$. Thus, $[\mathcal{F}_n(f_{i_0})]^k(\langle U_1,\dots,U_n\rangle)\cap\langle V_1,\dots,V_n\rangle\neq\emptyset$. Therefore, $\mathcal{F}_n(f_{i_0})$ is transitive.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is weakly mixing. Let $i_0\in\{1,\dots,m\}$ and let $\mathcal{U}_1,\mathcal{U}_2,\mathcal{V}_1$ and $\mathcal{V}_2$ be four nonempty open subsets of $\mathcal{F}_n(X_{i_0})$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1^1,\dots,U_n^1$, $U_1^2,\dots,U_n^2$, $V_1^1,\dots,V_n^1$, $V_1^2,\dots,V_n^2$ of X_{i_0} such that $\langle U_1^1,\dots,U_n^1\rangle\subseteq\mathcal{U}_1$, $\langle U_1^2,\dots,U_n^2\rangle\subseteq\mathcal{U}_2$, $\langle V_1^1,\dots,V_n^1\rangle\subseteq\mathcal{V}_1$ and $\langle V_1^2,\dots,V_n^2\rangle\subseteq\mathcal{V}_2$. For each $i\in\{1,\dots,m\}\setminus\{i_0\}$ and for every $j\in\{1,\dots,n\}$, let $W_i^j=X_i$, $T_i^j=X_i$, $F_i^j=X_i$, $L_i^j=X_i$, $W_{i_0}^j=U_j^1$, $T_{i_0}^j=U_j^2$, $F_{i_0}^j=V_j^1$ and $L_{i_0}^j=V_j^2$. Moreover, for all $j\in\{1,\dots,n\}$, let, $W_j=\prod_{i=1}^m W_i^j$, $T_j=\prod_{i=1}^m T_i^j$, $F_j=\prod_{i=1}^m F_i^j$ and $L_j=\prod_{i=1}^m L_i^j$. Then, $\langle W_1,\dots,W_n\rangle$, $\langle T_1,\dots,T_n\rangle$, $\langle F_1,\dots,F_n\rangle$ and $\langle L_1,\dots,L_n\rangle$ are nonempty open subsets of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. By hypothesis, $(\mathcal{F}_n(\prod_{i=1}^m f_i))^k(\langle W_1,\dots,W_n\rangle)\cap\langle F_1,\dots,F_n\rangle\neq\emptyset$ and $(\mathcal{F}_n(\prod_{i=1}^m f_i))^k(\langle T_1,\dots,T_n\rangle)\cap\langle L_1,\dots,L_n\rangle\neq\emptyset$. Thus, there exist $\{(x_1^j,\dots,x_m^j):j\in\{1,\dots,r\}\}\in\langle W_1,\dots,W_n\rangle$ and $\{(y_1^j,\dots,y_m^j):j\in\{1,\dots,p\}\}\in\langle T_1,\dots,T_n\rangle$ such that

$$\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\{(x_1^j,\dots,x_m^j):j\in\{1,\dots,r\}\})\in\langle F_1,\dots,F_n\rangle$$

and

$$\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\{(y_1^j,\dots,y_m^j):j\in\{1,\dots,p\}\})\in\langle L_1,\dots,L_n\rangle.$$

Thus, by Remark 3.1, parts (1) and (2), we have that $\{(f_1^k(x_1^j),\dots,f_m^k(x_m^j)):j\in\{1,\dots,r\}\}\in\langle F_1,\dots,F_n\rangle$ and $\{(f_1^k(y_1^j),\dots,f_m^k(y_m^j)):j\in\{1,\dots,p\}\}\in\langle L_1,\dots,L_n\rangle$. By Lemma 5.2, it follows that $\{f_{i_0}^k(x_{i_0}^1),\dots,f_{i_0}^k(x_{i_0}^r)\}\in$

$\langle V_1^1,\ldots,V_n^1\rangle$ and $\{f_{i_0}^k(y_{i_0}^1),\ldots,f_{i_0}^k(y_{i_0}^p)\}\in\langle V_1^2,\ldots,V_n^2\rangle$. Then $(\mathcal{F}_n(f_{i_0}))^k(\{x_{i_0}^1,\ldots,x_{i_0}^r\})\in\langle V_1^1,\ldots,V_n^1\rangle$ and $(\mathcal{F}_n(f_{i_0}))^k(\{y_{i_0}^1,\ldots,y_{i_0}^p\})\in\langle V_1^2,\ldots,V_n^2\rangle$. Moreover, $\{x_{i_0}^1,\ldots,x_{i_0}^r\}\in\langle U_1^1,\ldots,U_n^1\rangle$ and $\{y_{i_0}^1,\ldots,y_{i_0}^p\}\in\langle U_1^2,\ldots,U_n^2\rangle$. Hence, we have that $(\mathcal{F}_n(f_{i_0}))^k(\langle U_1^1,\ldots,U_n^1\rangle)\cap\langle V_1^1,\ldots,V_n^1\rangle\neq\emptyset$ and $(\mathcal{F}_n(f_{i_0}))^k(\langle U_1^2,\ldots,U_n^2\rangle)\cap\langle V_1^2,\ldots,V_n^2\rangle\neq\emptyset$. It follows that, for each $i\in\{1,2\}$, $(\mathcal{F}_n(f_{i_0}))^k(\mathcal{U}_i)\cap\mathcal{V}_i\neq\emptyset$. Finally, $\mathcal{F}_n(f_{i_0})$ is weakly mixing.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is totally transitive. Let $i_0\in\{1,\ldots,m\}$, let $s\in\mathbb{N}$, and let $\mathcal{U},\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X_{i_0})$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1,\ldots,U_n,V_1,\ldots,V_n$ of X_{i_0} such that $\langle U_1,\ldots,U_n\rangle\subseteq\mathcal{U}$ and $\langle V_1,\ldots,V_n\rangle\subseteq\mathcal{V}$. For each $i\in\{1,\ldots,m\}\setminus\{i_0\}$ and for every $j\in\{1,\ldots,n\}$, let $U_i^j=X_i$, $V_i^j=X_i$, $U_{i_0}^j=U_j$ and $V_{i_0}^j=V_j$. Moreover, for all $j\in\{1,\ldots,n\}$, let $U_j^{'}=\prod_{i=1}^m U_i^j$ and $V_j^{'}=\prod_{i=1}^m V_i^j$. It follows that $\langle U_1^{'},\ldots,U_n^{'}\rangle$ and $\langle V_1^{'},\ldots,V_n^{'}\rangle$ are nonempty open subsets of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. Then, since $(\mathcal{F}_n(\prod_{i=1}^m f_i))^s$ is transitive, we have that, there exists $k\in\mathbb{N}$ such that $([\mathcal{F}_n(\prod_{i=1}^m f_i)]^s)^k(\langle U_1^{'},\ldots,U_n^{'}\rangle)\cap\langle V_1^{'},\ldots,V_n^{'}\rangle\neq\emptyset$. Thus, there exists $\{(x_1^j,\ldots,x_m^j):j\in\{1,\ldots,l\}\}\in\langle U_1^{'},\ldots,U_n^{'}\rangle$ such that $\left[\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right]^{sk}(\{(x_1^j,\ldots,x_m^j):j\in\{1,\ldots,l\}\})\in\langle V_1^{'},\ldots,V_n^{'}\rangle$. In consequence, $\{(f_1^{sk}(x_1^j),\ldots,f_m^{sk}(x_m^j)):j\in\{1,\ldots,l\}\}\in\langle V_1^{'},\ldots,V_n^{'}\rangle$. Then, by Lemma 5.2, $\{f_{i_0}^{sk}(x_{i_0}^1),\ldots,f_{i_0}^{sk}(x_{i_0}^l)\}\in\langle V_1,\ldots,V_n\rangle$. Hence, $([\mathcal{F}_n(f_{i_0})]^s)^k(\{x_{i_0}^1,\ldots,x_{i_0}^l\})\in\langle V_1,\ldots,V_n\rangle$. Meanwhile, by Lemma 5.2, $\{x_{i_0}^1,\ldots,x_{i_0}^l\}\in\langle U_1,\ldots,U_n\rangle$. It follows that $([\mathcal{F}_n(f_{i_0})]^s)^k(\langle U_1,\ldots,U_n\rangle)\cap\langle V_1,\ldots,V_n\rangle\neq\emptyset$. Consequently, $([\mathcal{F}_n(f_{i_0})]^s)^k(\mathcal{U})\cap\mathcal{V}\neq\emptyset$. Therefore, $[\mathcal{F}_n(f_{i_0})]^s$ is transitive. Since $s\in\mathbb{N}$ is arbitrary, we have that $\mathcal{F}_n(f_{i_0})$ is totally transitive.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is strongly transitive. Let $i_0\in\{1,\ldots,m\}$ and let $\mathcal{U}$ be a nonempty open subset of $\mathcal{F}_n(X_{i_0})$. Thence, by [10, Lemma 4.2], there exist nonempty open subsets $U_1,\ldots,U_n$ of X_{i_0} such that $\langle U_1,\ldots,U_n\rangle\subseteq\mathcal{U}$. For each $i\in\{1,\ldots,m\}\setminus\{i_0\}$ and for every $j\in\{1,\ldots,n\}$, let $U_i^j=X_i$ and $U_{i_0}^j=U_j$. Moreover, for all $j\in\{1,\ldots,n\}$, let $U_j^{'}=\prod_{i=1}^m U_i^j$. Note that $\langle U_1^{'},\ldots,U_n^{'}\rangle$ is a nonempty open subset of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. By hypothesis, there exists $s\in\mathbb{N}$ such that $\mathcal{F}_n\left(\prod_{i=1}^m X_i\right)=\bigcup_{k=0}^s\left[\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right]^k(\langle U_1^{'},\ldots,U_n^{'}\rangle)$. Let $\{x_1,\ldots,x_r\}\in\mathcal{F}_n(X_{i_0})$. For each $i\in\{1,\ldots,m\}\setminus\{i_0\}$ and for every $j\in\{1,\ldots,r\}$, let $a_i^j\in X_i$ and let $a_{i_0}^j=x_j$. Then, for all $j\in\{1,\ldots,r\}$, let $x_j^{'}=(a_1^j,\ldots,a_m^j)$. Note that $\{x_1^{'},\ldots,x_r^{'}\}\in\mathcal{F}_n(\prod_{i=1}^m X_i)$. Thus, there exists $k\in\{0,\ldots,s\}$ such that $\{x_1^{'},\ldots,x_r^{'}\}\in[\mathcal{F}_n(\prod_{i=1}^m f_i)]^k(\langle U_1^{'},\ldots,U_n^{'}\rangle)$. Then, there exists $\{(y_1^j,\ldots,y_m^j):j\in\{1,\ldots,p\}\}\in\langle U_1^{'},\ldots,U_n^{'}\rangle$ such that $\left[\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right]^k(\{(y_1^j,\ldots,y_m^j):j\in\{1,\ldots,p\}\})=\{x_1^{'},\ldots,x_r^{'}\}$. By Remark 3.1, we have that $\{(f_1^k(y_1^j),\ldots,f_m^k(y_m^j)):j\in\{1,\ldots,p\}\}=\{x_1^{'},\ldots,x_r^{'}\}$. Thus, $[\mathcal{F}_n(f_{i_0})]^k(\{y_{i_0}^1,\ldots,y_{i_0}^p\})=\{x_1,\ldots,x_r\}$. On the other hand, by Lemma 5.2, $\{y_{i_0}^1,\ldots,y_{i_0}^p\}\in\langle U_1,\ldots,U_n\rangle$. Hence, $\{x_1,\ldots,x_r\}\in[\mathcal{F}_n(f_{i_0})]^k(\langle U_1,\ldots,U_n\rangle)$. Therefore, $\{x_1,\ldots,x_r\}\in\bigcup_{k=0}^s[\mathcal{F}_n(f_{i_0})]^k(\mathcal{U})$ and $\mathcal{F}_n(f_{i_0})$ is strongly transitive.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is chaotic. Then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is transitive and $Per(\mathcal{F}_n(\prod_{i=1}^m f_i))$ is dense in $\mathcal{F}_n(\prod_{i=1}^m X_i)$. Thus, for each $i\in\{1,\ldots,m\}$, $\mathcal{F}_n(f_i)$ is transitive and by Theorem 5.8, part (2), for every $i\in\{1,\ldots,m\}$, $Per(\mathcal{F}_n(f_i))$ is dense in $\mathcal{F}_n(X_i)$. Therefore, for all $i\in\{1,\ldots,m\}$, $\mathcal{F}_n(f_i)$ is chaotic.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is orbit-transitive. Then, there exists a transitive point $\{(x_1^j,\ldots,x_m^j):j\in\{1,\ldots,l\}\}$ of $\mathcal{F}_n(\prod_{i=1}^m f_i)$. Thus, by Theorem 5.6, we have that, for each $i\in\{1,\ldots,m\}$, $\{x_i^1,\ldots,x_i^l\}$ is a transitive point of $\mathcal{F}_n(f_i)$. Consequently, for every $i\in\{1,\ldots,m\}$, $\mathcal{O}(\{x_i^1,\ldots,x_i^l\},\mathcal{F}_n(f_i))$ is a dense subset in

$\mathcal{F}_n(X_i)$. Which implies that, for all $i \in \{1,\dots,m\}$, $\mathcal{F}_n(f_i)$ is orbit-transitive.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is strictly orbit-transitive. It follows that, there exists a transitive point $\{(f_1(x_1^j),\dots,f_m(x_m^j)) : j \in \{1,\dots,l\}\}$ of $\mathcal{F}_n(\prod_{i=1}^m f_i)$. By Theorem 5.6, for each $i \in \{1,\dots,m\}$, we have that $\{f_i(x_i^1),\dots,f_i(x_i^l)\}$ is a transitive point of $\mathcal{F}_n(f_i)$. Thus, for every $i \in \{1,\dots,m\}$, $\mathcal{F}_n(f_i)(\{x_i^1,\dots,x_i^l\})$ is a transitive point of $\mathcal{F}_n(f_i)$. Hence, for all $i \in \{1,\dots,m\}$, the subset $\mathcal{O}(\mathcal{F}_n(f_i)(\{x_i^1,\dots,x_i^l\}),\mathcal{F}_n(f_i))$ is dense in $\mathcal{F}_n(X_i)$. Therefore, for each $i \in \{1,\dots,m\}$, $\mathcal{F}_n(f_i)$ is strictly orbit-transitive.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is ω-transitive. By hypothesis, there exists $\{(x_1^j,\dots,x_m^j) : j \in \{1,\dots,l\}\} \in \mathcal{F}_n(\prod_{i=1}^m X_i)$ such that $\omega(\{(x_1^j,\dots,x_m^j) : j \in \{1,\dots,l\}\},\mathcal{F}_n(\prod_{i=1}^m f_i)) = \mathcal{F}_n(\prod_{i=1}^m X_i)$. Then, by Theorem 5.7, for each $i \in \{1,\dots,m\}$, $\omega(\{x_i^1,\dots,x_i^l\},\mathcal{F}_n(f_i)) = \mathcal{F}_n(X_i)$, which means that for every $i \in \{1,\dots,m\}$, $\mathcal{F}_n(f_i)$ is ω-transitive.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is TT_{++}. Let $i_0 \in \{1,\dots,m\}$ and let $\mathcal{U}$, $\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X_{i_0})$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1,\dots,U_n,V_1,\dots,V_n$ of X_{i_0} such that $\langle U_1,\dots,U_n\rangle_n \subseteq \mathcal{U}$ and $\langle V_1,\dots,V_n\rangle_n \subseteq \mathcal{V}$. For each $i \in \{1,\dots,m\}\backslash\{i_0\}$ and for every $j \in \{1,\dots,n\}$, let $U_i^j = X_i$, $V_i^j = X_i$, $U_{i_0}^j = U_j$ and $V_{i_0}^j = V_j$. Moreover, for all $j \in \{1,\dots,n\}$, let $U_j^{'} = \prod_{i=1}^m U_i^j$ and $V_j^{'} = \prod_{i=1}^m V_i^j$. Note that $\langle U_1^{'},\dots,U_n^{'}\rangle$ and $\langle V_1^{'},\dots,V_n^{'}\rangle$ are nonempty open subsets of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. By hypothesis, $n_{\mathcal{F}_n(\prod_{i=1}^m f_i)}(\langle U_1^{'},\dots,U_n^{'}\rangle,\langle V_1^{'},\dots,V_n^{'}\rangle)$ is infinite. On the other hand, by Lemma 5.3, we have that

$$n_{\mathcal{F}_n(\prod_{i=1}^m f_i)}(\langle U_1^{'},\dots,U_n^{'}\rangle,\langle V_1^{'},\dots,V_n^{'}\rangle) \subseteq n_{\mathcal{F}_n(f_{i_0})}(\langle U_1,\dots,U_n\rangle,\langle V_1,\dots,V_n\rangle).$$

Consequently, $n_{\mathcal{F}_n(f_{i_0})}(\langle U_1,\dots,U_n\rangle,\langle V_1,\dots,V_n\rangle)$ is infinite. Therefore, $\mathcal{F}_n(f_{i_0})$ is TT_{++}.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is Touhey. Let $i_0 \in \{1,\dots,m\}$ and let $\mathcal{U},\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X_{i_0})$. Thence, by [10, Lemma 4.2], there exist nonempty open subsets $U_1,\dots,U_n$, $V_1,\dots,V_n$ of X_{i_0} such that $\langle U_1,\dots,U_n\rangle \subseteq \mathcal{U}$ and $\langle V_1,\dots,V_n\rangle \subseteq \mathcal{V}$. For each $i \in \{1,\dots,m\}\backslash\{i_0\}$ and for every $j \in \{1,\dots,n\}$, let $U_i^j = X_i$, $V_i^j = X_i$, $U_{i_0}^j = U_j$ and $V_{i_0}^j = V_j$. Finally, for all $j \in \{1,\dots,n\}$, let $U_j^{'} = \prod_{i=1}^m U_i^j$ and $V_j^{'} = \prod_{i=1}^m V_i^j$. It follows that $\langle U_1^{'},\dots,U_n^{'}\rangle$ and $\langle V_1^{'},\dots,V_n^{'}\rangle$ are nonempty open subsets of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. Since $\mathcal{F}_n(\prod_{i=1}^m F_i)$ is Touhey, there exist a periodic point $\{(x_1^l,\dots,x_m^l) : r \leq n \text{ and } l \in \{1,\dots,r\}\} \in \langle U_1^{'},\dots,U_n^{'}\rangle$ and $k \in \mathbb{Z}_+$ such that $[\mathcal{F}_n(\prod_{i=1}^m f_i)]^k(\{(x_1^l,\dots,x_m^l) : r \leq n \text{ and } l \in \{1,\dots,r\}\}) \in \langle V_1,\dots,V_n\rangle$. By Remark 3.1, part (2), $\{(f_1^k(x_1^l),\dots f_m^k(x_m^l)) : r \leq n \text{ and } l \in \{1,\dots,r\}\} \in \langle V_1^{'},\dots,V_n^{'}\rangle$. Then, by Lemma 5.2, $\{f_{i_0}^k(x_{i_0}^1),\dots,f_{i_0}^k(x_{i_0}^r)\} \in \langle V_1,\dots,V_n\rangle \subseteq \mathcal{V}$. Thus, $[\mathcal{F}_n(f_{i_0})]^k(\{x_{i_0}^{'},\dots,x_{i_0}^r\}) \in \mathcal{V}$. On the other hand, since $\{(x_1^l,\dots,x_m^l) : r \leq n \text{ and } l \in \{1,\dots,r\}\} \in \langle U_1^{'},\dots,U_n^{'}\rangle$, by Lemma 5.2, $\{x_{i_0}^1,\dots,x_{i_0}^r\} \in \langle U_1,\dots,U_n\rangle$. Moreover, since $\{(x_1^l,\dots,x_m^l) : r \leq n \text{ and } l \in \{1,\dots,r\}\}$ is a periodic point of $\mathcal{F}_n(\prod_{i=1}^m f_i)$, by [4, Theorem 3.4], for each $l \in \{1,\dots,r\}$, $(x_1^l,\dots,x_m^l)$ is a periodic point of $\prod_{i=1}^m f_i$. Then, by Theorem 3.3, part (4), for each $l \in \{1,\dots,r\}$, $x_{i_0}^l$ is a periodic point of f_{i_0}. Thus, by [4, Theorem 3.4], $\{x_{i_0}^1,\dots,x_{i_0}^r\}$ is a periodic point of $\mathcal{F}_n(f_{i_0})$. Therefore, $\mathcal{F}_n(f_{i_0})$ is Touhey.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is an F-system. Then, $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is totally transitive and the subset $Per(\mathcal{F}_n(\prod_{i=1}^m f_i))$ is dense in $\mathcal{F}_n(\prod_{i=1}^m X_i)$. By [4, Theorem 3.4], $Per(\prod_{i=1}^m f_i)$ is dense in $\prod_{i=1}^m X_i$. In consequence, by Theorem 3.15, for each $i \in \{1,\dots,m\}$, $Per(f_i)$ is dense in X_i. Again, by [4, Theorem 3.4], for every $i \in \{1,\dots,m\}$, $Per(\mathcal{F}_n(f_i))$ is dense in $\mathcal{F}_n(X_i)$. On the other hand, by the third paragraph of this proof, we have that, for all $i \in \{1,\ldots,m\}$, $\mathcal{F}_n(f_i)$ is totally transitive. Therefore, for each $i \in \{1,\ldots,m\}$, $\mathcal{F}_n(f_i)$ is an F-system.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is backward minimal. Let $i_0 \in \{1,\ldots,m\}$ and let $\{x_1,\ldots,x_r\} \in \mathcal{F}_n(X_{i_0})$. For each $i \in \{1,\ldots,m\}\setminus\{i_0\}$ and for every $j \in \{1,\ldots,r\}$, let $y_i^j \in X_i$ and let $y_{i_0}^j = x_j$. Hence, $\{(y_1^j,\ldots,y_m^j) : j \in \{1,\ldots,r\}\} \in \mathcal{F}_n(\prod_{i=1}^m X_i)$. Since $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is backward minimal, the set $\{\mathcal{A} \in \mathcal{F}_n(\prod_{i=1}^m X_i) : [\mathcal{F}_n(\prod_{i=1}^m f_i)]^l(\mathcal{A}) = \{(y_1^j,\ldots,y_m^j) : j \in \{1,\ldots,r\}\}$, for some $l \in \mathbb{N}\}$, is dense in $\mathcal{F}_n(\prod_{i=1}^m X_i)$. Let $\mathcal{U}$ be a nonempty open subset of $\mathcal{F}_n(X_{i_0})$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1,\ldots,U_n$ of X_{i_0} such that $\langle U_1,\ldots,U_n\rangle \subseteq \mathcal{U}$. For each $i \in \{1,\ldots,m\}\setminus\{i_0\}$ and for every $j \in \{1,\ldots,n\}$, let $U_i^j = X_i$ and $U_{i_0}^j = U_j$. Finally, for all $j \in \{1,\ldots,n\}$, let $U_j' = \prod_{i=1}^m U_i^j$. Thus, $\langle U_1',\ldots,U_n'\rangle$ is a nonempty open subset of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. By hypothesis, there exist $\{(z_1^j,\ldots,z_m^j) : p \geq n \text{ and } j \in \{1,\ldots,p\}\} \in \langle U_1',\ldots,U_n'\rangle$ and $l \in \mathbb{N}$ such that $[\mathcal{F}_n(\prod_{i=1}^m f_i)]^l(\{(z_1^j,\ldots,z_m^j) : p \geq n \text{ and } j \in \{1,\ldots,p\}\}) = \{(y_1^j,\ldots,y_m^j) : j \in \{1,\ldots,r\}\}$. Meanwhile, by Lemma 5.2, $\{z_{i_0}^1,\ldots,z_{i_0}^p\} \in \langle U_1,\ldots,U_n\rangle$. Moreover, by Remark 3.1, parts (1) and (2), $\{(f_1^l(z_1^j),\ldots,f_m^l(z_m^j)) : p \geq n \text{ and } j \in \{1,\ldots,p\}\} = \{(y_1^j,\ldots,y_m^j) : j \in \{1,\ldots,r\}\}$. It follows that $\{f_{i_0}^l(z_{i_0}^1),\ldots,f_{i_0}^l(z_{i_0}^p)\} = \{y_{i_0}^1,\ldots,y_{i_0}^r\}$. Consequently, $[\mathcal{F}_n(f_{i_0})]^l(\{z_{i_0}^1,\ldots,z_{i_0}^p\}) = \{y_{i_0}^1,\ldots,y_{i_0}^r\}$. Therefore, the set $\{A \in \mathcal{F}_n(X_{i_0}) : [\mathcal{F}_n(f_i)]^l(A) = \{x_1,\ldots,x_r\}$, for some $l \in \mathbb{N}\}$ is dense in $\mathcal{F}_n(X_{i_0})$ and $\mathcal{F}_n(f_{i_0})$ is backward minimal.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is mild mixing. Let $i_0 \in \{1,\ldots,m\}$, let Y be a topological space and let $g : Y \to Y$ be a transitive function. By hypothesis, $\mathcal{F}_n(\prod_{i=1}^m f_i) \times g$ is transitive. Thus, by Theorem 6.12, $\mathcal{F}_n(f_{i_0}) \times g$ is transitive. Therefore, $\mathcal{F}_n(f_{i_0})$ is mild mixing.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is scattering. Let $i_0 \in \{1,\ldots,m\}$, let Y be a topological space and let $g : Y \to Y$ be a minimal function. By hypothesis, $\mathcal{F}_n(\prod_{i=1}^m f_i) \times g$ is transitive. Thus, by Theorem 6.12, $\mathcal{F}_n(f_{i_0}) \times g$ is transitive. Therefore, $\mathcal{F}_n(f_{i_0})$ is scattering. □

Theorem 6.14 *Let $X_1,\ldots,X_m$ be topological spaces, for each $i \in \{1,\ldots,m\}$, let $f_i : X_i \to X_i$ be a continuous function, and let $n \in \mathbb{N}$. If $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal, then, for every $i \in \{1,\ldots,m\}$, $\mathcal{F}_n(f_i)$ is minimal.*

Proof Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal. Let $i_0 \in \{1,\ldots,m\}$. By hypothesis, f_{i_0} is continuous. Hence, $\mathcal{F}_n(f_{i_0})$ is continuous. Thus, by [15, Proposition 6.2], it is sufficient to prove that for each $A \in \mathcal{F}_n(X_{i_0})$, $\mathrm{cl}_{\mathcal{F}_n(X_{i_0})}(\mathcal{O}(A,\mathcal{F}_n(f_{i_0}))) = \mathcal{F}_n(X_{i_0})$. Let $\{x_1,\ldots,x_r\} \in \mathcal{F}_n(X_{i_0})$ with $r \leq n$. For each $i \in \{1,\ldots,m\}\setminus\{i_0\}$ and for every $j \in \{1,\ldots,r\}$, let $y_i^j \in X_i$ and $y_{i_0}^j = x_j$. Thus, $\{(y_1^j,\ldots,y_m^j) : j \in \{1,\ldots,r\}\} \in \mathcal{F}_n(\prod_{i=1}^m X_i)$. Since $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal. We have that $\mathrm{cl}_{\mathcal{F}_n(\prod_{i=1}^m X_i)}\left(\mathcal{O}\left(\{(y_1^j,\ldots,y_m^j) : j \in \{1,\ldots,r\}\},\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)\right) = \mathcal{F}_n\left(\prod_{i=1}^m X_i\right)$. Thus, by Theorem 5.6, for all $i \in \{1,\ldots,m\}$ we have that, $\mathrm{cl}_{\mathcal{F}_n(X_i)}(\mathcal{O}(\{y_i^1,\ldots,y_i^r\},\mathcal{F}_n(f_i))) = \mathcal{F}_n(X_i)$. In particular, we have that, $\mathrm{cl}_{\mathcal{F}_n(X_{i_0})}(\mathcal{O}(\{y_{i_0}^1,\ldots,y_{i_0}^r\},\mathcal{F}_n(f_{i_0}))) = \mathcal{F}_n(X_{i_0})$. Consequently

$$\mathrm{cl}_{\mathcal{F}_n(X_{i_0})}(\mathcal{O}(\{x_1,\ldots,x_r\},\mathcal{F}_n(f_{i_0}))) = \mathcal{F}_n(X_{i_0}).$$

Since $\{x_1,\ldots,x_r\} \in \mathcal{F}_n(X_{i_0})$ is arbitrary, $\mathcal{F}_n(f_{i_0})$ is minimal. □

Theorem 6.15 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function, and let $n \in \mathbb{N}$. If $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is totally minimal, then, for every $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is totally minimal.*

Proof Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is totally minimal. Let $s \in \mathbb{N}$. By hypothesis, $[\mathcal{F}_n(\prod_{i=1}^m f_i)]^s$ is minimal. Then, by Remark 3.1, part (2), $\mathcal{F}_n(\prod_{i=1}^m f_i^s)$ is minimal. Then, by Theorem 6.14, for each $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i^s)$ is minimal. Again, by Remark 3.1, part (2), for every $i \in \{1, \ldots, m\}$, $[\mathcal{F}_n(f_i)]^s$ is minimal. Since $s \in \mathbb{N}$ is arbitrary, we have that, for all $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is totally minimal. □

By [4, Theorems 4.11, 4.12, 4.14, 4.15, 4.19, 5.1, 5.3, 5.6, 5.9], Theorem 6.3, and Theorem 6.13, we have the following result.

Theorem 6.16 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, let $n \in \mathbb{N}$, and let $\mathcal{M}$ be one of the following classes of functions: transitive, weakly mixing, totally transitive, strongly transitive, chaotic, orbit-transitive, strictly orbit-transitive, ω-transitive, TT_{++}, Touhey, an F-system, backward minimal, mild mixing or scattering. If $\mathcal{F}_n(\prod_{i=1}^m f_i) \in \mathcal{M}$, then, for every $i \in \{1, \ldots, m\}$, $f_i \in \mathcal{M}$.*

The converse of Theorem 6.16 is not true in general. Let us see a partly example of this in the following:

Example 6.17 *Let $f : [0, 2] \to [0, 2]$ be a function given by:*

$$f(x) = \begin{cases} 2x + 1, & 0 \le x \le \frac{1}{2}, \\ -2x + 3, & \frac{1}{2} \le x \le 1, \\ -x + 2, & 1 \le x \le 2. \end{cases}$$

In [8, Example 1], it is shown that f is transitive; however, $f \times f : [0, 2] \times [0, 2] \to [0, 2] \times [0, 2]$ is not transitive. If we suppose that $\mathcal{F}_n(f \times f)$ is transitive, by [4, Theorem 4.11], we have that $f \times f$ is transitive. Which is a contradiction. Therefore, $\mathcal{F}_n(f \times f)$ is not transitive.

By Theorems 6.3, 6.14, 6.15, and [4, Theorem 4.18], we have the following result.

Theorem 6.18 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function, and let $n \in \mathbb{N}$. Then the following hold:*

1. *If $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal, then, for every $i \in \{1, \ldots, m\}$, f_i is minimal.*

2. *If $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is totally minimal, then, for all $i \in \{1, \ldots, m\}$, f_i is totally minimal.*

By Theorems 3.14, 4.10, 6.6, and [4, Theorems 5.2, 5.4, 5.7], we obtain the following result.

Theorem 6.19 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, let $n \in \mathbb{N}$, and let $\mathcal{M}$ be one of the following classes of functions: transitive, totally transitive, chaotic, orbit-transitive, strictly orbit-transitive, ω-transitive, Touhey, an F-system, mild mixing or scattering. If for every $i \in \{1, \ldots, m\}$, X_i is $+$invariant over open subsets under f_i and $f_i \in \mathcal{M}$, then $\mathcal{F}_n(\prod_{i=1}^m f_i) \in \mathcal{M}$.*

Corollary 6.20 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, let $n \in \mathbb{N}$, and let $\mathcal{M}$ be one of the following classes of functions: transitive, totally transitive, chaotic, orbit-transitive, strictly orbit-transitive, ω-transitive, Touhey, an F-system, mild mixing, scattering or TT_{++}. If for every $i \in \{1, \ldots, m\}$, X_i is $+$invariant over open subsets under f_i and $\mathcal{F}_n(f_i) \in \mathcal{M}$, then $\mathcal{F}_n(\prod_{i=1}^m f_i) \in \mathcal{M}$.*

Theorem 6.21 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, and let $n \in \mathbb{N}$. If for every $i \in \{1, \ldots, m\}$, f_i is weakly mixing and continuous and X_i is $+$invariant over open subsets under f_i, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is weakly mixing.*

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, f_i is weakly mixing and continuous and that X_i is +invariant over open subsets under f_i. Then, by Theorem 4.10, $\prod_{i=1}^m f_i$ is weakly mixing. Even more, $\prod_{i=1}^m f_i$ is continuous. Thus, by [4, Theorem 4.13], we have that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is weakly mixing. □

Corollary 6.22 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function such that $\prod_{i=1}^m f_i$ is continuous, and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is weakly mixing and X_i is $+$invariant over open subsets under f_i, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is weakly mixing.*

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is weakly mixing, and that X_i is $+$invariant over open subsets under f_i and $\prod_{i=1}^m f_i$ is continuous. Then, by [4, Theorem 4.12], for each $i \in \{1, \ldots, m\}$, f_i is weakly mixing. Even more, for each $i \in \{1, \ldots, m\}$, f_i is continuous. Thus, by Theorem 6.21, $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is weakly mixing. □

Theorem 6.23 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function, and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, f_i is minimal and X_i is $+$invariant over open subsets under f_i, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal.*

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, f_i is minimal and that X_i is $+$invariant over open subsets under f_i. Then, by Proposition 4.11, $\prod_{i=1}^m f_i$ is minimal. Even more, $\prod_{i=1}^m f_i$ is continuous and by Theorem 3.14, $\prod_{i=1}^m X_i$ is +invariant over open subsets under $\prod_{i=1}^m f_i$. Thus, by Theorem 6.7, $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal. □

As a consequence of Theorem 6.23 and [4, Theorem 4.18], we have the following result.

Corollary 6.24 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function, and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is minimal and X_i is $+$invariant over open subsets under f_i, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal.*

As a consequence of Corollary 4.12, Theorem 3.14, and Proposition 6.8, we obtain the following.

Corollary 6.25 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function, and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, f_i is totally minimal and X_i is $+$invariant over open subsets under f_i, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is totally minimal.*

As a consequence of Theorem 6.3 and Corollary 6.25, we have:

Corollary 6.26 *Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is totally minimal and X_i is $+$invariant over open subsets under f_i, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is totally minimal.*

Acknowledgment

We are very grateful to the referee for their careful reading and valuable suggestions to the improvement of this paper.

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