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On connected tetravalent normal edge-transitive Cayley graphs of non-Abelian groups of order $5p^2$

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Abstract: Our aim in this paper is to investigate graph automorphism and group automorphism determining all connected tetravalent normal edge transitive Cayley graphs on non-Abelian groups of order $5p^2$ with respect to tetravalent sets and same-order elements, where p is a prime number and its Sylow p-subgroup is cyclic.

Key words: Cayley graph, normal edge-transitive, arc-transitive, graph automorphism, group automorphism

1. Introduction

There is an old definition of a graph attributed to Arthur Cayley, which is related to a group G, and a subset S of G, not including the identity element 1. The Cayley graph $Cay(G,S)$ is the graph with the vertex set $V(Cay(G,S)) = G$ and the edge set $E(Cay(G,S)) = \{(u,v)|vu^{-1} \in S\}$. We notice that the edge set can be identified with set of ordered pairs $\{(g,sg)|g \in G, s \in S\}$. If S is closed under taking inverse, $S = S^{-1}$, then $Cay(G,S)$ is an undirected graph. Obviously, the degree of each vertex is $|S|$ and $Cay(G,S)$ is connected if and only if $G = \langle S \rangle$.

A graph Γ is called vertex-transitive or edge-transitive if the automorphism group $\mathbb{A}ut(\Gamma)$, acts transitively on vertex-set or edge-set of Γ, respectively. Now, let $\Gamma = Cay(G,S)$.

For $g \in G$, let $\rho_g : G \to G$ given by $\rho_g(x) = xg$. By definition of Cayley graph, clearly $\rho_g \in \mathbb{A}ut(\Gamma)$. The set $\rho(G) = \{\rho_g \mid g \in G\}$ forms a subgroup (isomorphic to G) of $\mathbb{A}ut(\Gamma)$. In this way, Γ is vertex-transitive because $\rho(G) \leqslant \mathbb{A}ut(\Gamma)$, acting right regularly on the vertices of Γ, while Γ is not edge-transitive in general.

We employ the following notation and terminology. The notation $G = K \rtimes H$ is used to indicate that G is a semidirect product of K by H. We denote by $\mathbb{A}ut(G,S)$, the subgroup of $\mathbb{A}ut(G)$ consisting of all $\sigma \in \mathbb{A}ut(G)$ such that $\sigma(S) = S$. It is easy to see that $\mathbb{A}ut(G,S)$ is a subgroup of the automorphisms group of $Cay(G,S)$. $\mathbb{Z}_n$ denotes a cyclic group of order n, and $\mathbb{S}_4$ denotes the symmetric group on four letters. The distance of length i, between two vertices u and v, in a graph Γ is the number of i edges in a shortest path connecting them, denoted by $d(u,v) = i$. Let us introduce the $D_i(u) = \{v \in V(\Gamma) \mid d(u,v) = i\}$.

As far as the authors know, the concepts normal and normal edge transitive introduced by M.Y. Xu and C.E. Praeger in [9] and [7], respectively for the first time. A Cayley graph $\Gamma = Cay(G,S)$ is called normal if $\rho(G)$ is a normal subgroup of $\mathbb{A}ut(\Gamma)$, i.e. $N_{\mathbb{A}ut(\Gamma)}(\rho(G)) = \mathbb{A}ut(\Gamma)$; and Γ is called normal edge-transitive if

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$N_{\mathbb{A}ut(\Gamma)}(\rho(G))$ is transitive on the edges of Γ. These concepts play an important role in the theory of Cayley graphs.

Normal edge-transitive Cayley graphs on non-Abelian groups of order p^2, $3p^2$, $4p^2$ and modular groups of order $8n$, where p is prime and n is a natural number, were studied in [2, 5, 6, 8]. In this paper, motivated by [2, 6], we determine the structure of Cayley graphs on non-Abelian groups of orders $5p^2$ with cyclic Sylow p-subgroup with respect to tetravalent sets with same-order elements, where p is a prime number. Some results of normal edge-transitive Cayley graphs of $PGL(2,p)$, p prime, and Frobenius groups are given in [4] and [1], respectively.

We collect the main results in this paper into one theorem.

Theorem 1.1 (Main Theorem) *Let G be a finite group of order $5p^2$ with cyclic Sylow p-subgroup. There exists exactly three tetravalent subsets S_i of G, $1 \leqslant i \leqslant 3$, such that for each i, $G = \langle S_i \rangle$, all elements of S_i are of order 5 and one of the following holds.*

(1) *$S_1 = \{x, xy, x^{-1}, (xy)^{-1}\}$, and each element of S_1 has order 5; $\Gamma = Cay(G, S_1)$ is normal, normal edge-transitive and edge-transitive, but it is not arc-transitive. $\mathbb{A}ut(G, S_1) \cong \mathbb{Z}_2$ and $\mathbb{A}ut(\Gamma) \cong \rho(G) \rtimes \mathbb{Z}_2$.*

(2) *$S_2 = \{x^2, xy, x^{-2}, (xy)^{-1}\}$, and each element of S_2 has order 5; $\Gamma = Cay(G, S_2)$ is normal but it is not normal edge-transitive and arc-transitive. $\mathbb{A}ut(G, S_2)$ is trivial and $\mathbb{A}ut(\Gamma) \cong \rho(G)$.*

(3) *$S_3 = \{x^2, x^2y, x^{-2}, (x^2y)^{-1}\}$, and each element of S_3 has order 5; $\Gamma = Cay(G, S_3)$ is normal, normal edge transitive and edge transitive, but it is not arc-transitive; $\mathbb{A}ut(G, S_1) \cong \mathbb{Z}_2$ and $\mathbb{A}ut(\Gamma) \cong \rho(G) \rtimes \mathbb{Z}_2$.*

The tetravalent nonnormal Cayley graphs of order $5p^2$ is studied in Khazaei and Sharifi.*

2. Preliminary and some results

We review some facts whose proofs can be found in the literature. Here we keep fixed terminologies used in the first section.

Lemma 2.1 *([3, Lemma 2.1] or [7]) For a Cayley graph $\Gamma = Cay(G, S)$, we have $N_{\mathbb{A}ut(\Gamma)}(\rho(G)) = \rho(G) \rtimes \mathbb{A}ut(G, S)$.*

Therefore, Γ is normal edge-transitive when $\rho(G) \rtimes \mathbb{A}ut(G, S)$ is transitive on the edge-set of Γ.

The following lemma is essential in this paper.

Lemma 2.2 *([7, Proposition 1(c)]) Consider the Cayley graph $\Gamma = Cay(G, S)$. Then the following are equivalent:*

(i) Γ is normal edge-transitive;

(ii) $S = T \cup T^{-1}$, where T is an $\mathbb{A}ut(G, S)$-orbit in G;

Moreover, $\rho(G) \rtimes \mathbb{A}ut(G, S)$ is transitive on the arcs of Γ if and only if $\mathbb{A}ut(G, S)$ is transitive on S.

*Khazaei S, Sharifi H. Tetravalent non-normal Cayley graphs of order $5p^2$ (submitted).

Lemma 2.3 *([9], Proposition 1.5]) The following are equivalent*

(i) $\rho(G) \trianglelefteq A$;

(ii) $\mathbb{A}ut(\Gamma) = \rho(G) \rtimes \mathbb{A}ut(G,S)$;

(iii) $A_1 \leqslant \mathbb{A}ut(G,S)$.

Let G be a finite group of order $5p^2$. By Sylow theorems we can see that if the Sylow p-subgroup of G is cyclic then G is isomorphic to

$$\langle x, y | x^5 = y^{p^2} = 1, x^{-1}yx = y^k \rangle,$$

where $1 < k < p^2$, $p \nmid k$, $xyx^{-1} = y^{k^4}$ and $k^5 \equiv 1 \pmod{p^2}$.

By a simple computation we find $o(x^i y^j) = 5$, for $1 \leq i \leq 4$, $0 \leq j < p^2$.

Lemma 2.4 $| \mathbb{A}ut(G) | \leqslant p^3(p-1)$.

Proof Suppose that f is an automorphism of G. Therefore, for the generators x and y of G, $f(x)$ and $f(y)$ must be of order 5 and p^2, respectively. In fact, $f(x) \in \{x^i y^j \mid 1 \leqslant i < 5,\ 0 \leqslant j < p^2\}$ and $f(y) \in \{y^j \mid (j,p) = 1\}$. We claim that $f(x) = xy^j$, $0 \leqslant j < p^2$. We shall prove this claim by the following steps:

Step 1. $f(x) \neq x^2 y^j$, $0 \leqslant j < p^2$.

Suppose that $f(x) = x^2 y^j$ and $f(y) = y^{j'}$. On the other hand, $x^{-1}yx = y^k$. Thus, we have

$$f(y^k) = f(x^{-1}yx) = f(x)^{-1} f(y) f(x) = y^{-j} x^{-2} y^{j'} x^2 y^j$$

$$= y^{-j} x^{-1} (x^{-1} y^{j'} x) x y^j = y^{-j} x^{-1} (y^{kj'}) x y^j = y^{-j} y^{k^2 j'} y^j = y^{k^2 j'}.$$

Moreover,

$$f(y) = y^{j'} \Rightarrow f(y^k) = y^{kj'}.$$

Therefore,

$$y^{kj'} = y^{k^2 j'} \Rightarrow p^2 \mid (k^2 - k) j' \Rightarrow p^2 \mid (k^2 - k).$$

Since $p \nmid k$, we have $p^2 \mid k-1$, contradicting $1 < k < p^2$. Thus, $f(x) \neq x^2 y^j$ with $0 \leqslant j \leqslant p^2$.

Step 2. $f(x) \neq x^3 y^j$, $0 \leqslant j < p^2$.

Suppose that $f(x) = x^3 y^j$ for some $0 \leqslant j < p^2$. Similar to the first step, we conclude $p^2 \mid k(k^2 - 1)$, since $1 < k < p^2$, $p^2 \mid k^2 - 1$. That is a contradiction because if $p^2 \mid k^2 - 1$ is true then there are three cases. First, if $p | k+1$ and $p | k-1$, then this is invalid because p is odd. Secondly, if $p^2 | k+1$ then $p^2 = k+1$, due to $1 < k < p^2$. Besides, $k^5 \equiv 1$ (mode p^2). Thus, we have: $(p^2-1)^5 \equiv 1$ (mode p^2). However, this implies $-1 \equiv 1$ (mode p^2), which is impossible. Third case i.e. $p^2 | k-1$ does not occur when $1 < k < p^2$.

Step 3. $f(x) \neq x^4y^j$, $0 \leqslant j < p^2$. Suppose that $f(x) = x^4y^j$ for some $0 \leqslant j < p^2$. Again, similar to the first case, we have $p^2 \mid k(k^3-1)$. In this way, $p^2 \mid k^3-1$, makes a contradiction because if $p^2|k^3-1$ then $p^2|k^5-k^2$. Moreover, $k^5 \equiv 1$ (mode p^2). Thus, p^2 will divide k^2-1. However, in Step 2 we showed that this does not happen.

Therefore, there are p^2 cases for the image of f on x and the image of f on y has $\varphi(p^2)$ cases, where φ is the Euler function. Hence, all states are totally $p^3(p-1)$. This completes the proof. □

We are going to characterize the automorphism group of G. In the following lemma $\mathbb{Z}_{p^2}^{\times}$ denotes the multiplication group of the field of order p^2 with $\varphi(p^2)$ elements.

Lemma 2.5 $\mathbb{A}ut(G) \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{p^2}^{\times}$.

Proof Let $A = \{f \in \mathbb{A}ut(G) \mid f(y) = y, f(x) = xy^j, 0 \leqslant j < p^2\}$. It is easy to see that A is a subgroup of order p^2 of $\mathbb{A}ut(G)$. If $f \in A$ and $g \in \mathbb{A}ut(G)$ with property $f(y) = y, f(x) = xy^j$ and $g(y) = y^i, g(x) = xy^{j'}$, then

$$g^{-1}fg(y) = g^{-1}(f(y^i)) = g^{-1}(y^i) = y.$$

Hence, $g^{-1}fg \in A$. Therefore A is a normal subgroup of $\mathbb{A}ut(G)$. Suppose $f(x) = xy$, $f(y) = y$, then f is an element of order p^2 belonging A. Thus $A = \langle f \rangle$. We conclude that $A \cong \mathbb{Z}_{p^2}$.

Now, Let $B = \{f \in \mathbb{A}ut(G) \mid f(x) = x, f(y) = y^i, 1 \leqslant i < p^2, (i,p) = 1\}$. B is a subgroup of order $\varphi(p^2)$ such that intersection A and B is trivial. Obviously, B is isomorphic to $\mathbb{Z}_{p^2}^{\times}$, and the result is now immediate. □

We are interested in the Cayley graph $\Gamma = Cay(G,S)$ when $|S| = 4$, $G = \langle S \rangle$, and all elements of S are of order 5.

The elements of S are of the form $x^iy^j, 0 \leq i \leq 4, 0 \leq j < p^2$. Since $G = \langle S \rangle$ and inverse x^3y^j and x^4y^j are $x^2y^{j'}$ and $xy^{j'}$, respectively. We conclude that $S = S_i$ with $i \in \{1,2,3\}$, where, if $x^iy^j \in S$ then $(x^iy^j)^{-1} \in S$, with the following description:

$$S_1 = \{xy^j, xy^{j'}, (xy^j)^{-1}, (xy^{j'})^{-1}\}, j \not\equiv j' \ (mod\ p^2);$$

$$S_2 = \{xy^j, x^2y^{j'}, (xy^j)^{-1}, (x^2y^{j'})^{-1}\}, j' \not\equiv j(k+1) \ (mod\ p^2);$$

$$S_3 = \{x^2y^j, (x^2y^j)^{-1}, x^2y^{j'}, (x^2y^{j'})^{-1}\}, \quad (j \not\equiv j' \ (mod\ p^2));$$

So our task is then to study all $Cay(G,S_i)$, $1 \leqslant i \leqslant 3$.

3. $\Gamma_1 = Cay(G,S_1)$

We start with the following lemma.

Lemma 3.1 *S_1 is equivalent to $\{x, xy, x^{-1}, (xy)^{-1}\}$.*

Proof We know $S_1 = \{xy^j, xy^{j'}, (xy^j)^{-1}, (xy^{j'})^{-1}\}$ where $0 \le j, j' < p^2$, $j \not\equiv j'$ $(mod\ p^2)$. It is sufficient to let $f(x) = xy^j$ and $f(y) = y^{j'-j}$. Hence, $f(xy) = xy^j y^{j'-j} = xy^{j'}$, as wanted. □

From now on, we use this equivalent set for S_1. Clearly, $G = \langle S_1 \rangle$.

Theorem 3.2 $\mathbb{A}ut(G, S_1) \cong \mathbb{Z}_2$.

Proof Obviously, $\mathbb{A}ut(G, S_1) \leqslant \mathbb{S}_4$. On the other hand, $\mathbb{A}ut(G, S_1)$ does not have any element of order 3. Because, if $\sigma \in \mathbb{A}ut(G, S_1)$ is of order 3, then σ will fix an element of S_1. So, its inversion should also be kept fix by σ. Thus, two remaining elements of S_1 will be sent to each other by σ. Therefore, σ has a transposition in its decomposition to disjoint cycles. Consequently, the order of σ is even. Moreover, $\mathbb{A}ut(G, S_1)$ can not contain an element of order 4. Because, if there is such an element, then one of the following two cases happens; $\sigma_1 = (x, xy, x^{-1}, (xy)^{-1})$ or $\sigma_2 = (x, (xy)^{-1}, x^{-1}, xy)$. Assume that $\sigma(y) = y^i$ with $0 < i < p$. For the first case, $x^{-1} = \sigma_1(xy) = \sigma_1(x)\sigma_1(y) = xyy^i$, where $(i, p^2) = 1$. Therefore $x^{-2} = y^{i+1}$ which is a contradiction. And, for the second case, $x^{-1} = \sigma_2((xy)^{-1}) = \sigma_2(y^{-1}x^{-1}) = \sigma_2(y^{-1})\sigma_2(x^{-1})$. Therefore $x^{-2} = x^{-1}y^{-i}xy = y^{1-ki}$ which is again a contradiction. Thus, $\mathbb{A}ut(G, S_1) \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. By the proof of Lemma 2.4, if $\sigma \in \mathbb{A}ut(G, S_1)$, then $\sigma(x) = x$ or xy and $\sigma(xy) = x$ or xy. If $\sigma(x) = x$ then $\sigma(xy) = xy$. Thus, $\sigma(y) = y$ yields that σ is trivial. If $\sigma(x) = xy$ then $\sigma(xy) = x$. Therefore $x = \sigma(xy) = \sigma(x)\sigma(y) = xy\sigma(y)$. So, $\sigma(y) = y^{-1}$. Obviously, σ is an involution, and we are done.

□

Let $A = \mathbb{A}ut(\Gamma)$, and suppose that A_g is a stabilizer of g, when A acts on G. We have the following useful lemma.

Lemma 3.3 *If $\varphi \in A_g$ and one of the elements of $D_1(g)$ is fixed by φ, then φ fix all the elements of $D_1(g)$.*

Proof Let $C_1 = \{g, xg, x^2g, x^3g, x^4g\}$ and $C_2 = \{g, xyg, (xy)^2g, (xy)^3g, (xy)^4g\}$. It is straightforward to check that g is the only common vertex between these two cycles C_1 and C_2 of length 5, see Figure 1.

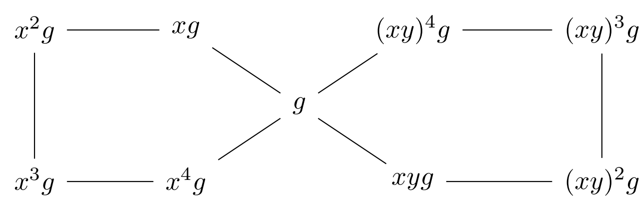

Figure 1. g is the only common vertex between these two cycles C_1 and C_2 of length 5.

If $\varphi(xg) = xg$ then $\varphi(x^ig) = x^ig$ for $1 \leqslant i \leqslant 4$. Because $x^4g \in D_1(g)$ and φ preserve distance, so we have:

$$\varphi\left(x^4g\right) \in \varphi\left(D_1\left(g\right)\right) = D_1\left(\varphi\left(g\right)\right) = D_1\left(g\right) = \left\{xg, x^4g, xyg, (xy)^{-1}g\right\}$$

$$= \left\{\varphi\left(xg\right), x^4g, xyg, (xy)^{-1}g\right\}$$

Since φ is one-to-one, so $\varphi\left(x^4g\right) \in \left\{x^4g, xyg, (xy)^{-1}g\right\}$. On the other hand, $\left(x^2g, x^3g\right)$ is an edge and $x^2g \in D_1\left(xg\right)$, $x^3g \in D_1\left(x^4g\right)$, so there exists a member of $\varphi\left(D_1\left(xg\right)\right)$ that is adjacent with a member of

$\varphi\left(D_1\left(x^4g\right)\right)$. Now, if $\varphi\left(x^4g\right)=xyg$ or $(xy)^{-1}g$, these members do not exist. Thus $\varphi\left(x^4g\right)=x^4g$. x^2g and x^3g are only vertices of $D_1\left(xg\right)$ and $D_1\left(x^4g\right)$ that are adjacent, so φ will fix them. Similarly, If $\varphi(xyg)=xyg$, then $\varphi((xy)^ig)=(xy)^ig$ for $1\leqslant i\leqslant 4$, also, if $\varphi(xg)=x^4g$, then $\varphi(x^ig)=x^{-i}g$ for $1\leqslant i\leqslant 4$. Then we have two cases for φ:

Case 1. If $\varphi(xg)=xg$ and $\varphi(xyg)=(xy)^{-1}g$, then since

$$xy^{-k}g\in D_1\left(x^2g\right)=\varphi\left(D_1\left(x^2g\right)\right),$$

we have

$$\varphi\left(xy^{-k}g\right)\in\left\{xg,x^3g,xy^{-k}g,x^3y^{k^2}g\right\}=$$

$$\left\{\varphi\left(xg\right),\varphi\left(x^3g\right),xy^{-k}g,x^3y^{k^2}g\right\}.$$

Furthermore, φ is one-to-one, thus we have

$$\varphi\left(xy^{-k}g\right)\in\left\{xy^{-k}g,x^3y^{k^2}g\right\}.$$

Now since $y^{-k}g$ is adjacent with $xy^{-k}g$ and φ preserves distance, so we have:

$$\varphi\left(y^{-k}g\right)\in D_1\left(xy^{-k}g\right)\cup D_1\left(x^3y^{k^2}g\right)=A$$

where $A=\left\{y^{-k}g,x^2y^{-k}g,y^{-k-1}g,x^4y^{k^2}g,x^2y^{k^2}g,x^4y^{k^3+k^2}g\right\}$.
On the other hand $y^{-k}g\in D_3\left(xyg\right)$, so $\varphi\left(y^{-k}g\right)\in\varphi\left(D_3\left(xyg\right)\right)=D_3\left((xy)^{-1}g\right)$. Now since $A\cap D_3\left((xy)^{-1}g\right)=\emptyset$. Consequently, this case is a contradiction.

Case 2. If $\varphi\left(xg\right)=x^4g$ and $\varphi\left(xyg\right)=xyg$, then let

$$\sigma:=\rho_{g^{-1}xg}\circ\varphi\circ\rho_{g^{-1}xg}.$$

Clearly $\varphi\circ\sigma\left(g\right)=g$, $\varphi\circ\sigma\left(xg\right)=xg$. $x^2y^kg\in D_1\left(xg\right)$, so we have

$$\varphi\left(x^2y^kg\right)\in D_1\left(\varphi\left(xg\right)\right)=D_1\left(x^4g\right)=\left\{g,x^3g,y^{k^4}g,x^3y^{-k^3}g\right\}=\left\{\varphi\left(g\right),\varphi\left(x^2g\right),y^{k^4}g,x^3y^{-k^3}g\right\}.$$

Since φ is one-to-one, so $\varphi\left(x^2y^kg\right)\in\left\{y^{k^4}g,x^3y^{-k^3}g\right\}$. Now if $\varphi\left(x^2y^kg\right)=x^3y^{-k^3}g$, then $\varphi\circ\sigma\left(xyg\right)=(xy)^{-1}g$, which is a contradiction by Case 1. Thus $\varphi\left(x^2y^kg\right)=y^{k^4}g$. Clearly,

$$y^kg\in D_2\left(x^2y^kg\right)\cap D_2\left((xy)^2g\right)$$

So $\varphi\left(y^k g\right) \in D_2\left(y^{k^4} g\right) \cap D_2\left((xy)^2 g\right)$, which is a contradiction.

□

Result 3.4 *If $\varphi \in \mathbb{A}ut(\Gamma_1)_g$ and φ fixes an element of $D_1(g)$, then $\varphi = id$.*

Proof Since graph is connected, it suffices to show that for every integer $i \geqslant 1$, the statement:

$$g' \in D_i\left(g\right) \Rightarrow \varphi\left(g'\right) = g'$$

holds. By Lemma 3.3 the statement is true for $i = 1$. Now assume that the statement is true for $1 \leqslant i \leqslant n$, and we will show that the statement holds for $n+1$. Let $g' \in D_{n+1}\left(g\right)$. Hence, there is a sequence of adjacent vertices

$$g = {g'}_0, {g'}_1, ..., {g'}_{n-1}, {g'}_n, {g'}_{n+1}$$

Clearly, $g'_{n-1} \in D_{n-1}(g)$ and $g'_n \in D_n(g)$. Therefore by hypothesis, $\varphi\left(g'_{n-1}\right) = g'_{n-1}$ and $\varphi\left(g'_n\right) = g'_n$. By applying Lemma 3.3 for $g := g'_n$ and the fact $\varphi\left(g'_{n-1}\right) = g'_{n-1}$. We conclude that $\varphi\left(g'_{n+1}\right) = g'_{n+1}$, or equivalently $\varphi\left(g'\right) = g'$. □

Lemma 3.5 *If $\varphi \in Aut(\Gamma_1)_1$ and $\varphi(x) = x^{-1}$ and let $k \geqslant 1$. Then*

$$\varphi\left(s_1^{i_1} s_2^{j_1} ... s_1^{i_k} s_2^{j_k}\right) = s_1^{-i_1} s_2^{-j_1} ... s_1^{-i_k} s_2^{-j_k}$$

where $s_1, s_2 \in \{x, xy\}$, $s_1 \neq s_2$ and $1 \leqslant i_l, j_l \leqslant 4$ for $1 \leqslant l \leqslant k$.

Proof Let $\sigma := \rho_x \varphi \rho_x$. We have $\sigma \circ \varphi(1) = 1$ and $\sigma \circ \varphi(x) = x$. So, by Lemma 3.3, $\sigma \circ \varphi = id$ and $\varphi^2 = id$. Consequently, $\sigma = \varphi$ and $\varphi = \varphi^{-1}$. Therefore for every $g \in G$, we have $\varphi\left(g\right) = \varphi\left(gx\right)x$, so $\varphi\left(gx\right) = \varphi\left(g\right)x^{-1}$. On the other hand, similar to the above, $\rho_{xy}\varphi\rho_{xy} = \varphi$. Thus for every $g \in G$, we conclude that $\varphi\left(gxy\right) = \varphi\left(g\right)\left(xy\right)^{-1}$. Since $G = \langle x, xy\rangle$, the proof is completed. □

Lemma 3.6 *If $\varphi \in \mathbb{A}ut(\Gamma_1)_1$, then $\varphi(x) \neq x^{-1}$.*

Proof Suppose that $\varphi(x) = x^{-1}$. By previous lemma we have

$$\varphi\left(y^{k^4}\right) = \varphi\left(\left(xy\right)x^{-1}\right) = \left(xy\right)^{-1}x = y^{-1}.$$

On the other hand:

$$\varphi\left(y^{k^4}\right) = \varphi\left(x^{-1}\left(xy\right)\right)^{k^4} = \left(x^{-1}\left(xy\right)^{-1}\right)^{k^4} = \left(xy^{-1}x^{-1}\right)^{k^4} = y^{-k^8} = y^{-k^3}.$$

So $y^{-1} = y^{-k^3}$ and $p^2 \left| k^3 - 1 \right.$, which is a contradiction.

□

Lemma 3.7 *If $\varphi \in \mathbb{A}ut(\Gamma_1)_1$, then $\varphi(x) \neq (xy)^{-1}$.*

Proof Suppose that $\varphi(x) = (xy)^{-1}$. If σ is a non-trivial element of $\mathbb{A}ut(G, S_1)$, then $\sigma \circ \varphi \in \mathbb{A}ut(\Gamma_1)_1$ and $\sigma \circ \varphi(x) = x^{-1}$, which is in contradiction with Lemma 3.6. □

Result 3.8 *If* $\varphi \in \mathbb{A}ut(\Gamma_1)_1$ *and* $\varphi(x) = xy$*, then* $\varphi = \sigma$*.*

Proof According to the Corollary 3.4 and $\varphi \circ \sigma(xy) = xy$, the corollary is established. □

Theorem 3.9 $\mathbb{A}ut(\Gamma_1)_1 \cong \mathbb{Z}_2$.

Proof The theorem now follows via Corollaries 3.4, 3.8 and Lemmas 3.6 and 3.7. □

Eventually, Theorems 3.2, 3.9 and Lemmas 2.2, 2.3, cover case(1) of the main theorem.

4. $\Gamma_2 = Cay(G, S_2)$

Lemma 4.1 S_2 *is equivalent to* $\{x^2, xy, x^{-2}, (xy)^{-1}\}$*.*

Proof We know that $S_2 = \{xy^j, x^2y^{j'}, (xy^j)^{-1}, (x^2y^{j'})^{-1}\}$, where $j' \not\equiv j(k+1)\ (mod\ p^2)$ and $0 \leqslant j, j' < p^2$. It is sufficient to set $f \in \mathbb{A}ut(G)$ as follows, $f(x) = xy^\alpha$ and $f(y) = y^\beta$, where α is one of the following states

$$\begin{cases} \alpha = \frac{j'}{2}(k^4 - k^3 + k^2 - k + 1), & \text{if } j' \text{ is even ;} \\ \alpha = \frac{p^2+j'}{2}(k^4 - k^3 + k^2 - k + 1), & \text{if } j' \text{ is odd.} \end{cases}$$

and $\beta = j - \alpha$. □

From now on, we use this equivalent set for S_2. Clearly, $G = \langle S_2 \rangle$.

Theorem 4.2 $\mathbb{A}ut(G, S_2)$ *is trivial.*

Proof If $f \in \mathbb{A}ut(G, S_2)$, then by proof of Lemma 2.4, we know $f(x^2) = x^2$ and $f(xy) = xy$. On the other hand, for some suitable i, j that $1 \leqslant i, j < p^2$ and $(j, p) = 1$ we have $f(x) = xy^i$ and $f(y) = y^j$. Therefore, $x^2 = f(x^2) = f(x)^2 = xy^ixy^i = x^2y^{i(k+1)}$, and hence $p^2 \mid i(k+1)$. Moreover, $xy = f(xy) = xy^iy^j = xy^{i+j}$, so $p^2 \mid i + j - 1$. Clearly, $(p, k+1) = 1$; otherwise, for some integer r, we have $k = rp - 1$, and so, $p^2 \mid k^5 - 1$, thus $1 \equiv k^5 (mod\ p^2) \equiv 5rp - 1$. Hence, $p^2 \mid 5rp - 2$, which is a contradiction. Therefore, $p^2 \mid i$. So, $i = 0$. Now since, $p^2 \mid i + j - 1$, thus $p^2 \mid j - 1$. Therefore, $j = 1$, and thus f is trivial. □

Lemma 4.3 *If* $\varphi \in A_g$ *and one of the elements of* $D_2(g)$ *is fixed by* φ *then* φ *will fix all the elements of* $D_2(g)$*.*

Proof Since $\varphi(g) = g$, so $\varphi(D_2(g)) = D_2(g)$. On the other hand, g is the only common vertex between these two cycles of length 5, as in Figure 2.

If $\varphi(xyg) = xyg$, similar to Lemma 3.3, we conclude that $\varphi((xy)^ig) = (xy)^ig$ for $1 \leqslant i \leqslant 4$ and if $\varphi(x^2g) = x^2g$, then $\varphi(x^ig) = x^ig$ for $1 \leqslant i \leqslant 4$. With this information, we consider two cases:

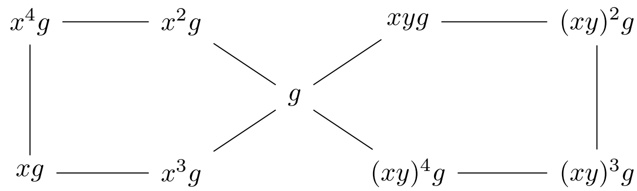

Figure 2. g is the only common vertex between these two cycles of length 5.

Case 1. If $\varphi(x^2g) = x^2g$ and contrary to the claim $\varphi(xyg) = (xy)^4g$, then we have $y^{k^2+k}g \in D_3(xg) \cap D_2((xy)^3g)$; since φ preserves distance, $\varphi(y^{k^2+k}g) \in D_3(xg) \cap D_2((xy)^2g)$. So we have:

$$\varphi(y^{k^2+k}g) \in \{(xy)^4g, xy^{k+1}g\} = \left\{\varphi\left(xyg\right), xy^{k+1}g\right\}.$$

Since φ is one-to-one, we have $\varphi\left(y^{k^2+k}g\right) = xy^{k+1}g$.

So $\varphi(y^{k^2+k}g) = xy^{k+1}g$.

As before, since $xy^{k+1}g \in D_3(xg) \cap D_2((xy)^2g)$, we have

$$\varphi(xy^{k+1}g) \in D_3(xg) \cap D_2((xy)^3g) = \{xyg, y^{k^2+k}g\}.$$

Clearly $\varphi(xy^{k+1}g) \neq xyg$. Therefore $\varphi(xy^{k+1}g) = y^{k^2+k}g$. On the other hand,

$$x^3y^{k+1}g \in D_1\left(xy^{k+1}g\right) \cap D_2\left((xy)^2g\right).$$

But $D_1\left(y^{k^2+k}g\right) \cap D_2\left((xy)^3g\right) = \emptyset$. This contradiction shows that this case will not happen.

Case 2. If $\varphi\left(x^2g\right) = x^3g$ and contrary to the claim $\varphi(xyg) = xyg$, putting $\sigma := \rho_{g^{-1}xg} \circ \varphi \circ \rho_{g^{-1}xg}$, then clearly $\varphi \circ \sigma\left(g\right) = g$, $\varphi \circ \sigma\left(x^2g\right) = x^2g$. So by previous case $\varphi \circ \sigma\left(xyg\right) = xyg$. On the other hand, similar to Lemma 3.3, $\varphi\left(x^2y^kg\right) \in \left\{x^3y^{-k^3}g, y^{k^4}g\right\}$. If $\varphi\left(x^2y^kg\right) = x^3y^{-k^3}g$, then $\varphi \circ \sigma\left(xyg\right) = (xy)^{-1}g$, which is impossible regarding to Case 1. So $\varphi\left(x^2y^kg\right) = y^{k^4}g$. On the other hand,

$$y^{k^2+k}g \in D_2\left(x^2y^kg\right) \cap D_2\left((xy)^3g\right).$$

Thus we have

$$\varphi\left(y^{k^2+k}g\right) \in D_2\left(y^{k^4}g\right) \cap D_2\left((xy)^3g\right),$$

which is a contradiction.

□

Result 4.4 *If $\varphi \in \mathbb{A}ut(\Gamma_2)_g$ and φ fixed an element of $D_2(g)$, then $\varphi = id$.*

Proof Similar to Corollary 3.4, it follows immediately from Lemma 4.3 and graph connectivity. □

Lemma 4.5 *If* $\varphi \in \mathbb{A}ut(\Gamma_2)_1$, *then* $\varphi(x^2) \neq x^3$.

Proof On the contrary let $\varphi(x^2) = x^3$. By Corollary 4.4, $\varphi(xy) = (xy)^4$. It is easy to see that the conditions of Lemma 3.5 are established; so we have

$$\varphi\left(y^{k^4}\right) = \varphi\left((xy)\,x^2x^2\right) = (xy)^{-1}x^{-2}x^{-2} = y^{-1}x^{-1}x = y^{-1}.$$

On the other hand,

$$\varphi\left(y^{k^4}\right) = \varphi\big(x^2x^2\,(xy)\big)^{k^4} = \left(x^3x^3(xy)^{-1}\right)^{k^4} =$$

$$\left(xy^{-1}x^{-1}\right)^{k^4} = \left(y^{-k^4}\right)^{k^4} = y^{-k^3}.$$

So $y^{-1} = y^{-k^3}$ and $p^2 \,\big|\, k^3 - 1$ which is a contradiction. □

Lemma 4.6 *If* $\varphi \in \mathbb{A}ut(\Gamma_2)_1$, *then* $\varphi(x^2) \neq xy$.

Proof Suppose that $\varphi(x^2) = xy$ and let $\sigma := {\rho_{(xy)}}^{-1}\varphi\rho_{x^2}$. One can see that $\sigma\circ\varphi^{-1}(1) = 1$, $\sigma\circ\varphi^{-1}(xy) = xy$. So by Corollary 4.4, $\sigma = \varphi$. Thus for every $g \in G$, we have $\varphi\left(gx^2\right) = \varphi(g)\,xy$. Since $\varphi(S_2) = S_2$, we have $\varphi(xy) \in \left\{x^2, x^3\right\}$. If $\varphi(xy) = x^3$, then $\varphi^2 \in Aut(\Gamma_2)_1$ and $\varphi^2\left(x^2\right) = x^3$, which is in contradiction with Lemma 4.5. If $\varphi(xy) = x^2$, since $\varphi^2 = id$, for every $g \in G$, $\varphi(\varphi(g)\,xy) = \varphi^2\left(gx^2\right) = gx^2$ and $\varphi(\varphi(g)\,xy) = gx^2$. Now by replacing $\varphi(g)$ by g, we have $\varphi(gxy) = \varphi(g)\,x^2$. Therefore,

$$\varphi\left(y^{k^4}\right) = \varphi\left((xy)\,x^{-1}\right) = \varphi\left((xy)\,x^2x^2\right) = x^2\,(xy)\,(xy) = x^4y^{k+1}.$$

On the other hand, $y^{k^4} = (xy)^{-1}x^2x^2x^2\,(xy)\,x^{-2}\,(xy)$. Thus,

$$\varphi\left(y^{k^4}\right) = x^{-2}(xy)^3x^2(xy)^{-1}x^2 = x^4y^{k^4+k^3-k+1}.$$

So $x^4y^{k^4+k^3-k+1} = x^4y^{k+1}$ and $p^2 \mid k^4 + k^3 - 2k$. In this way,

$$p^2 \big| k^5 + k^4 - 2k^2 \to p^2 \,\big|\, \left(k^2-1\right)^2.$$

Thus $p \,\big|\, k^2 - 1$ which is a contradiction. So, the assertion is true.

□

Lemma 4.7 *If* $\varphi \in \mathbb{A}ut(\Gamma_2)_1$, *then* $\varphi(x^2) \neq x^4y^{-k^4}$.

Proof If $\varphi(x^2) = (xy)^4$, then $\varphi((xy)^4)$ has two cases:

(I) $\varphi((xy)^4) = x^3$, which leads to a contradiction by Lemma 4.5, due to $\varphi^2(x^2) = x^3$.

(II) $\varphi((xy)^4) = x^2$; let $\sigma := \rho_{(xy)^{-1}}\varphi\rho_{x^3}$. One can show that $\sigma \circ \varphi(1) = 1$ and $\sigma \circ \varphi\left((xy)^{-1}\right) = (xy)^{-1}$. Thus, by Corollary 4.4, we have $\sigma = \varphi$. Therefore, for every $g \in G$, $\varphi\left(gx^2\right) = \varphi(g)(xy)^{-1}$. Similarly we have $\rho_{x^2}\varphi\rho_{xy} = \varphi$. So for every $g \in G$, $\varphi(gxy) = \varphi(g)x^3$. Now we have

$$\varphi\left(y^{k^4}\right) = \varphi\left((xy)x^{-1}\right) = \varphi\left((xy)x^2x^2\right) = x^3(xy)^{-2} = xy^{k^2+k+1}.$$

Furthermore, we have

$$\varphi\left(y^{k^4}\right) = \varphi\left((xy)^{-1}x^2x^2x^2(xy)x^{-2}(xy)\right) =$$

$$x^2(xy)^{-3}x^3(xy)x^3 = xy^{2k^3+k^2}.$$

We conclude

$$p^2 \mid 2k^3 - k - 1 \overset{\times k^4}{\Rightarrow} p^2 \mid (k^2 - 1)^2$$

which is a contradiction.

□

Theorem 4.8 $\mathbb{A}ut(\Gamma_2)_1 \cong id$.

Proof The theorem now follows via Corollary 4.4 and Lemmas 4.5, 4.6 and 4.7. □

Therefore, proof of the second part of main theorem can be seen by applying Theorem 4.2, 4.8, and Lemmas 2.2 and 2.3.

5. $\Gamma_3 = Cay(G, S_3)$

Lemma 5.1 *S_3 is equivalent to* $\{x^2, x^2y, x^{-2}, (x^2y)^{-1}\}$.

Proof We know $S_3 = \{x^2y^j, x^2y^{j'}, (x^2y^j)^{-1}, (x^2y^{j'})^{-1}\}$, where $j \not\equiv j'(mod p^2)$. Let $f \in \mathbb{A}ut(G)$, such that $f(x) = xy^i$ and $f(y) = y^d$, where $0 \leqslant i < p^2$ and $(d, p^2) = 1$. We consider:

$$\begin{cases} i = \frac{j}{2}(k^4 - k^3 + k^2 - k + 1), & \text{if } j \text{ is even;} \\ i = \frac{p^2+j}{2}(k^4 - k^3 + k^2 - k + 1), & \text{if } j \text{ is odd.} \end{cases}$$

and $d = j' - j$. Therefore, $(d, p) = 1$ and this completes the proof. □

From now on, we use this equivalent set for S_3. Clearly, $G = \langle S_3 \rangle$.

Theorem 5.2 $\mathbb{A}ut(G, S_3) \cong \mathbb{Z}_2$.

Proof Suppose that $f \in \mathbb{A}ut(G, S_3)$. According to the proof of Lemma 2.4, we have two cases for $f(x^2)$:

Case1. $f(x^2) = x^2$.

Assuming that $f(x) = xy^i$, where $0 \leqslant i < p^2$, we have $x^2 = f(x^2) = f(x)^2 = xy^ixy^i = x^2y^{i(k+1)}$ which implies $p^2|i(k+1)$. As we have already shown $p \nmid k+1$, so $p^2|i$. Thus, $f(x) = x$. On the other hand, $f(x^2y) = x^2y$. Therefore, $x^2y = f(x^2y) = f(x)^2f(y) = x^2f(y)$ which implies $f(y) = y$, so $f = id$.

Case2. $f(x^2) = x^2y$.

By the first case, since $(p, k+1) = 1$, there exist integers r and i, such that $i(k+1) + rp^2 = 1$. Thus, $p^2|i(k+1) - 1$. From this $f(x^2y) = x^2$, we have $x^2y^{i(k+1)+j} = x^2$, for some integer j. Therefore, $p^2|i(k+1) + j$ which implies $p^2|i(k+1) + j$ implies $p^2|j+1$, where $0 \leqslant j < p^2$. So, $j = p^2 - 1$. Thus $f(x) = xy^{k^4+k^2+1} = (x^2y)^3$ and $f(y) = y^{p^2-1}$. Hence, $o(f) = 2$, and the result now follows.

□

Lemma 5.3 *If $\varphi \in A_g$ and one of the elements of $D_1(g)$ is fixed by φ, then φ fixes all the elements of $D_1(g)$.*

Proof g is the only common vertex between the two cycles of the length five, as in Figure 3.

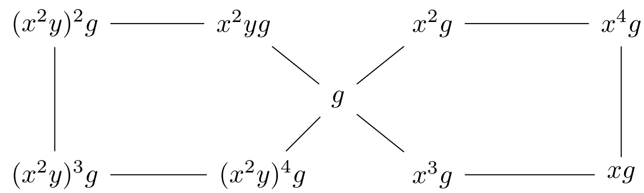

Figure 3. g is the only common vertex between these two cycles of length 5.

If $\varphi(x^2g) = x^2g$, then $\varphi(x^ig) = x^ig$ for $1 \leqslant i \leqslant 4$. Since $x^3g \in D_1(g)$ and φ preserves distance, we conclude that,

$$\begin{aligned}\varphi\left(x^3g\right) \in \varphi\left(D_1(g)\right) &= D_1(\varphi(g)) = D_1(g) = \left\{x^2g, x^3g, x^2yg, \left(x^2y\right)^{-1}g\right\} \\ &= \left\{\varphi\left(x^2g\right), x^3g, x^2yg, x^3y^{-k^3}g\right\}\end{aligned}$$

Since φ is one-to-one, so $\varphi\left(x^3g\right) \in \left\{x^3g, x^2yg, x^3y^{-k^3}g\right\}$. On the other hand, $\left(x^4g, xg\right)$ is an edge, $x^4g \in D_1\left(x^2g\right)$ and $xg \in D_1\left(x^3g\right)$, so there exists a vertex in $\varphi\left(D_1\left(x^2g\right)\right)$ that is adjacent with a vertex in $\varphi\left(D_1\left(x^3g\right)\right)$. Now if $\varphi\left(x^3g\right) = x^2yg$ or $x^3y^{-k^3}g$, these members do not exist. Thus $\varphi\left(x^3g\right) = x^3g$. Now x^4g and xg are only vertices $D_1\left(x^2g\right)$ and $D_1\left(x^3g\right)$ that are adjacent, so φ fixes them. Exactly similar to the above,
if $\varphi(x^2yg) = x^2yg$, then $\varphi((x^2y)^ig) = (x^2y)^ig$, for $1 \leqslant i \leqslant 4$,if $\varphi(x^3g) = x^2g$, then $\varphi(x^ig) = x^{-i}g$ for $1 \leqslant i \leqslant 4$ and if $\varphi(\left(x^2y\right)^{-1}g) = x^2yg$, then $\varphi((x^2y)^ig) = (x^2y)^{-i}g$, for $1 \leqslant i \leqslant 4$.

With this information, we consider two cases

Case 1. Suppose $\varphi(x^2g) = x^2g$. If $\varphi(x^2yg) = (x^2y)^4g$, then since φ preserves distance:

$$y^kg \in D_2(xg) \cap D_3((x^2y)^4g) \Longrightarrow \varphi(y^kg) \in D_2(xg) \cap D_3(x^2yg).$$

On the other hand, $D_2(xg) \cap D_3(x^2yg) = \{y^{k^3}g, xy^{-k}g\}$. If $\varphi(y^kg) = y^{k^3}g$, then $1 = d(y^{k^3}g, x^3g) = d(y^kg, x^3g) > 1$ which is a contradiction. And if, $\varphi(y^kg) = xy^{-k}g$, then $1 = d(xy^{-k}g, x^3g) = d(y^kg, x^3g) > 1$ which is impossible. So this case does not happen.

Case 2. Suppose $\varphi(x^2yg) = x^2yg$. If $\varphi(x^2g) = x^3g$, then

$$y^kg \in D_2(xg) \cap D_3((x^2y)^4g) \Longrightarrow \varphi(y^kg) \in D_2(x^4g) \cap D_3((x^2y)^4g)$$

Consequently, $\varphi(y^kg) \in \{x^4y^{k^2}g, y^{-1}g\}$. If $\varphi(y^kg) = x^4y^{k^2}g$, then $1 = d(x^4y^{k^2}g, x^2g) = d(y^kg, x^3g) > 1$, also, if $\varphi(y^kg) = y^{-1}g$, then $1 = d(y^{-1}g, x^2g) = d(y^kg, x^3g) > 1$. These contradictions show that this case also will not happen.

□

Result 5.4 *If $\varphi \in \mathbb{A}ut(\Gamma_3)_g$ and φ fixes an element of $D_1(g)$, then $\varphi = id$.*

Proof Similar to Corollary 3.4 ,it follows immediately from Lemma 5.3 and graph connectivity. □

Lemma 5.5 *If $\varphi \in \mathbb{A}ut(\Gamma_3)_g$, then $\varphi(x^2g) \neq x^3g$.*

Proof Suppose that $\varphi(x^2g) = x^3g$ and let $\sigma := \rho_{g^{-1}xg}\varphi\rho_{g^{-1}xg}$. Then $\sigma \circ \varphi\,(g) = g$, $\sigma \circ \varphi\left(x^2g\right) = x^2g$. So by Corollary 5.4 and the fact that $\varphi^2 = id$, we have:

$$\sigma \circ \varphi = id \Rightarrow \sigma = \varphi.$$

Thus for every $g' \in G$, $\varphi\left(g'x\right) = \varphi\left(g'\right)x^4$. Now let $\psi := \rho_{g^{-1}x^2yg}\varphi\rho_{g^{-1}x^2yg}$, then $\psi\,(g) = g$, $\psi\left(x^2yg\right) = \left(x^2yg\right)^4$, $\psi \circ \varphi = id$ and for every $g' \in G$, we have

$$\varphi\left(g'x^2y\right) = \varphi\left(g'\right)\left(x^2y\right)^4 = \varphi\left(g'\right)x^3y^{-k^3}.$$

Now we have

$$\varphi\,(y) = \varphi\left(x^3\left(x^2y\right)\right) = x^2x^3y^{-k^3} = y^{-k^3}.$$

Since $\varphi^2 = id$, we have $\varphi\left(y^{-k^3}\right) = y$. On the other hand,

$$\varphi\left(y^{-k^3}\right) = \varphi(y)^{-k^3} = \left(y^{-k^3}\right)^{-k^3} = y^k.$$

Hence $p^2\,|k-1$ which is impossible. □

Lemma 5.6 *If $\varphi \in \mathbb{A}ut(\Gamma_3)_1$ such that $\varphi(x^2) = x^2y$ and $f \in \mathbb{A}ut(G, S_3)$ is not trivial then $\varphi = f$.*

Proof We know, $f(x) = xy^{k^4+k^2+1}, f(y) = y^{p^2-1}$ and $f \circ \varphi(x^2) = x^2$. Thus, by Corollary 5.4, $f \circ \varphi = id$. On the other hand, since order of f is 2, $f = \varphi$. □

Lemma 5.7 *If $\varphi \in \mathbb{A}ut(\Gamma_3)_1$, then $\varphi(x^2) \neq (x^2y)^4$.*

Proof Otherwise, if $\varphi(x^2) = (x^2y)^4$, then $f \circ \varphi(x^2) = x^3$. But this is against of Lemma 5.5. □

As an application of Theorem 5.2, we offer the following theorem.

Theorem 5.8 $\mathbb{A}ut(\Gamma_3)_1 \cong \mathbb{Z}_2$.

Proof The statement is true by Corollary 5.4 and Lemmas 5.5, 5.6, and 5.7 □

Therefore, by Theorems 5.2 and 5.8, and Lemmas 2.2, 2.3, the third part of main theorem is proved.

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References

[1] Corr BP, Praeger CE. Normal edge-transitive Cayley graphs of Frobenius groups. Journal of Algebraic Combinatorics 2015; 42 (3): 803-827. doi: 10.1007/s10801-015-0603-4

[2] Darafsheh MR, Abdollahi M. On connected tetravalent Cayley graphs of a non-Abelian group of order $3p^2$. Algebra Colloquium 2017; 24 (3): 467-480. doi: 10.1142/S100538671700030X

[3] Godsil CD. On the full automorphism group of a graph. Combinatorica 1981; 1 (3): 243-256. doi: 10.1007/BF02579330

[4] Hua XH, Xu SJ, Deng YP. Tetravalent edge-transitive Cayley graphs of $PGL(2,p)$. Acta Mathematicae Applicatae Sinica, English Series 2013; 29 (4): 837-842. doi: 10.1007/s10255-013-0260-0

[5] Li CH, Liu Z, Lu ZP. The edge-transitive tetravalent Cayley graphs of square-free order. Discrete Mathematics 2012; 312 (12-13): 1952-1967. doi: 10.1016/j.disc.2012.03.013

[6] Pakravash Y, Iranmanesh A. Normal edge-transitive Cayley graphs on non-Abelian groups of order $4p^2$, where p is a prime number. Bulletin of the Iranian Mathematical Society 2017; 43 (3): 951-974.

[7] Praeger CE. Finite normal edge-transitive cayley graphs. Bulletin of the Australian Mathematical Society 1999; 60 (2): 207-220. doi: 10.1017/S0004972700036340

[8] Sharifi H, Darafsheh MR. On tetravalent normal edge-transitive Cayley graphs on the modular group. Turkish Journal of Mathematics 2017; 41 (5): 1308-1312. doi:10.3906/mat-1604-109

[9] Xu MY. Automorphism groups and isomorphisms of Cayley digraphs. Discrete Mathematics 1998; 182 (1-3): 309-319. doi: 10.1016/S0012-365X(97)00152-0