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Some results of $K-$frames and their multipliers

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Abstract: K-frames are strong tools for the reconstruction of elements from range of a bounded linear operator K on a separable Hilbert space $\mathcal{H}$. In this paper, we study some properties of K-frames and introduce the K-frame multipliers. We also focus on representing elements from the range of K by K-frame multipliers.

Key words: K-frame, K-dual, K-left inverse, K-right inverse, multiplier

1. Introduction, notation, and motivation

For the first time, frames in Hilbert space were offered by Duffin and Schaeffer in 1952 and were brought to life by Daubechies et al. [18]. A frame allows each element in the underlying space to be written as a linear combination of the frame elements, but linear independence between the frame elements is not required. This fact has a key role in applications such as signal processing, image processing, coding theory, and more. For more details and applications of ordinary frames see [2, 3, 7–16, 20]. K-frames which have recently been introduced by Găvruţa are a generalization of frames, in the meaning that the lower frame bound only holds for range of a linear and bounded operator K in a Hilbert space [19].

A sequence $m := \{m_i\}_{i\in I}$ of complex scalars is called seminormalized if there exist constants a and b such that $0 < a \leq |m_i| \leq b < \infty$, for all $i \in I$. For two sequences $\Phi := \{\varphi_i\}_{i\in I}$ and $\Psi := \{\psi_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ and a sequence m of complex scalars, the operator $\mathbb{M}_{m,\Phi,\Psi} : \mathcal{H} \to \mathcal{H}$ given by

$$\mathbb{M}_{m,\Phi,\Psi} f = \sum_{i\in I} m_i \langle f, \psi_i\rangle \varphi_i, \qquad (f \in \mathcal{H})$$

is called a multiplier. The sequence m is called the symbol. If Φ and Ψ are Bessel sequences for $\mathcal{H}$ and $m \in \ell^\infty$, then $\mathbb{M}_{m,\Phi,\Psi}$ is well-defined, $\mathbb{M}^*_{m,\Phi,\Psi} = \mathbb{M}_{\overline{m},\Psi,\Phi}$ and $\|\mathbb{M}_{m,\Phi,\Psi}\| \leq \sqrt{B_\Phi B_\Psi}\|m\|_\infty$ where B_Φ and B_Ψ are Bessel bounds of Φ and Ψ, respectively [4]. Frame multipliers have many applications in psychoacoustical modeling and denoising [6, 24]. Moreover, several generalizations of multipliers are proposed [5, 21, 22].

It is important to detect the inverse of a multiplier if it exists [8, 23]. Our aim is to introduce K-frame multipliers and apply them to reconstruct elements from the range of K.

Throughout this paper, we suppose that $\mathcal{H}$ is a separable Hilbert space, I a countable index set, and $I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$. For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote by $B(\mathcal{H}_1, \mathcal{H}_2)$ the collection of

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all bounded linear operators between $\mathcal{H}_1$ and $\mathcal{H}_2$, and we abbreviate $B(\mathcal{H},\mathcal{H})$ by $B(\mathcal{H})$. The range and null spaces of $K \in B(\mathcal{H})$ is denoted by $R(K)$ and $N(U)$, respectively. Moreover, π_V is the orthogonal projection of $\mathcal{H}$ onto a closed subspace $V \subseteq \mathcal{H}$. The pseudo-inverse of operator $U \in B(\mathcal{H}_1,\mathcal{H}_2)$ is a bounded operator in $B(\mathcal{H}_2,\mathcal{H}_1)$ and denoted by $U^\dagger$ such that [16]

$$UU^\dagger U = U,\ R(U^\dagger) = N(U)^\perp,\ N(U^\dagger) = R(U)^\perp.$$

We end this section by the following proposition.

Proposition 1.1 *[19] Let $L_1 \in B(\mathcal{H}_1,\mathcal{H})$ and $L_2 \in B(\mathcal{H}_2,\mathcal{H})$ be two bounded operators. The following statements are equivalent:*

1. *$R(L_1) \subseteq R(L_2)$.*
2. *$L_1L_1^* \leq \lambda^2 L_2L_2^*$ for some $\lambda \geq 0$.*
3. *there exists a bounded operator $X \in B(\mathcal{H}_1,\mathcal{H}_2)$ so that $L_1 = L_2X$.*

2. K-frames

Let $\mathcal{H}$ be a separable Hilbert space, a sequence $F := \{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.1}$$

Clearly if $K = I_{\mathcal{H}}$, then F is an ordinary frame. The constants A and B in (2.1) are called lower and upper bounds of F, respectively. The Bessel sequence $\{f_i\}_{i\in I}$ is called A-*tight*, if $A\|K^*f\|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2$. Moreover, if $\|K^*f\|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2$ we call F a Parseval K-frame. Obviously every K-frame is a Bessel sequence; hence, similar to ordinary frames the synthesis operator can be defined as $T_F : l^2 \to \mathcal{H}$; $T_F(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$. It is a bounded operator and its adjoint which is called the analysis operator given by $T_F^*(f) = \{\langle f, f_i\rangle\}_{i\in I}$. Finally, the frame operator is given by $S_F : \mathcal{H} \to \mathcal{H}$; $S_F f = T_F T_F^* f = \sum_{i\in I}\langle f, f_i\rangle f_i$. Many properties of ordinary frames do not hold for K-frames, for example, the frame operator of a K-frame is not invertible in general. It is worthwhile to mention that if K has close range then S_F from $R(K)$ onto $S_F(R(K))$ is an invertible operator [25]. In particular,

$$B^{-1}\|f\| \leq \|S_F^{-1}f\| \leq A^{-1}\|K^\dagger\|^2\|f\|, \quad (f \in S_F(R(K))), \tag{2.2}$$

where $K^\dagger$ is the pseudoinverse of K. For further information in K-frames refer to [1, 19, 25]. Suppose $\{f_i\}_{i\in I}$ is a Bessel sequence. Define $K : \mathcal{H} \to \mathcal{H}$ by $Ke_i = f_i$ for all $i \in I$ where $\{e_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}$. By using [16, Lemma 3.3.6] K has a unique extension to a bounded operator on $\mathcal{H}$, so $\{f_i\}_{i\in I}$ is a K-frame for $\mathcal{H}$ by Corollary 3.7 of [25]. Thus, every Bessel sequence is a K-frame for some bounded operator K. Moreover, every frame sequence $\{f_i\}_{i\in I}$ can be considered as a K-frame. In fact, let $\{f_i\}_{i\in I}$ be a frame sequence with bounds A and B, respectively and $K = \pi_{\mathcal{H}_0}$, where $H_0 = \overline{span}_{i\in I}\{f_i\}$, then for every $f \in \mathcal{H}$

$$\begin{aligned} A\|K^*f\|^2 &\leq \sum_{i\in I} |\langle K^*f, f_i\rangle|^2 \\ &= \sum_{i\in I} |\langle f, \pi_{\mathcal{H}_0} f_i\rangle|^2 \\ &= \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2. \end{aligned}$$

In the following proposition we study K-frames with respect to their synthesis operator. The proof is similar in spirit to that of [19, Theorem 4].

Proposition 2.1 *A sequence $F = \{f_i\}_{i\in I}$ is a K-frame if and only if*

$$T_F : \ell^2 \to R(T_F); \quad \{c_i\}_{i\in I} \mapsto \sum_{i\in I} c_i f_i,$$

is a well-defined operator and $R(K) \subseteq R(T_F)$.

Proof First, suppose that F is a K-frame. Then T_F is well defined and bounded by [16, Theorem 5.4.1]. Moreover, the lower K-frame condition follows that

$$\begin{aligned} A\langle KK^*f, f\rangle &= A\|K^*f\|^2 \\ &\leq \|T_F^*f\|^2 = \langle T_F T_F^* f, f\rangle. \end{aligned}$$

Applying Proposition 1.1 yields

$$R(K) \subseteq R(T_F).$$

For the opposite direction, suppose that T_F is a well-defined operator from ℓ^2 to $R(T_F)$. Then [16, Lemma 3.1.1] shows that F is a Bessel sequence. Assume that $T_F^\dagger : R(T_F) \to \ell^2$ is the pseudoinverse of T_F. Since $R(K) \subseteq R(T_F)$, for every $f \in \mathcal{H}$ we obtain

$$Kf = T_F T_F^\dagger K f.$$

This follows that

$$\begin{aligned} \|K^*f\|^4 &= |\langle K^*f, K^*f\rangle|^2 \\ &= \left|\left\langle K^*(T_F^\dagger)^* T_F^* f, K^*f\right\rangle\right|^2 \\ &\leq \|K^*\|^2 \|T_F^\dagger\|^2 \|T_F^* f\|^2 \|K^*f\|^2. \end{aligned}$$

Hence,

$$\frac{1}{\|T_F^\dagger\|^2\|K\|^2}\|K^*f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

□

Definition 2.2 *Let $\{f_i\}_{i\in I}$ be a Bessel sequence. A Bessel sequence $\{g_i\}_{i\in I}\subseteq\mathcal{H}$ is called a K-dual of $\{f_i\}_{i\in I}$ if*

$$Kf=\sum_{i\in I}\langle f,g_i\rangle\pi_{R(K)}f_i,\quad (f\in\mathcal{H}). \tag{2.3}$$

An approach to the K-duals of a K-frame can be found in [1]. Notice that K-duals of [1] satisfy (2.3). In addition, the K-duals introduced by (2.3) covers a larger class than the K-duals of [1].

Lemma 2.3 *[1] If $G:=\{g_i\}_{i\in I}$ is a K-dual of a Bessel sequence $F:=\{f_i\}_{i\in I}$ in $\mathcal{H}$ with Bessel bounds B_F and B_G, respectively. Then $\{g_i\}_{i\in I}$ is a K^*-frame with lower bound B_F^{-1} and $\left\{\pi_{R(K)}f_i\right\}_{i\in I}$ is a K-frame for $\mathcal{H}$ with bounds B_G^{-1} and B_F, respectively.*

Using (2.3) and the similar argument in [1, Proposition 2.3] we can represent a K-dual for every K-frame.

Proposition 2.4 *Let $K\in B(\mathcal{H})$ have closed range and $F=\{f_i\}_{i\in I}$ be a K-frame with bounds A and B, respectively. Then $\left\{K^*(S_F\mid_{R(K)})^{-1}\pi_{S_F(R(K))}f_i\right\}_{i\in I}$ is a K-dual of F with the bounds B^{-1} and $BA^{-1}\|K\|^2\|K^\dagger\|^2$, respectively.*

Proof First note that $S_F|_{R(K)}:R(K)\to S_F(R(K))$ is invertible by (2.2). It follows that $\left\{K^*(S_F\mid_{R(K)})^{-1}\pi_{S_F(R(K))}f_i\right\}_{i\in I}$ is a Bessel sequence. Moreover,

$$\begin{aligned}\left\langle S_F|_{R(K)}f,g\right\rangle &= \left\langle\sum_{i\in I}\left\langle\pi_{R(K)}f,f_i\right\rangle f_i,g\right\rangle\\ &= \left\langle f,\sum_{i\in I}\left\langle g,f_i\right\rangle\pi_{R(K)}f_i\right\rangle,\end{aligned}$$

for all $f\in R(K)$ and $g\in S_F(R(K))$. Thus,

$$(S_F\mid_{R(K)})^*g=\sum_{i\in I}\langle g,f_i\rangle\pi_{R(K)}f_i. \tag{2.4}$$

Thus,

$$\begin{aligned}Kf &= (S_F\mid_{R(K)})^{-1}S_F\mid_{R(K)}Kf\\ &= (S_F|_{R(K)})^*\left((S_F\mid_{R(K)})^{-1}\right)^*Kf\\ &= \sum_{i\in I}\left\langle\left((S_F\mid_{R(K)})^{-1}\right)^*Kf,f_i\right\rangle\pi_{R(K)}f_i\\ &= \sum_{i\in I}\left\langle f,K^*(S_F\mid_{R(K)})^{-1}\pi_{S_F(R(K))}f_i\right\rangle\pi_{R(K)}f_i,\end{aligned}$$

for all $f\in\mathcal{H}$. Thus, $\left\{K^*(S_F\mid_{R(K)})^{-1}\pi_{S_F(R(K))}f_i\right\}_{i\in I}$ is a K-dual of F and with the lower bound of B^{-1}, by the last lemma. On the other hand, by using (2.2) we have

$$\begin{aligned}\sum_{i\in I}\left|\langle f, K^*(S_F\mid_{R(K)})^{-1}\pi_{S_F(R(K))}f_i\rangle\right|^2 &\leq B\left\|\left((S_F\mid_{R(K)})^{-1}\right)^* Kf\right\|^2\\ &\leq B\left\|(S_F\mid_{R(K)})^{-1}\right\|^2\|Kf\|^2\\ &\leq BA^{-1}\left\|K^\dagger\right\|^2\|K\|^2\|f\|^2;\end{aligned}$$

for all $f\in\mathcal{H}$. This completes the proof. □

The K-dual $\left\{K^*(S_F\mid_{R(K)})^{-1}\pi_{S_F(R(K))}f_i\right\}_{i\in I}$ of $F=\{f_i\}_{i\in I}$, introduced in the above proposition, is called the canonical K-dual of F and represented by $\widetilde{F}$ for brevity.

The relation between discrete frame bounds and its canonical dual bounds does not hold for K-frames, see the following example.

Example 2.5 *Let* $F=\left\{(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\right\}$ *in* $\mathcal{H}=\mathbb{C}^2$ *and* K *be the orthogonal projection onto the subspace spanned by* e_1*, where* $\{e_1,e_2\}$ *is the orthonormal basis of* $\mathbb{C}^2$*. For all* $f=(a,b)\in\mathbb{C}^2$ *we obtain*

$$1\|K^*f\|^2\leq\sum_{i=1}^{3}|\langle f,f_i\rangle|^2=\frac{3}{2}(a^2+b^2)-ab\leq 2\|f\|^2.$$

One can see that $S_F(R(K))=span(\frac{3}{2},\frac{-1}{2})$*. Hence,*

$$\widetilde{F}=\left\{(\frac{-4}{5\sqrt{2}},0),(\frac{2}{5\sqrt{2}},0),(\frac{-4}{5\sqrt{2}},0)\right\}.$$

Therefore,

$$\sum_{i=1}^{3}\left|\left\langle f,\widetilde{f}_i\right\rangle\right|^2=\frac{36}{50}\|Kf\|^2.$$

In discrete frames, every frame and its canonical dual are dual of each other. However, it is not true for K-frames in general. In Example 2.5, we obtain $S_{\widetilde{F}}\mid_{R(K^*)}=span(\frac{36}{50},0)$. It requires easy computations to see that

$$K(S_{\widetilde{F}}\mid_{R(K^*)})^{-1}\pi_{S_{\widetilde{F}}(R(K^*))}(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=(\frac{50^2}{36^2\sqrt{2}},0)\neq(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}).$$

Proposition 2.6 *Let* $K\in B(\mathcal{H})$ *have closed range and* $F=\{f_i\}_{i\in I}$ *be a* K*-frame. Then* $\left\{K^*\pi_{R(K)}f_i\right\}_{i\in I}$ *is a* K*-dual for* $\left\{(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))}f_i\right\}_{i\in I}$*.*

Proof Applying (2.3) we have

$$K^*f=\sum_{i\in I}\langle f,\pi_{R(K)}f_i\rangle\widetilde{f}_i,$$

for all $f \in \mathcal{H}$. On the other hand, $KK^{\dagger}$ is a projection on $R(K)$. Thus,

$$\begin{aligned} Kf &= \pi_{R(K)}(K^{\dagger})^* K^* K f \\ &= \sum_{i\in I} \left\langle Kf, \pi_{R(K)} f_i \right\rangle \pi_{R(K)}(K^{\dagger})^* \widetilde{f}_i \\ &= \sum_{i\in I} \left\langle Kf, \pi_{R(K)} f_i \right\rangle \pi_{R(K)}(S_F|_{R(K)})^{-1} \pi_{S_F(R(K))} f_i. \end{aligned}$$

for all $f \in \mathcal{H}$. □

Theorem 2.7 *Let $F = \{f_i\}_{i\in I}$ be a K-frame and $\sum_{i\in I} \left\langle f, \widetilde{f}_i \right\rangle f_i$ has a representation $\sum_{i\in I} c_i f_i$ for some coefficients $\{c_i\}_{i\in I}$, where $f \in \mathcal{H}$. Then*

$$\sum_{i\in I} |c_i|^2 = \sum_{i\in I} |\langle f, \widetilde{f}_i \rangle|^2 + \sum_{i\in I} |c_i - \langle f, \widetilde{f}_i \rangle|^2.$$

Proof First we claim that $K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))}S_F\left((S_F|_{R(K)})^{-1}\right)^* K$ is the frame operator of the canonical K-dual of F. Indeed,

$$\begin{aligned} S_{\widetilde{F}} f &= \sum_{i\in I} \left\langle f, \widetilde{f}_i \right\rangle \widetilde{f}_i \\ &= \sum_{i\in I} \left\langle f, K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))} f_i \right\rangle K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))} f_i \\ &= K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))} \sum_{i\in I} \left\langle \left((S_F|_{R(K)})^{-1}\right)^* Kf, f_i \right\rangle f_i \\ &= K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))} S_F \left((S_F|_{R(K)})^{-1}\right)^* K, \qquad (f \in \mathcal{H}) \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{i\in I} \left| \left\langle f, \widetilde{f}_i \right\rangle \right|^2 &= \left\langle S_{\widetilde{F}} f, f \right\rangle \\ &= \left\langle K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))} S_F \left((S_F|_{R(K)})^{-1}\right)^* Kf, f \right\rangle. \end{aligned}$$

Using the assumption we have

$$\begin{aligned} \sum_{i\in I} c_i f_i &= \sum_{i\in I} \left\langle f, \widetilde{f}_i \right\rangle f_i \\ &= \sum_{i\in I} \left\langle f, K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))} f_i \right\rangle f_i \\ &= S_F \left((S_F|_{R(K)})^{-1}\right)^* Kf. \end{aligned}$$

Thus,

$$\begin{aligned}\left\langle \sum_{i\in I} c_i f_i, \left((S_F|_{R(K)})^{-1}\right)^* Kf \right\rangle &= \left\langle S_F \left((S_F|_{R(K)})^{-1}\right)^* Kf, \left((S_F|_{R(K)})^{-1}\right)^* Kf \right\rangle \\ &= \left\langle K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))} \left((S_F|_{R(K)})^{-1}\right)^* Kf, f \right\rangle \\ &= \left\langle S_{\widetilde{F}} f, f \right\rangle.\end{aligned}$$

Similarly,

$$\left\langle \left((S_F|_{R(K)})^{-1}\right)^* Kf, \sum_{i\in I} c_i f_i \right\rangle = \left\langle S_{\widetilde{F}} f, f \right\rangle.$$

This follows that

$$\begin{aligned}\sum_{i\in I}\left| c_i - \left\langle f, \widetilde{f}_i \right\rangle \right|^2 &= \sum_{i\in I}\left| c_i - \left\langle \left((S_F|_{R(K)})^{-1}\right)^* Kf, f_i \right\rangle \right|^2 \\ &= \sum_{i\in I}|c_i|^2 - \sum_{i\in I} c_i \left\langle f_i, \left((S_F|_{R(K)})^{-1}\right)^* Kf \right\rangle \\ &\quad - \sum_{i\in I}\overline{c_i}\left\langle \left((S_F|_{R(K)})^{-1}\right)^* Kf, f_i \right\rangle + \sum_{i\in I}\left|\left\langle \left((S_F|_{R(K)})^{-1}\right)^* Kf, f_i \right\rangle\right|^2 \\ &= \sum_{i\in I}|c_i|^2 - \sum_{i\in I} c_i \left\langle f_i, \left((S_F|_{R(K)})^{-1}\right)^* Kf \right\rangle \\ &\quad - \sum_{i\in I}\overline{c_i}\left\langle \left((S_F|_{R(K)})^{-1}\right)^* Kf, f_i \right\rangle + \left\langle S_{\widetilde{F}} f, f \right\rangle \\ &= \sum_{i\in I}|c_i|^2 - 2\left\langle K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))} S_F \left((S_F|_{R(K)})^{-1}\right)^* Kf, f \right\rangle \\ &\quad + \sum_{i\in I}\left|\left\langle f, \widetilde{f}_i \right\rangle\right|^2 \\ &= \sum_{i\in I}|c_i|^2 - \sum_{i\in I}\left|\left\langle f, \widetilde{f}_i \right\rangle\right|^2.\end{aligned}$$

□

As a consequence of [16, Theorem 2.5.3] we obtain the following result.

Corollary 2.8 *Let* $\{f_i\}_{i\in I}$ *be a* K*-frame with the synthesis operator* $T_F : \ell^2 \to \mathcal{H}$*. Then for every* $f \in \mathcal{H}$ *we have*

$$T_F^{\dagger}(S_F \left((S_F|_{R(K)})^{-1}\right)^* Kf) = \{\langle f, \widetilde{f}_i \rangle\}_{i\in I},$$

where $T_F^{\dagger}$ *is the pseudoinverse of* T_F*.*

3. K-frame multiplier

In this section, we introduce the notion of multiplier for K-frames, when $K \in B(\mathcal{H})$. Many properties of ordinary frame multipliers may not hold for K-frame multipliers. Similar differences can be observed between frames and K-frames, see [25].

Definition 3.1 *Let $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be two Bessel sequences and let the symbol $m = \{m_i\}_{i\in I} \in \ell^\infty$. An operator $\mathcal{R} : \mathcal{H} \to \mathcal{H}$ is called a K-right inverse of $\mathbb{M}_{m,\Phi,\Psi}$ if*

$$\mathbb{M}_{m,\Phi,\Psi}\mathcal{R}f = Kf, \qquad (f \in \mathcal{H}),$$

and $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ is called a K-left inverse of $\mathbb{M}_{m,\Phi,\Psi}$ if

$$\mathcal{L}\mathbb{M}_{m,\Phi,\Psi}f = Kf, \qquad (f \in \mathcal{H}).$$

Moreover, a K-inverse is a mapping in $B(\mathcal{H})$ that is both a K-left and a K-right inverse.

By using Proposition 1.1, we give some sufficient and necessary conditions for the K-right invertibility of multipliers. Moreover, similar to ordinary frames, the K-dual systems are investigated by K-right inverse (resp. K-left inverse) of K-frame multipliers.

Proposition 3.2 *Let $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be two Bessel sequences and $m \in \ell^\infty$. The following statements are equivalent:*

1. *$R(K) \subset R(\mathbb{M}_{m,\Phi,\Psi})$.*
2. *$KK^* \leq \lambda^2 \mathbb{M}_{m,\Phi,\Psi}\mathbb{M}^*_{m,\Phi,\Psi}$ for some $\lambda \geq 0$.*
3. *$\mathbb{M}_{m,\Phi,\Psi}$ has a K-right inverse.*

Now, we can show that a K-dual of a K-frame fulfills the lower frame condition.

Lemma 3.3 *Let $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be two Bessel sequences and $m \in \ell^\infty$.*

1. *If $\mathbb{M}_{m,\Phi,\Psi} = K$, then Φ and Ψ are K- frame and K^*-frame, respectively. In particular, if $\mathbb{M}_{1,\Phi,\Psi} = K$, then Ψ is a K-dual of Φ.*
2. *If $\mathbb{M}_{m,\Phi,\Psi}$ has a K-right (resp. K-left) inverse, then Φ (resp. Ψ) is K-frame (resp. K^*-frame).*

Proof (1) Let $\mathbb{M}_{m,\Phi,\Psi} = K$. Then

$$\begin{aligned} \|K^*f\|^4 &= |\langle \mathbb{M}_{m,\Phi,\Psi}K^*f, f\rangle|^2 \\ &= \left|\sum_{i\in I} m_i \langle K^*f, \psi_i\rangle \langle \varphi_i, f\rangle\right|^2 \\ &\leq \sup_{i\in I}|m_i| \|K^*f\|^2 B_\Psi \sum_{i\in I} |\langle \varphi_i, f\rangle|^2 , \end{aligned}$$

for every $f \in \mathcal{H}$. Therefore, Φ is K-frame. Similarly, Ψ is a K^*-frame. In fact,

$$\begin{aligned}\|Kf\|^4 &= \left|\left\langle \mathbb{M}^*_{m,\Phi,\Psi}Kf, f\right\rangle\right|^2 \\ &= \left|\sum_{i\in I}\overline{m_i}\langle Kf,\varphi_i\rangle\langle\psi_i,f\rangle\right|^2 \\ &\leq \sup_{i\in I}|m_i|\|Kf\|^2 B_\Phi \sum_{i\in I}|\langle\psi_i,f\rangle|^2 .\end{aligned}$$

In particular,

$$\begin{aligned}Kf &= \sum_{i\in I}\langle f,\psi_i\rangle\varphi_i \\ &= \sum_{i\in I}\langle f,\psi_i\rangle\pi_{R(K)}\varphi_i.\end{aligned}$$

(2) Let $\mathcal{R}$ be a K-right inverse of $\mathbb{M}_{m,\Phi,\Psi}$. Then

$$\begin{aligned}\|K^*f\|^2 &= \|\mathcal{R}^*\mathbb{M}^*_{m,\Phi,\Psi}f\|^2 \\ &= \|\mathcal{R}^*\mathbb{M}_{\overline{m},\Psi,\Phi}f\|^2 \\ &\leq \|\mathcal{R}^*\|^2\left\|\sum_{i\in I}m_i\langle f,\varphi_i\rangle\psi_i\right\|^2 \\ &\leq \sup_{i\in I}|m_i|\,\|\mathcal{R}\|^2\,B_\Psi\sum_{i\in I}|\langle f,\varphi_i\rangle|^2;\end{aligned}$$

The other case is similar. □

In what follows, we discuss K-left and K-right invertibility of a multiplier.

Theorem 3.4 *Let $\Phi = \{\varphi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be two Bessel sequences. Moreover, let $\mathcal{L}$ (resp. $\mathcal{R}$) be a K-left (resp. K-right) inverse of $\mathbb{M}_{1,\pi_{R(K)}\Phi,\Psi}$ (resp. $\mathbb{M}_{1,\Phi,\pi_{R(K^*)}\Psi}$). Then $\mathcal{L}K$ (resp. $K\mathcal{R}$) is in the form of multipliers.*

Proof It is obvious to check that $\mathcal{L}\Phi$ is a Bessel sequence. Moreover, note that Ψ is a K-dual of $\mathcal{L}\pi_{R(K)}\Phi$. Indeed,

$$\begin{aligned}Kf &= \mathcal{L}\mathbb{M}_{1,\pi_{R(K)}\Phi,\Psi}f \\ &= \sum_{i\in I}\langle f,\psi_i\rangle\,\mathcal{L}\pi_{R(K)}\varphi_i \\ &= \sum_{i\in I}\langle f,\psi_i\rangle\,\pi_{R(K)}\mathcal{L}\pi_{R(K)}\varphi_i, \qquad (f\in\mathcal{H}).\end{aligned}$$

Now, if $\Phi^\dagger$ is any K-dual of Φ, then

$$\begin{aligned}\mathbb{M}_{1,\mathcal{L}\pi_{R(K)}\Phi,\Phi^\dagger} f &= \sum_{i\in I} \left\langle f, \varphi_i^\dagger \right\rangle \mathcal{L}\pi_{R(K)}\varphi_i \\ &= \mathcal{L}\sum_{i\in I} \left\langle f, \varphi_i^\dagger \right\rangle \pi_{R(K)}\varphi_i = \mathcal{L}Kf,\end{aligned}$$

for all $f \in \mathcal{H}$. For the statement for $\mathcal{R}$ we have

$$\begin{aligned}K^* f &= \mathcal{R}^*\mathbb{M}^*_{1,\Phi,\pi_{R(K^*)}\Psi} f \\ &= \mathcal{R}^*\mathbb{M}_{1,\pi_{R(K^*)}\Psi,\Phi} f \\ &= \sum_{i\in I} \langle f, \varphi_i\rangle \mathcal{R}^*\pi_{R(K^*)}\psi_i \\ &= \sum_{i\in I} \langle f, \varphi_i\rangle \pi_{R(K^*)}\mathcal{R}^*\pi_{R(K^*)}\psi_i.\end{aligned}$$

Therefore, Φ is a K^*-dual of $\mathcal{R}^*\pi_{R(K^*)}\Psi$. Furthermore, every K^*-dual Ψ^d of Ψ yields

$$\begin{aligned}\mathbb{M}_{1,\Psi^d,\mathcal{R}^*\pi_{R(K^*)}\Psi} f &= \sum_{i\in I} \left\langle f, \mathcal{R}^*\pi_{R(K^*)}\psi_i\right\rangle \psi_i^d \\ &= \sum_{i\in I} \left\langle \mathcal{R}f, \pi_{R(K^*)}\psi_i\right\rangle \psi_i^d \\ &= (K^*)^*\mathcal{R}f = K\mathcal{R}f.\end{aligned}$$

□

A sequence $F = \{f_i\}_{i\in I}$ of $\mathcal{H}$ is called a minimal K-frame whenever it is a K-frame and for each $\{c_i\}_{i\in I} \in \ell^2$ such that $\sum_{i\in I} c_i f_i = 0$ then $c_i = 0$ for all $i \in I$. A minimal K-frame and its canonical K-dual are not biorthogonal in general. To see this, let $\mathcal{H} = \mathbb{C}^4$ and $\{e_i\}_{i=1}^4$ be the standard orthonormal basis of $\mathcal{H}$. Define $K : \mathcal{H} \to \mathcal{H}$ by

$$K\sum_{i=1}^4 c_i e_i = c_1e_1 + c_1e_2 + c_2e_3.$$

Then $K \in B(\mathcal{H})$ and the sequence $F = \{e_1, e_2, e_3\}$ is a minimal K-frame with the bounds $A = \frac{1}{8}$ and $B = 1$. It is easy to see that $\widetilde{F} = \{e_1, e_1, e_2\}$ is the canonical K-dual of F and $\langle f_1, \widetilde{f_2}\rangle \neq 0$. However, every minimal Bessel sequence; therefore, every minimal K-frame has a biorthogonal sequence in $\mathcal{H}$ by Lemma 5.5.3 of [16]. It is worthwhile to mention that a minimal K-frame may have more than one biorthogonal sequence in $\mathcal{H}$, but it is unique in $\overline{span}_{i\in I}\{f_i\}$.

Let $\Phi = \{\varphi_i\}_{i\in I}$ be a K-frame and $\Psi = \{\psi_i\}_{i\in I}$ a minimal K^*-frame. Then $\mathbb{M}_{1,\pi_{R(K)}\Phi,\Psi}$ (resp. $\mathbb{M}_{1,\Psi,\pi_{R(K)}\Phi}$) has a K-right inverse (resp. K^*-left inverse) in the form of multipliers. Indeed, if $G := \{g_i\}_{i\in I}$

is a biorthogonal sequence for minimal K^*-frame Ψ, then

$$\begin{aligned}\mathbb{M}_{1,\pi_{R(K)}\Phi,\Psi}\mathbb{M}_{1,G,\widetilde{\Phi}}f &= \sum_{i,j\in I}\langle f,\widetilde{\varphi}_i\rangle\langle g_i,\psi_j\rangle\pi_{R(K)}\varphi_j\\ &= \sum_{i\in I}\langle f,\widetilde{\varphi}_i\rangle\pi_{R(K)}\varphi_i = Kf,\end{aligned}$$

for all $f\in\mathcal{H}$. Similarly,

$$\begin{aligned}\mathbb{M}_{1,\widetilde{\Phi},G}\mathbb{M}_{1,\Psi,\pi_{R(K)}\Phi}f &= \sum_{i,j\in I}\left\langle f,\pi_{R(K)}\varphi_i\right\rangle\left\langle\psi_i,g_j\right\rangle\widetilde{\varphi}_j\\ &= \sum_{i\in I}\left\langle f,\pi_{R(K)}\varphi_i\right\rangle\widetilde{\varphi}_i = K^*f.\end{aligned}$$

We use the following lemma for the invertibility of operators, whose proof is left to the reader.

Lemma 3.5 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $T\in B(\mathcal{H}_1,\mathcal{H}_2)$ be invertible. Suppose $U\in B(\mathcal{H}_1,\mathcal{H}_2)$ such that $\|T-U\|<\|T^{-1}\|^{-1}$. Then U is also invertible.*

In the rest of this section we state a sufficient condition for the K-right invertibility of $\mathbb{M}_{m,\Psi,\Phi}$, whenever Ψ is a perturbation of Φ.

Theorem 3.6 *Let $\Phi=\{\varphi_i\}_{i\in I}$ be a K-frame with bounds A and B, respectively, and $\Psi=\{\psi_i\}_{i\in I}$ be a Bessel sequence such that*

$$\left(\sum_{i\in I}|\langle f,\psi_i-\varphi_i\rangle|^2\right)^{\frac{1}{2}} < \frac{aA}{b\sqrt{B}\|K^\dagger\|^2}\|f\|, \qquad (f\in R(K)), \tag{3.1}$$

where $m=\{m_i\}_{i\in I}$ is a seminormalized sequence with bounds a and b, respectively. Then

1. *The sequence Ψ has a K-dual. In particular, it is a K-frame.*
2. *$\mathbb{M}_{\overline{m},\Psi,\Phi}$ has a K-right inverse in the form of multipliers.*

Proof (1) Obviously $\Phi^d:=\{\sqrt{m_i}\varphi_i\}_{i\in I}$ is a K-frame for $\mathcal{H}$ with bounds aA and bB, respectively. Denote its frame operator by S_{Φ^d}. Due to (2.2) we obtain $\left\|S_{\Phi^d}^{-1}\right\|\leq\frac{\|K^\dagger\|^2}{aA}$. Moreover, (3.1) follows that

$$\begin{aligned}\|\mathbb{M}_{m,\Phi,\Psi}f-S_{\Phi^d}f\| &= \left\|\sum_{i\in I}m_i\left\langle f,\psi_i-\varphi_i\right\rangle\varphi_i\right\|\\ &\leq b\left(\sum_{i\in I}|\langle f,\psi_i-\varphi_i\rangle|^2\right)^{\frac{1}{2}}\sqrt{B}\\ &< \frac{aA}{\|K^\dagger\|^2}\|f\|\\ &< \frac{1}{\left\|S_{\Phi^d}^{-1}\right\|}\|f\|,\end{aligned}$$

for all $f \in R(K)$. Then $\mathbb{M}_{m,\Phi,\Psi}$ has an inverse on $R(K)$, denoted by $\mathbb{M}^{-1}$, by using Lemma 3.5. Moreover, for $\mathbb{M}_{m,\Phi,\Psi}$ on $R(K)$ we have

$$\begin{aligned} \langle(\mathbb{M}_{m,\Phi,\Psi})^* f, g\rangle &= \langle f, \mathbb{M}_{m,\Phi,\Psi}\pi_{R(K)}g\rangle \\ &= \left\langle f, \sum_{i\in I} m_i \langle \pi_{R(K)}g, \psi_i\rangle \varphi_i \right\rangle \\ &= \left\langle f, \sum_{i\in I} m_i \langle g, \pi_{R(K)}\psi_i\rangle \varphi_i \right\rangle \\ &= \left\langle \sum_{i\in I} \overline{m_i} \langle f, \varphi_i\rangle \pi_{R(K)}\psi_i, g \right\rangle, \end{aligned}$$

for all $f \in \mathbb{M}_{m,\Phi,\Psi}(R(K))$ and $g \in R(K)$. Using this fact, we obtain that

$$\begin{aligned} Kf &= (\mathbb{M}^{-1}\mathbb{M}_{m,\Phi,\Psi})^* Kf \\ &= \mathbb{M}^*_{m,\Phi,\Psi}\pi_{\mathbb{M}_{m,\Phi,\Psi}(R(K))}(\mathbb{M}^{-1})^* Kf \\ &= \sum_{i\in I} \overline{m_i} \langle \pi_{\mathbb{M}_{m,\Phi,\Psi}(R(K))}(\mathbb{M}^{-1})^* Kf, \varphi_i\rangle \pi_{R(K)}\psi_i \\ &= \sum_{i\in I} \langle f, K^*\mathbb{M}^{-1}\pi_{\mathbb{M}_{m,\Phi,\Psi}(R(K))} m_i\varphi_i\rangle \pi_{R(K)}\psi_i. \end{aligned}$$

Hence, $\{K^*\mathbb{M}^{-1}_{m,\Phi,\Psi}\pi_{\mathbb{M}_{m,\Phi,\Psi}(R(K))} m_i\varphi_i\}_{i\in I}$ is a K-dual of $\Psi := \{\psi_i\}_{i\in I}$.

(2) The above computations shows that $(\mathbb{M}^{-1})^* K$ is a K-right inverse of $\mathbb{M}_{\overline{m},\Psi,\Phi}$. Indeed,

$$\begin{aligned} Kf &= (\mathbb{M}^{-1}\mathbb{M}_{m,\Phi,\Psi})^* Kf \\ &= \mathbb{M}^*_{m,\Phi,\Psi}(\mathbb{M}^{-1})^* Kf = \mathbb{M}_{\overline{m},\Psi,\Phi}(\mathbb{M}^{-1})^* Kf. \end{aligned}$$

On the other hand, for every K-dual Φ^d of Φ we have

$$\begin{aligned} \mathbb{M}_{1,(\mathbb{M}^{-1})^*\pi_{R(K)}\Phi,\Phi^d} f &= \sum_{i\in I} \langle f, \varphi_i^d\rangle (\mathbb{M}^{-1})^*\pi_{R(K)}\varphi_i \\ &= (\mathbb{M}^{-1})^* Kf, \end{aligned}$$

for all $f \in \mathcal{H}$. This completes the proof. □

The next theorem determines a class of multipliers which are K-right invertible and whose K-right inverse can be written as a multiplier.

Theorem 3.7 *Let $\Psi = \{\psi_i\}_{i\in I}$ be a K-frame and $\Phi = \{\varphi_i\}_{i\in I}$ a K^*-frame. Then the following assertions hold.*

1. *If $R(T^*_\Psi) \subseteq R(T^*_\Phi K^*)$, then $\mathbb{M}_{1,\pi_{R(K)}\Psi,K\Phi}$ has a K-right inverse in the form of multipliers.*
2. *If $R(T^*_\Phi K^*) \subseteq R(T^*_\Psi)$, then $\mathbb{M}_{1,\pi_{R(K)}\Psi,K\Phi}$ has a K^*-left inverse in the form of multipliers.*

3. *If* $R(T_\Phi^* K^*) = R(T_\Psi^*)$*, then* $\mathbb{M}_{1,\pi_{R(K)}\Psi,K\Phi}$ *has* K*-right inverse and* K^**-left inverse in the form of multipliers.*

Proof

(1) One can see that the sequence $(K^\dagger)^* S_\Phi^{-1}\pi_{S_\Phi R(K^*)}\Phi$ denoted by $\Phi^\dagger$ is a Bessel sequence. Then for all $f \in \mathcal{H}$ we have

$$\begin{aligned}
\mathbb{M}_{1,\Phi^\dagger,\widetilde{\Psi}} f &= \sum_{i\in I} \left\langle f, K^* S_\Psi^{-1}\pi_{S_\Psi R(K)}\psi_i \right\rangle (K^\dagger)^* S_\Phi^{-1}\pi_{S_\Phi R(K^*)}\varphi_i \\
&= (K^\dagger)^* S_\Phi^{-1}\pi_{S_\Phi R(K^*)} \sum_{i\in I} \left\langle (S_\Psi^{-1})^* K f, \psi_i \right\rangle \varphi_i \\
&= (K^\dagger)^* S_\Phi^{-1}\pi_{S_\Phi R(K^*)} T_\Phi T_\Psi^* (S_\Psi^{-1})^* K f.
\end{aligned}$$

Applying Proposition 1.1, there exists $X \in B(\mathcal{H})$ so that $T_\Psi^* = T_\Phi^* K^* X$. Since Φ is a K^*-frame we obtain

$$S_\Phi^{-1}\pi_{S_\Phi R(K^*)} S_\Phi K^* = S_\Phi^{-1} S_\Phi K^* = K^*.$$

Moreover, $KK^\dagger K = K$ and (2.4) follow that

$$\begin{aligned}
\mathbb{M}_{1,\pi_{R(K)}\Psi,K\Phi}\mathbb{M}_{1,\Phi^\dagger,\widetilde{\Psi}} &= \pi_{R(K)} T_\Psi T_\Phi^* K^* (K^\dagger)^* S_\Phi^{-1}\pi_{S_\Phi R(K^*)} T_\Phi T_\Psi^* (S_\Psi^{-1})^* K \\
&= \pi_{R(K)} T_\Psi T_\Phi^* K^* (K^\dagger)^* S_\Phi^{-1}\pi_{S_\Phi R(K^*)} T_\Phi T_\Phi^* K^* X (S_\Psi^{-1})^* K \\
&= \pi_{R(K)} T_\Psi T_\Phi^* K^* (K^\dagger)^* K^* X (S_\Psi^{-1})^* K \\
&= \pi_{R(K)} T_\Psi T_\Psi (S_\Psi^{-1})^* K \\
&= \left(S_\Psi|_{R(K)}\right)^* (S_\Psi^{-1})^* K = K.
\end{aligned}$$

(2) One can see that $S_\Phi^{-1}\pi_{S_\Phi R(K^*)}\Phi$ and $S_\Psi^{-1}\pi_{S_\Psi R(K)}\Psi$ denoted by $\Phi^\ddagger$ and $\Psi^\ddagger$, respectively, are Bessel sequences in $\mathcal{H}$. Thus, for all $f \in R(K)$ we obtain

$$\mathbb{M}_{1,\Phi^\ddagger,\Psi^\ddagger} f = S_\Phi^{-1}\pi_{S_\Phi R(K^*)} T_\Phi T_\Psi^* \left(S_\Psi^{-1}\right)^* f.$$

There is an operator $X \in B(\mathcal{H})$ such that $T_\Phi^* K^* = T_\Psi^* X$ by Proposition 1.1. Therefore,

$$\begin{aligned}
\mathbb{M}_{1,\Phi^\ddagger,\Psi^\ddagger}\mathbb{M}_{1,\pi_{R(K)}\Psi,K\Phi} &= S_\Phi^{-1}\pi_{S_\Phi R(K^*)} T_\Phi T_\Psi^* \left(S_\Psi^{-1}\right)^* \pi_{R(K)} T_\Psi T_\Phi^* K^* \\
&= S_\Phi^{-1}\pi_{S_\Phi R(K^*)} T_\Phi T_\Psi^* \left(S_\Psi^{-1}\right)^* \pi_{R(K)} T_\Psi T_\Psi^* X \\
&= S_\Phi^{-1}\pi_{S_\Phi R(K^*)} T_\Phi T_\Psi^* \left(S_\Psi^{-1}\right)^* \left(S_\Psi|_{R(K)}\right)^* X \\
&= S_\Phi^{-1}\pi_{S_\Phi R(K^*)} T_\Phi T_\Psi^* X \\
&= S_\Phi^{-1}\pi_{S_\Phi R(K^*)} T_\Phi T_\Phi^* K^* \\
&= S_\Phi^{-1}\pi_{S_\Phi R(K^*)} S_\Phi K^* = K^*.
\end{aligned}$$

□

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