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Bounds for a new subclass of bi-univalent functions subordinate to the Fibonacci numbers

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Abstract: In this investigation, by using a relation of subordination, we define a new subclass of analytic bi-univalent functions associated with the Fibonacci numbers. Moreover, we survey the bounds of the coefficients for functions in this class.

Key words: Bi-univalent functions, Fibonacci numbers, subordination.

1. Introduction and background

Let $\mathbb{C}$ be the complex plane and let $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, the open unit disc. Further, by $\mathcal{A}$ we represent the class of functions analytic in $\mathbb{U}$, satisfying the condition

$$f(0) = f'(0) - 1 = 0.$$

Thus each function f in $\mathcal{A}$ has a Taylor series representation

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1.1}$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{U}$. The Carathéodory class, consisting of the functions p analytic in $\mathbb{U}$ satisfying $p(0) = 1$ and $\Re p(z) > 0$, is usually denoted by $\mathcal{P}$. Indeed, $p \in \mathcal{P}$ has a representation

$$p(z) = 1 + x_1 z + x_2 z^2 + x_3 z^3 + \cdots \quad (x_1 > 0)$$

with coefficients satisfying $|x_n| \leq 2$ $(n \in \mathbb{N})$ (see [13], [7]).

We now recall that the analytic function f is said to be *subordinate* to the analytic function g (indicated as $f \prec g$), if there exists a Schwarz function

$$\varpi(z) = \sum_{n=1}^{\infty} \mathfrak{c}_n z^n \quad (\varpi(0) = 0, \ |\varpi(z)| < 1),$$

analytic in $\mathbb{U}$ such that

$$f(z) = g(\varpi(z)) \qquad (z \in \mathbb{U}).$$

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For the function $\varpi(z)$ we know that $|\mathfrak{c}_n| < 1$ (see [6]).

We next turn to the Koebe-One Quarter Theorem which ensures that every univalent function $f \in \mathcal{A}$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$ and $f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f);\quad r_0(f) \geq \frac{1}{4}\right)$, where

$$\begin{aligned} g(w) = f^{-1}(w) = w \; &- a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 \\ &- \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \end{aligned} \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

Bi-univalent functions have been studied since the mid-1990s, and thousands of research papers have been written about them (see e.g., [4, 11, 12] and see also the references cited therein). After that, bounds for the first few coefficients $|a_2|$, $|a_3|$ of various subclasses of bi-univalent functions have been obtained by a number of sequels to [15] including (among others) [1, 9, 10, 16]. However, in the literature, there are only a few works (by making use of the Faber polynomial expansions) determining the general coefficient bounds $|a_n|$ for bi-univalent functions ([2, 3, 8, 14]). Hence, determination of the bounds for each of

$$|a_n| \qquad (\, n \in \mathbb{N} \backslash \{1, 2\}\,;\quad \mathbb{N} = \{1, 2, 3, ...\})$$

is still an open problem for functions in the class Σ.

By using a relation of subordination, we define a new subclass of bi-univalent functions associated with the Fibonacci numbers.

Definition 1.1 *A function $f \in \Sigma$ is said to be in the class*

$$W_\Sigma\left(\mu, \rho; \widetilde{\mathfrak{p}}\right) \qquad (\mu \geq 0, \rho \geq 0;\; z, w \in \mathbb{U})$$

if the following subordination relationships are satisfied:

$$\left[(1 - \mu + 2\rho)\frac{f(z)}{z} + (\mu - 2\rho)f'(z) + \rho z f''(z)\right] \prec \widetilde{\mathfrak{p}}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

and

$$\left[(1 - \mu + 2\rho)\frac{g(w)}{w} + (\mu - 2\rho)g'(w) + \rho w g''(w)\right] \prec \widetilde{\mathfrak{p}}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2},$$

where $g = f^{-1}$ and $\tau = \frac{1-\sqrt{5}}{2}$ ≈ -0.618.

It is interesting to note that the special values of μ and ρ lead the class $W_\Sigma\left(\mu, \rho; \widetilde{\mathfrak{p}}\right)$ to various subclasses, we illustrate the following subclasses:

1. For $\mu = 1 + 2\rho$, we get the class $W_\Sigma\left(1 + 2\rho, \rho; \widetilde{\mathfrak{p}}\right) = W_\Sigma\left(\rho; \widetilde{\mathfrak{p}}\right)$. A function $f \in \Sigma$ is said to be in the class

   $$W_\Sigma\left(\rho; \widetilde{\mathfrak{p}}\right) \qquad (\rho \geq 0;\; z, w \in \mathbb{U})$$

   if the following subordinations are satisfied:

   $$\left[f'(z) + \rho z f''(z)\right] \prec \widetilde{\mathfrak{p}}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}$$

and

$$[g'(w) + \rho w g''(w)] \prec \widetilde{\mathfrak{p}}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}.$$

2. For $\rho = 0$, we obtain the class $W_\Sigma\left(\mu,0;\widetilde{\mathfrak{p}}\right) = W_\Sigma\left(\mu;\widetilde{\mathfrak{p}}\right)$. A function $f \in \Sigma$ is said to be in the class

$$W_\Sigma\left(\mu;\widetilde{\mathfrak{p}}\right) \quad (\mu \geq 0;\ z,w \in \mathbb{U})$$

if the following subordinations are satisfied:

$$\left[(1-\mu)\frac{f(z)}{z} + \mu f'(z)\right] \prec \widetilde{\mathfrak{p}}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$$

and

$$\left[(1-\mu)\frac{g(w)}{w} + \mu g'(w)\right] \prec \widetilde{\mathfrak{p}}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}.$$

3. For $\rho = 0$ and $\mu = 1$, we get the class $W_\Sigma\left(1,0;\widetilde{\mathfrak{p}}\right) = W_\Sigma\left(\widetilde{\mathfrak{p}}\right)$. A function $f \in \Sigma$ is said to be in the class

$$W_\Sigma\left(\widetilde{\mathfrak{p}}\right) \quad (z,w \in \mathbb{U})$$

if the following subordinations are satisfied:

$$f'(z) \prec \widetilde{\mathfrak{p}}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$$

and

$$g'(w) \prec \widetilde{\mathfrak{p}}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}.$$

Remark 1.2 *The function $\widetilde{\mathfrak{p}}(z)$ is not univalent in $\mathbb{U}$, it is univalent in the disc $|z| < \frac{3-\sqrt{5}}{2} \approx 0.38$. Observe that $\widetilde{\mathfrak{p}}(0) = \widetilde{\mathfrak{p}}\left(-\frac{1}{2\tau}\right)$ and $\widetilde{\mathfrak{p}}\left(e^{\pm i \arccos(1/4)}\right) = \frac{\sqrt{5}}{5}$. Also, it can be written as*

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|}$$

which indicates that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section (see for details [5, 17]).

Additionally, Dziok et al. [5] indicate a connection between the function $\widetilde{\mathfrak{p}}(z)$ and the Fibonacci numbers. Let $\{F_n\}$ be the sequence of Fibonacci numbers

$$F_{n+2} = F_n + F_{n+1} \qquad (n \in \mathbb{N}_0 = \{0,1,2,\ldots\})$$

with $F_0 = 0,\ F_1 = 1$, then

$$F_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \quad \tau = \frac{1-\sqrt{5}}{2}.$$

If we set

$$\widetilde{\mathfrak{p}}\,(z) \quad = \quad 1+\sum_{n=1}^{\infty}\widetilde{\mathfrak{p}}_n z^n = 1+(F_0+F_2)\tau z+(F_1+F_3)\tau^2 z^2$$

$$+\sum_{n=3}^{\infty}(F_{n-3}+F_{n-2}+F_{n-1}+F_n)\tau^n z^n,$$

then we arrive at

$$\widetilde{\mathfrak{p}}_n = \begin{cases} \tau & (n=1) \\ 3\tau^2 & (n=2) \\ \tau\widetilde{\mathfrak{p}}_{n-1}+\tau^2\widetilde{\mathfrak{p}}_{n-2} & (n=3,4,\ldots) \end{cases}. \tag{1.3}$$

2. Inequalities for the Taylor–Maclaurin coefficients

In this part, we offer to get the upper bounds on the Taylor–Maclaurin coefficients and obtain the Fekete–Szegö inequalities for functions in the bi-univalent function class $W_{\Sigma}\left(\mu,\rho;\widetilde{\mathfrak{p}}\right)$.

Theorem 2.1 *Let the function f given by (1.1) be in the class $W_{\Sigma}\left(\mu,\rho;\widetilde{\mathfrak{p}}\right)$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{\left|(1+\mu)^2+\left[(1+2\mu+2\rho)-3\,(1+\mu)^2\right]\tau\right|}},$$

$$|a_3| \leq \frac{\tau^2}{(1+\mu)^2}+\frac{|\tau|}{1+2\mu+2\rho},$$

for any real number η,

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{1+2\mu+2\rho}, & |\eta-1| \leq \frac{\left|(1+\mu)^2+\left[(1+2\mu+2\rho)-3(1+\mu)^2\right]\tau\right|}{(1+2\mu+2\rho)|\tau|} \\ \frac{|1-\eta|\tau^2}{\left|(1+\mu)^2+\left[(1+2\mu+2\rho)-3(1+\mu)^2\right]\tau\right|}, & |\eta-1| \geq \frac{\left|(1+\mu)^2+\left[(1+2\mu+2\rho)-3(1+\mu)^2\right]\tau\right|}{(1+2\mu+2\rho)|\tau|} \end{cases}.$$

Proof Suppose that $f \in W_{\Sigma}\left(\mu,\rho;\widetilde{\mathfrak{p}}\right)$. Firstly, let $p \prec \widetilde{\mathfrak{p}}$. Then, by the relation of subordination, for the analytic functions $\mathfrak{u},\mathfrak{v}$ such that $\mathfrak{u}(0)=\mathfrak{v}(0)=0,\ |\mathfrak{u}(z)|<1, |\mathfrak{v}(w)|<1\ \ (z,w\in\mathbb{U})$, we can write

$$\left[(1-\mu+2\rho)\frac{f(z)}{z}+(\mu-2\rho)f'(z)+\rho z f''(z)\right]=\widetilde{\mathfrak{p}}\,(\mathfrak{u}(z)) \tag{2.1}$$

and

$$\left[(1-\mu+2\rho)\frac{g(w)}{w}+(\mu-2\rho)g'(w)+\rho w g''(w)\right]=\widetilde{\mathfrak{p}}\,(\mathfrak{v}(w))\,. \tag{2.2}$$

Next, define the functions p_1 and p_2 by

$$p_1(z)=\frac{1+\mathfrak{u}(z)}{1-\mathfrak{u}(z)}=1+x_1 z+x_2 z^2+\cdots,$$

$$p_2(w) = \frac{1+\mathfrak{v}(w)}{1-\mathfrak{v}(w)} = 1 + y_1 w + y_2 w^2 + \cdots .$$

Since $\mathfrak{u}$ and $\mathfrak{v}$ are Schwarz functions, p_1 and p_2 are analytic functions in $\mathbb{U}$ (with $p_1(0) = p_2(0) = 1$), we obtain the equations

$$\mathfrak{u}(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[x_1 z + \left(x_2 - \frac{x_1^2}{2}\right) z^2\right] + \cdots ,$$

$$\mathfrak{v}(w) = \frac{p_2(w)-1}{p_2(w)+1} = \frac{1}{2}\left[y_1 w + \left(y_2 - \frac{y_1^2}{2}\right) w^2\right] + \cdots$$

lead to

$$\widetilde{\mathfrak{p}}\left(\mathfrak{u}(z)\right) = 1 + \frac{\widetilde{\mathfrak{p}}_1 x_1}{2} z + \left[\frac{1}{2}\left(x_2 - \frac{x_1^2}{2}\right)\widetilde{\mathfrak{p}}_1 + \frac{x_1^2}{4}\widetilde{\mathfrak{p}}_2\right] z^2 + \cdots ,$$

$$\widetilde{\mathfrak{p}}\left(\mathfrak{v}(w)\right) = 1 + \frac{\widetilde{\mathfrak{p}}_1 y_1}{2} w + \left[\frac{1}{2}\left(y_2 - \frac{y_1^2}{2}\right)\widetilde{\mathfrak{p}}_1 + \frac{y_1^2}{4}\widetilde{\mathfrak{p}}_2\right] w^2 + \cdots .$$

Now, upon comparing the corresponding coefficients in (2.1) and (2.2), we get

$$(1+\mu)a_2 = \frac{\widetilde{\mathfrak{p}}_1 x_1}{2}, \tag{2.3}$$

$$(1+2\mu+2\rho)a_3 = \frac{1}{2}\left(x_2 - \frac{x_1^2}{2}\right)\widetilde{\mathfrak{p}}_1 + \frac{x_1^2}{4}\widetilde{\mathfrak{p}}_2, \tag{2.4}$$

$$-(1+\mu)a_2 = \frac{\widetilde{\mathfrak{p}}_1 y_1}{2}, \tag{2.5}$$

$$(1+2\mu+2\rho)\left(2a_2^2 - a_3\right) = \frac{1}{2}\left(y_2 - \frac{y_1^2}{2}\right)\widetilde{\mathfrak{p}}_1 + \frac{y_1^2}{4}\widetilde{\mathfrak{p}}_2. \tag{2.6}$$

From equations (2.3) and (2.5), one can easily find that

$$x_1 = -y_1, \tag{2.7}$$

$$2(1+\mu)^2 a_2^2 = \frac{\widetilde{\mathfrak{p}}_1^2}{4}(x_1^2 + y_1^2). \tag{2.8}$$

If we add (2.4) to (2.6), we obtain

$$2(1+2\mu+2\rho)a_2^2 = \frac{\widetilde{\mathfrak{p}}_1}{2}\left(x_2 + y_2\right) + \frac{\left(\widetilde{\mathfrak{p}}_2 - \widetilde{\mathfrak{p}}_1\right)}{4}\left(x_1^2 + y_1^2\right). \tag{2.9}$$

By making the use of (2.8) in (2.9), we have

$$a_2^2 = \frac{\widetilde{\mathfrak{p}}_1^3\left(x_2 + y_2\right)}{4\left\{(1+2\mu+2\rho)\widetilde{\mathfrak{p}}_1^2 - (1+\mu)^2(\widetilde{\mathfrak{p}}_2 - \widetilde{\mathfrak{p}}_1)\right\}} \tag{2.10}$$

which yields

$$|a_2| \leq \frac{|\tau|}{\sqrt{\left|(1+\mu)^2 + \left[(1+2\mu+2\rho) - 3(1+\mu)^2\right]\tau\right|}}.$$

Next, if we subtract (2.6) from (2.4), we obtain

$$2(1+2\mu+2\rho)\left(a_3 - a_2^2\right) = \frac{\widetilde{\mathfrak{p}}_1}{2}\left(x_2 - y_2\right). \tag{2.11}$$

Then, in view of (2.8), the equation (2.11) becomes

$$a_3 = \frac{\widetilde{\mathfrak{p}}_1^2(x_1^2 + y_1^2)}{8(1+\mu)^2} + \frac{\widetilde{\mathfrak{p}}_1\left(x_2 - y_2\right)}{4(1+2\mu+2\rho)}.$$

By using triangle inequality for the modulus, we obtain

$$|a_3| \leq \frac{\tau^2}{(1+\mu)^2} + \frac{|\tau|}{1+2\mu+2\rho}.$$

Notice that from (2.10) and (2.11), we can compute that

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{(1-\eta)^2\widetilde{\mathfrak{p}}_1^3\left(x_2+y_2\right)}{4\left\{(1+2\mu+2\rho)\widetilde{\mathfrak{p}}_1^2 - (1+\mu)^2(\widetilde{\mathfrak{p}}_2 - \widetilde{\mathfrak{p}}_1)\right\}} + \frac{\widetilde{\mathfrak{p}}_1\left(x_2 - y_2\right)}{4(1+2\mu+2\rho)} \\ &= \frac{\widetilde{\mathfrak{p}}_1}{4}\left[\left(h(\eta) + \tfrac{1}{1+2\mu+2\rho}\right)x_2 + \left(h(\eta) - \frac{1}{1+2\mu+2\rho}\right)y_2\right], \end{aligned}$$

where

$$h(\eta) = \frac{(1-\eta)\widetilde{\mathfrak{p}}_1^2}{(1+2\mu+2\rho)\widetilde{\mathfrak{p}}_1^2 - (1+\mu)^2(\widetilde{\mathfrak{p}}_2 - \widetilde{\mathfrak{p}}_1)}.$$

This enables us to conclude that

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{\left|\widetilde{\mathfrak{p}}_1\right|}{1+2\mu+2\rho}, & 0 \leq |h(\eta)| \leq \dfrac{1}{1+2\mu+2\rho} \\ |h(\eta)|\left|\widetilde{\mathfrak{p}}_1\right|, & |h(\eta)| \geq \dfrac{1}{1+2\mu+2\rho} \end{cases}.$$

Theorem 3 is proved. □

3. Consequences and observations

In this investigation, we studied the analytic bi-univalent function class

$$W_\Sigma\left(\mu, \rho; \widetilde{\mathfrak{p}}\right) \quad (\mu \geq 0, \rho \geq 0;\ z, w \in \mathbb{U})$$

associated with the Fibonacci numbers. For functions belonging to this class, we have derived Taylor–Maclaurin coefficient inequalities and the celebrated Fekete–Szegö problem. The geometric properties of the function class $W_\Sigma\left(\mu, \rho; \widetilde{\mathfrak{p}}\right)$ vary according to the values according to the parameters included. This approach has been extended to find more examples of bi-univalent functions with the Fibonacci numbers.

Corollary 3.1 *Let the function f given by (1.1) be in the class $W_\Sigma\left(\rho;\widetilde{\mathfrak{p}}\right)$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{\left|4\left(1+\rho\right)^2+3\left[\left(1+2\rho\right)-4\left(1+\rho\right)^2\right]\tau\right|}},$$

$$|a_3| \leq \frac{|\tau|}{3(1+2\rho)}+\frac{\tau^2}{4(1+\rho)^2},$$

for any real number η,

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{3(1+2\rho)}, & |\eta-1| \leq \frac{\left|4(1+\rho)^2+3\left[(1+2\rho)-4(1+\rho)^2\right]\tau\right|}{3(1+2\rho)|\tau|} \\ \frac{|1-\eta|\tau^2}{\left|4(1+\rho)^2+3\left[(1+2\rho)-4(1+\rho)^2\right]\tau\right|}, & |\eta-1| \geq \frac{\left|4(1+\rho)^2+3\left[(1+2\rho)-4(1+\rho)^2\right]\tau\right|}{3(1+2\rho)|\tau|} \end{cases}.$$

Corollary 3.2 *Let the function f given by (1.1) be in the class $W_\Sigma\left(\mu;\widetilde{\mathfrak{p}}\right)$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{\left|\left(1+\mu\right)^2+\left[\left(1+2\mu\right)-3\left(1+\mu\right)^2\right]\tau\right|}},$$

$$|a_3| \leq \frac{|\tau|}{1+2\mu}+\frac{\tau^2}{(1+\mu)^2},$$

for any real number η,

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{1+2\mu}, & |\eta-1| \leq \frac{\left|(1+\mu)^2+\left[(1+2\mu)-3(1+\mu)^2\right]\tau\right|}{(1+2\mu)|\tau|} \\ \frac{|1-\eta|\tau^2}{\left|(1+\mu)^2+\left[(1+2\mu)-3(1+\mu)^2\right]\tau\right|}, & |\eta-1| \geq \frac{\left|(1+\mu)^2+\left[(1+2\mu)-3(1+\mu)^2\right]\tau\right|}{(1+2\mu)|\tau|} \end{cases}.$$

Corollary 3.3 *Let the function f given by (1.1) be in the class $W_\Sigma\left(\widetilde{\mathfrak{p}}\right)$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{|4-9\tau|}},$$

$$|a_3| \leq \frac{|\tau|}{3}+\frac{\tau^2}{4},$$

for any real number η,

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{3}, & |\eta-1| \leq \dfrac{|4-9\tau|}{3\,|\tau|} \\ \dfrac{|1-\eta|\,\tau^2}{|4-9\tau|}, & |\eta-1| \geq \dfrac{|4-9\tau|}{3\,|\tau|} \end{cases}.$$

If we restrict our considerations for a given univalent function $\widetilde{\mathfrak{p}}(z)$ in $\mathbb{U}$, we can examine mapping problems for other regions of the complex z-plane. Thus, one can define different subclasses of the function class which we have studied in this paper.

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