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On solvability of inverse problem for one equation of fourth order

Aysel RAMAZANOVA[1,*], **Yashar MEHRALİYEV**[1,2]

[1]Department of Nonlinear Optimization, Faculty of Mathematics, University Duisburg-Essen, Essen, Germany
[2]Department of Mechanics-Mathematics, Faculty of Mathematics, Baku State University, Baku, Azerbaijan

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Abstract: The work is devoted to study the existence and uniqueness of the classical solution of the inverse boundary value problem of determining the lowest coefficient in one fourth order equation. The original problem is reduced to an equivalent problem. The existence and uniqueness of the integral equation are proved by means of the contraction mappings principle, and we obtained that this solution is unique for a boundary value problem. Further, using these facts, we prove the existence and uniqueness of the classical solution for this problem.

Key words: Inverse boundary value problem, fourth order equation, Fourier method, classical solution

1. Introduction

Modern problems of natural science lead to study qualitatively new problems, a vivid example of which is the class of nonlocal problems for partial differential equations. The study of such problems is caused by both theoretical interest and practical necessity. There are many cases where the needs of practice lead to the problem of determining the coefficients or the right side of a differential equation from some known data from its solution. Such problems are called inverse problems of mathematical physics. If the the solution and the right-hand side of equations are unknown, then the inverse problem will be linear; if the solution and at least one of the coefficients are unknown, then the inverse problem will be nonlinear.

Among non-local problems, of great interest are problems with integral conditions. Such integral conditions appear in the mathematical modeling of phenomena associated with a physical plasma [18], the spread of heat [2, 6], and the process of moisture transfer in capillary-simple media [7], issues of demography and mathematical biology, as well as in the study of some inverse problems of mathematical physics. Questions of solvability of problems with non-local integral conditions for partial differential equations are studied in the papers [4, 8, 11]. Inverse problems with an integral redefinition condition for partial differential equations were studied in [5, 9, 13–17]. The purpose of this work is to prove the uniqueness and existence of solutions of the inverse boundary value problem for a single fourth-order equation with integral condition.

2. Problem statement and its reduction to an equivalent task

Consider for equation [3]

$$u_{tt}(x,t) - u_{xxxx}(x,t) = a(t)u(x,t) + f(x,t), \tag{2.1}$$

*Correspondence: aysel.ramazanova@uni-due.de

2010 *AMS Mathematics Subject Classification:* 35L57, 65M32 in the domain $D_T = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ an inverse boundary problem with boundary conditions

$$u(x,0) = \varphi(x), u_t(x,T) = \psi(x)(0 \le x \le 1), \tag{2.2}$$

with periodic conditions

$$u(0,t) = u(1,t), u_x(0,t) = u_x(1,t), u_{xx}(0,t) = u_{xx}(1,t)(0 \le t \le T), \tag{2.3}$$

with nonlocal integral condition

$$\int_0^1 u(x,t)dx = 0(0 \le t \le T), \tag{2.4}$$

and with an additional condition

$$u(x_0,t) = h(t)(0 \le t \le T), \tag{2.5}$$

where $x_0 \in (0,1)$ –-fixed number, $f(x,t)$, $\varphi(x), \psi(x), h(t)$ -–given functions, $u(x,t)$ and $a(t)$–- desired functions. Denote

$$C^{2,4}(D_T) = \left\{u(x,t) : u(x,t) \in C^2(D_T), u_{xxx}(x,t), u_{xxxx}(x,t) \in C(D_T)\right\}.$$

Definition 2.1 *The classical solution of the inverse boundary value problem(2.1)-(2.5) is the pair* $\{u(x,t), a(t)\}$ *of functions* $u(x,t) \in C^{2,4}(D_T)$ *and* $a(t) \in C[0,T]$ *satisfying equation (2.1) in* D_T *, condition (2.2) in* $[0,1]$ *and conditions (2.3)-(2.5) in* $[0,T]$.

For investigating problem (2.1)-(2.5), firstly we consider the following problem:

$$y''(t) = a(t)y(t)(0 \le t \le T), \tag{2.6}$$

$$y(0) = 0, y'(T) = 0, \tag{2.7}$$

where $a(t) \in C[0,T]$-given function, $y = y(t)$-unknown function, and if $y(t)$ is the solution of problem (2.6),(2.7)then $y(t)$ is continuous on $[0,T]$ together with all derivatives contained in equation (2.6) and satisfying conditions (2.6),(2.7) in the ordinary sense. The following lemma is proved:

Lemma 2.2 *[12] Let function* $a(t) \in C[0,T]$ *such that*

$$\|a(t)\|_{C[0,T]} \le R = const,$$

and

$$\frac{1}{2}T^2 R < 1,$$

where R *is a constant. Then problem (2.6),(2.7) has only a trivial solution.*

Along with the inverse boundary problem (2.1) - (2.5), we consider the following auxiliary inverse boundary value problem. It is required to define a pair $\{u(x,t), a(t)\}$ of functions $u(x,t)$ in $C^{2,4}(D_T)$ and $a(t)$ in $C[0,T]$ from (2.1)-(2.3) and

$$u_{xxx}(0,t) = u_{xxx}(1,t)(0 \le t \le T), \tag{2.8}$$

$$h''(t) - u_{xxxx}(x_0,t) = a(t)h(t) + f(x_0,t)(0 \le t \le T). \tag{2.9}$$

The following theorem is valid:

Theorem 2.3 *Let* $\varphi(x), \psi(x) \in C[0,1], h(t) \in C^2[0,T], h(t) \neq 0$, $(0 \le t \le T)$, $f(x,t) \in C(D_T)$, $\int_0^1 f(x,t)dx = 0 (0 \le x \le 1)$ *and the consistency conditions*

$$\int_0^1 \varphi(x)dx = 0, \int_0^1 \psi(x)dx = 0,$$

$$\varphi(x_0) = h(0), \psi(x_0) = h'(T)$$

be satisfied. Then the following statements are valid:

1. *Each classical solution* $\{u(x,t), a(t)\}$ *of the problem (2.1)-(2.5) is also a solution of problem (2.1) - (2.3), (2.8), (2.9);*
2. *Each solution* $\{u(x,t), a(t)\}$ *of (2.1) - (2.3), (2.8), (2.9) is a classical solution of the problem (2.1) - (2.5), if*

$$\frac{1}{2}T^2 \|a(t)\|_{C[0,T]} < 1.$$

Proof Let $\{u(x,t), a(t)\}$ be a solution of problem (2.1)-(2.5). Integrating the equation (2.1) over x from 0 to 1, we have:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx - (u_{xxx}(1,t) - u_{xxx}(0,t)) =$$

$$= a(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx (0 \le t \le T). \tag{2.10}$$

Assuming that $\int_0^1 f(x,t)dx = 0 (0 \le t \le T)$, with considering (2.4), we easily come to fulfillment (2.8). Further, considering $h(t) \in C^2[0,T]$ and differentiating twice (2.5), we obtain:

$$u_{tt}(x_0,t) = h''(t)(0 \le t \le T). \tag{2.11}$$

From (2.1) we get:

$$u_{tt}(x_0,t) - u_{xxxx}(x_0,t) = a(t)u(x_0,t) + f(x_0,t)(0 \le t \le T). \tag{2.12}$$

Hence, taking into account (2.5) and (2.11), we come to fulfillment (2.9). Now, suppose that $\{u(x,t), a(t)\}$ is a solution of problem (2.1)-(2.3), (2.8), (2.9). Then from (2.8) and (2.10) we get:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx - a(t)\int_0^1 u(x,t)dx = 0 \ \ (0 \le t \le T). \tag{2.13}$$

From (2.2) and $\int_0^1 \varphi(x)dx = 0$, $\int_0^1 \psi(x)dx = 0$, we have:

$$\int_0^1 u(x,0)dx = \int_0^1 \varphi(x)dx = 0, \int_0^1 u_t(x,T)dx = \int_0^1 \psi(x)dx = 0. \tag{2.14}$$

Since, by Lemma 2.1, the problem (2.13), (2.14) has only a trivial solution, $\int_0^1 u(x,t)dx = 0$, i.e. fulfilled conditions (2.4).

Now, from (2.9) and (2.14), we obtained:

$$\frac{d^2}{dt^2}(u(x_0,t) - h(t)) = a(t)(u(x_0,t) - h(t))(0 \le t \le T). \tag{2.15}$$

Further, due to (2.2) and $\varphi(x_0) = h(0), \psi(x_0) = h'(T)$, we have:

$$\begin{cases} u(x_0,0) - h(0) & = \varphi(x_0) - h(0) = 0, \\ u_t(x_0,T) - h'(T) & = \psi(x_0) - h'(T) = 0. \end{cases} \tag{2.16}$$

From (2.15) and (2.16), due to Lemma 2.1,we conclude that the condition (2.5) is satisfied. The theorem is proved. □

3. Investigation of the existence and uniqueness of a classical solution of an inverse boundary value problem

Suppose that the data of the problem (2.1)-(2.3),(2.8),(2.9) satisfy the following conditions:

1. $\varphi(x) \in C^4[0,1], \varphi^{(5)}(x) \in L_2(0,1), \varphi(0) = \varphi(1), \varphi'(0) = \varphi'(1), \varphi''(0) = \varphi''(1), \varphi'''(0) = \varphi'''(1), \varphi^{(4)}(0) = \varphi^{(4)}(1)$;
2. $\psi(x) \in C^2[0,1], \psi^{(3)}(x) \in L_2(0,1), \psi(0) = \psi(1), \psi'(0) = \psi'(1), \psi''(0) = \psi''(1)$;
3. $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T), f_{xxx}(x,t) \in L_2(D_T),, f(0,t) = f(1,t)$, $f_x(0,t) = f_x(1,t), f_{xx}(0,t) = f_{xx}(1,t)(0 \le t \le T)$;
4. $h(t) \in C^2[0,T], h(t) \ne 0(0 \le t \le T)$.

Obviously, [1],

$$1, \cos\lambda_1 x, \sin\lambda_1 x, ..., \cos\lambda_k x, \sin\lambda_k x, ... \tag{3.1}$$

is a basis in $L_2(0,1)$, where $\lambda_k = 2k\pi(k = 0,1,...)$. Since the system (3.1) forms basis in $L_2(0,1)$, it is obvious that for each solution $\{u(x,t), a(t)\}$ problems (2.1)-(2.3),(2.8),(2.9) first komponent $u(x,t)$ has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x(\lambda_k = 2\pi k), \tag{3.2}$$

where

$$u_{10}(t)=\int_0^1 u(x,t)dx, u_{1k}(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx(k=1,2,...),$$

$$u_{2k}(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx(k=1,2,\ldots).$$

Then, applying the formal scheme of the Fourier method, to determine the desired coefficients $u_{1k}(t)(k=0,1,\ldots)$, $u_{2k}(t)(k=1,2,\ldots)$ funtions $u(x,t)$, from (2.1) and (2.3) we obtained:

$$u''_{10}(t)=F_{10}(t;u,a)(0\leq t\leq T), \tag{3.3}$$

$$u''_{ik}(t)-\lambda_k^4 u_{ik}(t)=F_{ik}(t;u,a)(0\leq t\leq T;i-1,2;k=1,2,\ldots), \tag{3.4}$$

$$u_{10}(0)=\varphi_{10}, u'_{10}(T)=\psi_{10}, \tag{3.5}$$

$$u_{ik}(0)=\varphi_{ik}, u'_{ik}(T)=\psi_{ik}(i=1,2;k=1,2,\ldots), \tag{3.6}$$

where

$$F_{1k}(t;u,a)=f_{1k}(t)+a(t)u_{1k}(t)(k=0,1,...),$$

$$f_{10}(t)=\int_0^1 f(x,t)dx, f_{1k}(t)=2\int_0^1 f(x,t)\cos\lambda_k x dx\ \ (k=0,1,\ldots),$$

$$\varphi_{10}=\int_0^1\varphi(x)dx, \psi_{10}=2\int_0^1\psi(x)dx,$$

$$\varphi_{1k}(t)=2\int_0^1\varphi(x)\cos\lambda_k x dx, \psi_{1k}(t)=2\int_0^1\psi(x)\cos\lambda_k x dx\ \ (k=0,1,\ldots),$$

$$F_{2k}(t)=a(t)u_{2k}(t)+f_{2k}(t), f_{2k}(t)=2\int_0^1 f(x,t)\cos\lambda_k x dx\ \ (k=0,1,\ldots),$$

$$\varphi_{2k}(t)=2\int_0^1\varphi(x)\sin\lambda_k x dx, \psi_{2k}(t)=2\int_0^1\psi(x)\sin\lambda_k x dx\ \ (k=0,1,\ldots),$$

Further, from (3.3)-(3.6) we find:

$$u_{10}(t)=\varphi_{10}+\psi_{10}t+\int_0^T G_0(t,\tau)F_{10}(\tau;u,a)d\tau, \tag{3.7}$$

$$u_{ik}(t)=\frac{ch(\lambda_k^2(T-t))}{ch(\lambda_k^2T)}\varphi_{ik}+\frac{sh(\lambda_k^2t)}{\lambda_k^2ch(\lambda_k^2T)}\psi_{ik}+$$

$$+\frac{1}{\lambda_k^2}\int_0^T G_k(t,\tau)F_{ik}(\tau;u,a)d\tau(i=1,2;k=1,2,...),\tag{3.8}$$

where

$$G_0(t,\tau)=\begin{cases}-t,t\in[0,\tau],\\ -\tau,t\in[\tau,T],\end{cases}$$

$$G_k(t,\tau)=\begin{cases}\frac{sh(\lambda_k^2(T-(t-\tau)))-sh(\lambda_k^2(T+t-\tau))}{2ch(\lambda_k^2T)},t\in[0,\tau],\\ \frac{sh(\lambda_k^2(T-(t-\tau)))-sh(\lambda_k^2(T-(t+\tau)))}{2ch(\lambda_k^2T)},t\in[\tau,T].\end{cases}$$

After substitution of expressions $u_{1k}(t)(k=0,1,...)$ and $u_{2k}(t)(k=1,2,...)$ in (3.6), determining the components of the solution $u(x,t)$ of the problem (2.1)-(2.3),(2.8),(2.9), we get:

$$u(x,t)=\varphi_{10}+\psi_{10}t+\int_0^T G_0(t,\tau)F_{10}(\tau;u,a)d\tau+$$

$$+\sum_{k=1}^{\infty}\left\{\frac{ch(\lambda_k^2(T-t))}{ch(\lambda_k^2T)}\varphi_{1k}+\frac{sh(\lambda_k^2t)}{\lambda_k^2ch(\lambda_k^2T)}\psi_{1k}+\right.$$

$$\left.+\frac{1}{\lambda_k^2}\int_0^T G_k(t,\tau)F_{1k}(\tau;u,a)d\tau\right\}\cos\lambda_kx+$$

$$+\sum_{k=1}^{\infty}\left\{\frac{ch(\lambda_k^2(T-t))}{ch(\lambda_k^2T)}\varphi_{2k}+\frac{sh(\lambda_k^2t)}{\lambda_k^2ch(\lambda_k^2T)}\psi_{2k}+\right.$$

$$\left.+\frac{1}{\lambda_k^2}\int_0^T G_k(t,\tau)F_{2k}(\tau;u,a)d\tau\right\}\sin\lambda_kx.\tag{3.9}$$

Now, from (2.8), considering (3.2), we have:

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(x_0,t)-\right.$$

$$\left.-\sum_{k=1}^{\infty}\lambda_k^4u_{1k}(t)\cos\lambda_kx_0+\sum_{k=1}^{\infty}\lambda_k^4u_{2k}(t)\sin\lambda_kx_0\right\}.\tag{3.10}$$

In order to get the equation for the second component of the solution of the problem (2.1)-(2.3),(2.8),(2.9), we substitute the expression (3.8) in (3.10):

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(x_0,t)-\sum_{k=1}^{\infty}\lambda_k^4\left[\frac{ch(\lambda_k^2(T-t))}{ch(\lambda_k^2T)}\varphi_{1k}+\right.\right.$$

$$+\frac{sh(\lambda_k^2 t)}{\lambda_k^2 ch(\lambda_k^2 T)}\psi_{1k}+\frac{1}{\lambda_k^2}\int\limits_0^T G_k(t,\tau)F_{1k}(\tau;u,a)d\tau\Bigg]\cos\lambda_k x_0-$$

$$-\sum_{k=1}^{\infty}\lambda_k^4\left[\frac{ch(\lambda_k^2(T-t))}{ch(\lambda_k^2 T)}\varphi_{2k}+\frac{sh(\lambda_k^2 t)}{\lambda_k^2 ch(\lambda_k^2 T)}\psi_{2k}+\right.$$

$$\left.+\frac{1}{\lambda_k^2}\int\limits_0^T G_k(t,\tau)F_{2k}(\tau;u,a)d\tau\right]\sin\lambda_k x_0\Bigg\}. \qquad (3.11)$$

Thus, the solution of problem (2.1)-(2.3),(2.8),(2.9) is reduced to the solution of system(3.9), (3.11) for the unknown functions $u(x,t)$ and $a(t)$. Using the definition of the solution of the problem (2.1)-(2.3),(2.8),(2.9), we prove the following lemma.

Lemma 3.1 *If $\{u(x,t),a(t)\}$ –any classical solution of problem (2.1)-(2.3),(2.8),(2.9), then the functions $u_{10}(t)=\int\limits_0^1 u(x,t)dx$, $u_{1k}(t)=2\int\limits_0^1 u(x,t)\cos\lambda_k x dx$, $u_{2k}=2\int\limits_0^1 u(x,t)\sin\lambda_k x dx$ $(k=1,2,\ldots)$ satisfy system (3.7), (3.8).*

Remark 3.2 *From Lemma 3.1 it follows that to prove the uniqueness of the solution of the problem (2.1)-(2.3),(2.8),(2.9) enough to prove the uniqueness of the solution of the problem (3.9), (3.11).*

Now, in order to study the problem (2.1)-(2.3),(2.8),(2.9) we consider the following spaces:

1. We denote by $B_{2,T}^5$ [10] ,a consisting of all functions $u(x,t)$ of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_{1k}(t)\cos\lambda_k x dx+\sum_{k=0}^{\infty}u_{2k}(t)\sin\lambda_k x dx \ \ (\lambda_k=2\pi k),$$

considered in D_T , where each of the functions form $u_{1k}(t)(k=0,1,...)$ and $u_{2k}(t)(k=1,2,...)$ are continuous on $[0,T]$ and

$$J_T(u)\equiv\|u_{10}(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_{1k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}<+\infty.$$

The norm in this set is defined as follows:

$$\|u(x,t)\|_{B_{2,T}^5}=J_T(u).$$

2. The spaces E_T^5 denote the space consisting of a topological product

$$B_{2,T}^5\times C[0,T].$$

The norm of element $z + \{u, a\}$ is determined by the formula

$$\|z\|_{E_T^5} = \|u(x,t)\|_{B_{2,T}^5} + \|a(t)\|_{C[0,T]}.$$

It is obvious that $B_{2,T}^5$ and E_T^5 are Banach spaces.

Now in the space E_T^5 consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\},$$

where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t)\sin\lambda_k x,$$

$$\Phi_2(u,a) = \tilde{a}(t),$$

where $\tilde{u}_{10}(t), \tilde{u}_{ik}(t)(i = 1,2; k = 1,2,...)$ and $\tilde{a}(t)$ are equal to the right hand sides of (3.7), (3.8) (3.11). Now with the help of easy transformations we find:

$$\|\tilde{u}_{10}(t)\|_{C[0,T]} \le |\varphi_{10}| + T|\psi_{10}| +$$

$$+2T\sqrt{T}\left(\int_0^T |f_{10}(\tau)|^2 d\tau\right)^{\frac{1}{2}} + 2T^2\|a(t)\|_{C[0,T]}\|u_{10}(t)\|_{C[0,T]}, \tag{3.12}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|\tilde{u}_{ik}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le 2\left(\sum_{k=1}^{\infty}\left(\lambda_k^5|\varphi_{ik}|\right)^2\right)^{\frac{1}{2}} +$$

$$+2\left(\sum_{k=1}^{\infty}\left(\lambda_k^3|\psi_{ik}|\right)^2\right)^{\frac{1}{2}} + 4\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3|f_{ik}(\tau)|\right)^2\right) +$$

$$+4T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|u_{ik}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} (i = 1,2), \tag{3.13}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|h''(t) - f(x_0,t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} \times\right.$$

$$\times\sum_{i=1}^{2}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^5|\varphi_{ik}|)^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_{ik}|)^2\right)^{\frac{1}{2}} + \sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3|f_{ik}(\tau)|)^2 d\tau\right)^{\frac{1}{2}} +\right.$$

$$\left.\left. + T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{3.14}$$

Further, from (3.12)-(3.14) we have:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}} \leq A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{3.15}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{3.16}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\|\psi(x)\|_{L_2(0,1)} + 2T\sqrt{T}\|f(x,t)\|_{L_2(D_T)} +$$

$$+2\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + 2\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)} + 4\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\,,$$

$$B_1(T) = 2T^2 + 4T.$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|h''(t) - f(x_0,t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} \times\right.$$

$$\left.\times\, 2\left[\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + \left\|\psi^{(3)}(x)\right\|_{L_2(0,1)} + 2\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T) = 2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}} T.$$

From inequalities (3.15)-(3.16) we conclude:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{3.17}$$

where

$$A(T) = A_1(T) + A_2(T), B(T) = B_1(T) + B_2(T).$$

So, we can prove the following theorem:

Theorem 3.3 *Let all conditions 1.-4. be fulfilled and*

$$(A(T) + 2)^2 B(T) < 1. \tag{3.18}$$

Then the problem (2.1)-(2.3),(2.8),(2.9) has a unique solution in the sphere $K = K_R(\|z\|_{E^3_T} \leq R = A(T) + 2)$ *of the space* E^5_T.

Proof In the space E^5_T consider the equation

$$z = \Phi z, \tag{3.19}$$

where $z = \{u, a\}$, the components $\Phi_i(u,a)(i = 1,2)$, of the operator $\Phi(u,a)$, are determined by the right hand sides of equations (3.9) and (3.11). Consider the operator $\Phi(u,a)$ in the sphere $K = K_R$ from E^5_T. Similar to (3.17) we obtained that for any $z, z_1, z_2 \in K_R$ the following estimate are valid:

$$\|\Phi z\|_{E^3_T} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} +$$

$$+A(T)+B(T)\left(A(T)+2\right)^2, \tag{3.20}$$

$$\|\Phi z_1-\Phi z_s\|_{E_T^3}\leq$$

$$\leq B(T)TR(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3}. \tag{3.21}$$

Then from (3.20) and (3.21), with considering (3.18),it follows that operator Φ acts in the sphere $K=K_R$ and it is contraction mapping. Therefore, in the sphere $K=K_R$ the operator Φ has a unique fixed point $\{u,a\}$, that is a solution of equation (3.19). The function $u(x,t)$, as the element of the space $B_{2,T}^5$, has continuous derivatives $u(x,t), u_x(x,t), u_{xx}(x,t), u_{xxx}(x,t)$ and $u_{xxxx}(x,t)$ in D_T.

From (3.4) it is easy to see that

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u_{ik}''(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq\sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times$$

$$\times\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\|\|f_x(x,t)+a(t)u_x(x,t)\|\|_{L_2(0,1)}\right](i=1,2).$$

Then it follows that $u_{tt}(x,t)$ is continuous in D_T. It is easy to verify that (2.1) and conditions (2.2), (2.3), (2.8) and (2.9) are satisfied in the ordinary sense. Consequently, it is a solution to problem (2.1)-(2.3), (2.8),(2.9), and, by virtue of the lemma 3.1, it is unique. Theorem is proved. □

Using the theorem 2.2 we proved the following Lemma.

Theorem 3.4 *Let all the conditions of the Theorem 3.2*

$$\int_0^1 f(x,t)dx=0(0\leq t\leq T),\frac{1}{2}(A(T)+2)T^2<1,$$

and condition of approval

$$\int_0^1\varphi(x)dx=0,\int_0^1\psi(x)dx=0,\varphi(x_0)=h(0),\psi(x_0)=h'(T)$$

be satisfied. Then, problem (2.1)-(2.5) has in the sphere $K=K_R(\|z\|_{E_{T,T}^3}\leq R=A(T)+2)$ *from* E_T^5 *unique classical solution.*

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