The formulization of the intrinsic metric on the added Sierpinski triangle by using the code representations

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Abstract: To formulate the intrinsic metrics by using the code representations of the points on the classical fractals is an important research area since these formulas help to prove many geometrical and structural properties of these fractals. In various studies, the intrinsic metrics on the code set of the Sierpinski gasket, the Sierpinski tetrahedron, and the Vicsek (box) fractal are explicitly formulated. However, in the literature, there are not many works on the intrinsic metric that is obtained by the code representations of the points on fractals. Moreover, as seen in the studies on this subject, the contraction coefficients of the associated iterated function systems (IFSs) are the same for each fractal. In this paper, we define the intrinsic metric formula on the added Sierpinski triangle, whose IFS has different contraction factors, by using the code representations of the points of it. Finally, we give several geometrical properties of this fractal by using the intrinsic metric formula.

Key words: Fractal, added Sierpinski triangle, intrinsic metric, code representation

1. Introduction

The Cantor set, Sierpinski gasket, Koch curve, Sierpinski carpet, Vicsek fractal, Sierpinski tetrahedron, and Menger sponge are some of the fundamental examples of the classical fractals [2, 3, 8, 11, 14]. These sets have strong self-similarity, which means that every neighborhood of any point contains a copy of the entire shape (see Figure 1). There have been different ways to define the intrinsic metrics on the classical fractals such as the classical Sierpinski Gasket, the discrete Sierpinski Gasket, the Sierpinski Carpet, and the Vicsek fractals in the last two decades (for details see [5–7, 9, 13, 17, 24, 25]). Using the code representations of the points on the classical Sierpinski gasket, the Vicsek fractal, the Sierpinski tetrahedron, and mod-3 Sierpinski gasket $SG(3)$, the intrinsic metric formulas are given explicitly in [1, 15, 16, 22]. Moreover, the intrinsic metric formulas on isosceles and scalene Sierpinski triangles are defined in [19]. As a result, many geometrical and topological properties are investigated in [18–20, 22]. For example, the code representations of the points on the Sierpinski gasket according to the number of geodesics is also classified in [23]. Then the intrinsic metric formula is given for the n-dimensional Sierpinski gasket and thus the number of geodesics on them are investigated in [10]. The intrinsic metric formulas are reformulated on the code set of the equilateral Sierpinski propeller, which is self-similar but not strong self-similar, in [12]. Furthermore, some dynamical systems on the Sierpinski gasket and the Sierpinski tetrahedron are defined and whether these dynamical systems would be chaotic is investigated

by using these intrinsic metrics in [1] and [21], respectively. As seen in these studies, these intrinsic formulas help to show many properties of these fractals.
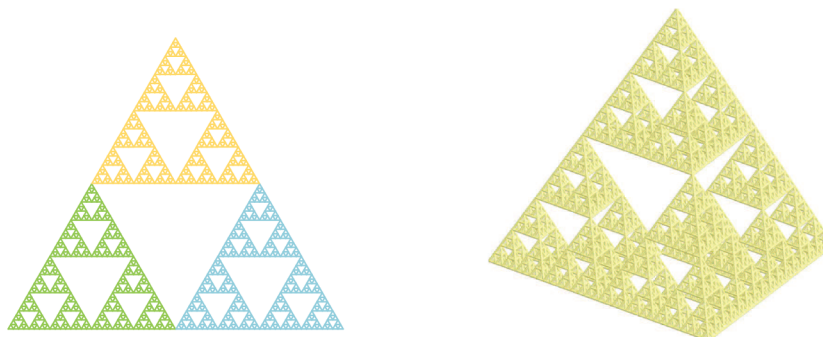

Figure 1. The Sierpinski gasket and the Sierpinski tetrahedron, respectively.

The main aim of the paper is to give the intrinsic metric formula for a different fractal, which we call the added Sierpinski triangle. Note that the associated iterated function systems (IFSs) of fractals such as the Sierpinski gasket, mod-3 Sierpinski gasket $SG(3)$, Sierpinski carpet, Sierpinski tetrahedron, and Box fractal have the same contraction factors, but the contraction factors of the IFS of the added Sierpinski triangle are not same. This paper will thus be the first work giving the intrinsic metric formula defined by the code representation of points on a fractal whose IFS consists of different contraction factors. Because of the different contraction coefficients, there are some difficulties in formulating the intrinsic metric on the code set of this fractal. For a better understanding, we first the give the code representations of the points on the added Sierpinski triangle in Proposition 2.1. Then we mention Cases 1, 2, 3, 4, 5, and 6, which include all states of the construction of the intrinsic metric on this set. We thus formulate the intrinsic metric in Theorem 3.3. In Lemma 3.5, we give a useful abbreviation for this formula. In Propositions 3.4, 3.6, and 3.8, we investigate some geometrical properties of this fractal. Examples 3.9 and 3.10 also show how this formula is used in different cases.

In the following section, we first explain the construction of the added Sierpinski triangle and then investigate the code representations of points on this set.

2. The code representations of the points and the construction of the added Sierpinski triangle

The construction of the added Sierpinski triangle is actually similar to the construction of the Sierpinski gasket. The only difference is that smaller triangles are added instead of removed triangles in each step, which is the reason why we call it the added Sierpinski triangle. For the construction of this fractal, we consider an equilateral triangle (with edge length 1) as the initial set. We mark the midpoints of each edge of the triangle and obtain four smaller triangles. Then we remove the middle triangle and mark the midpoints of the removed triangle. Combining these points, we obtain a smaller triangle. Obviously, the middle triangle has edge length two times smaller than the edge length of the remaining three triangles (see step 1 in Figure 2). We denote this structure by $\widetilde{T_1}$. In the second step, we repeat this process for each four new triangles and we get $\widetilde{T_2}$. Continuing this process infinitely, we get the added Sierpinski triangle (see the last step in Figure 2). We denote this fractal by $\widetilde{S}$. Therefore, we have $\bigcap_{i=0}^{\infty} \widetilde{T_i} = \widetilde{S}$. This fractal can also be obtained as the attractor of an iterated function

system (IFS). Let $\{\mathbb{R}^2; f_0, f_1, f_2, f_3\}$ be an iterated function system where

$$
\begin{aligned}
f_0(x,y) &= \left(\frac{x}{4} + \frac{3}{8}, \frac{y}{4} + \frac{\sqrt{3}}{8}\right), \\
f_1(x,y) &= \left(\frac{x}{2}, \frac{y}{2}\right), \\
f_2(x,y) &= \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right), \\
f_3(x,y) &= \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4}\right),
\end{aligned}
$$

with the contraction factors $\frac{1}{4}, \frac{1}{2}, \frac{1}{2}$, and $\frac{1}{2}$, respectively. Since

$$
F(\widetilde{S}) = \bigcup_{i=0}^{3} f_i(\widetilde{S}) = \widetilde{S},
$$

the attractor of the IFS above is the added Sierpinski triangle. This IFS satisfies the open set condition and thus the unique real solution of the Moran equation,

$$
\frac{1}{4^s} + \frac{1}{2^s} + \frac{1}{2^s} + \frac{1}{2^s} = 1,
$$

gives the fractal dimension of this set. Solving this equation, we compute the dimension s as

$$
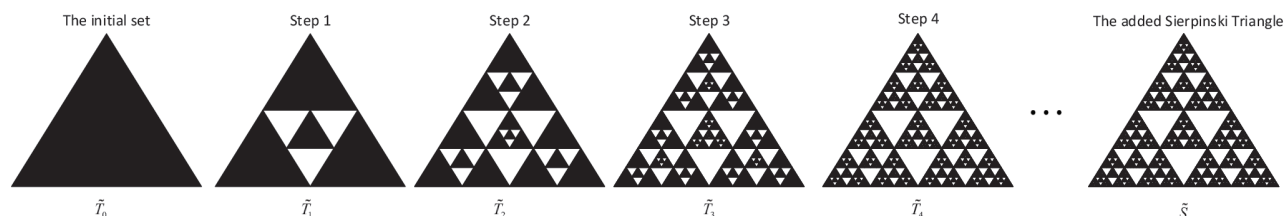s = \frac{\ln\left(\frac{3+\sqrt{13}}{2}\right)}{\ln 2}.
$$

Figure 2. The construction of the added Sierpinski triangle.

Now we show the code representations of the points on the added Sierpinski triangle and we constitute the code sets thanks to these code representations. As seen in Figure 3, we denote the middle part of $\widetilde{S}$ by $\widetilde{S}_0$ (with purple color), the left-bottom part of $\widetilde{S}$ by $\widetilde{S}_1$ (with green color), the right-bottom part of $\widetilde{S}$ by $\widetilde{S}_2$ (with blue color), and the upper part of $\widetilde{S}$ by $\widetilde{S}_3$ (with yellow color). Hence, we get

$$
\widetilde{S} = \widetilde{S}_0 \cup \widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{S}_3.
$$

We similarly denote the middle part of $\widetilde{S}_{a_1}$ by $\widetilde{S}_{a_1 0}$, the left-bottom part of $\widetilde{S}_{a_1}$ by $\widetilde{S}_{a_1 1}$, the right-bottom part of $\widetilde{S}_{a_1}$ by $\widetilde{S}_{a_1 2}$, and the upper part of $\widetilde{S}_{a_1}$ by $\widetilde{S}_{a_1 3}$ where $a_1 \in \{0, 1, 2, 3\}$. Therefore, we obtain

$$
\widetilde{S}_{a_1} = \widetilde{S}_{a_1 0} \cup \widetilde{S}_{a_1 1} \cup \widetilde{S}_{a_1 2} \cup \widetilde{S}_{a_1 3}.
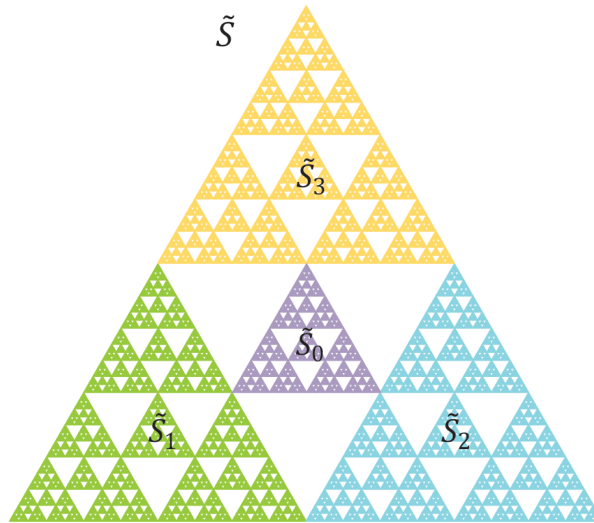$$

Figure 3. The code sets of the added Sierpinski triangle.

Let $\sigma = a_1 a_2 \ldots a_{k-1}$ where $a_i \in \{0, 1, 2, 3\}$ for $i = 0, 1, 2, \ldots, k-1$. Generally, we express the middle part of $\widetilde{S}_\sigma$ by $\widetilde{S}_{\sigma 0}$, the left-bottom part of $\widetilde{S}_\sigma$ by $\widetilde{S}_{\sigma 1}$, the right-bottom part of $\widetilde{S}_\sigma$ by $\widetilde{S}_{\sigma 2}$, and the upper part of $\widetilde{S}_\sigma$ by $\widetilde{S}_{\sigma 3}$. We thus have

$$\widetilde{S}_\sigma = \widetilde{S}_{\sigma 0} \cup \widetilde{S}_{\sigma 1} \cup \widetilde{S}_{\sigma 2} \cup \widetilde{S}_{\sigma 3}$$
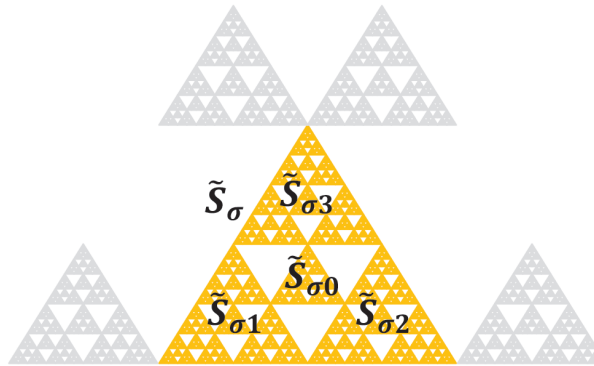
(see Figure 4).

Figure 4. The added Sierpinski subtriangle $\widetilde{S}_\sigma$.

Therefore, for the subtriangles $\widetilde{S}_{a_1}, \widetilde{S}_{a_1 a_2}, \widetilde{S}_{a_1 a_2 a_3}, \ldots, \widetilde{S}_{a_1 a_2 a_3 \ldots a_k}, \ldots$ of $\widetilde{S}$, we have $\widetilde{S}_{a_1} \supset \widetilde{S}_{a_1 a_2} \supset \widetilde{S}_{a_1 a_2 a_3} \supset \ldots \supset \widetilde{S}_{a_1 a_2 a_3 \ldots a_k} \ldots \supset \ldots$. There exists a point $A \in \widetilde{S}$ such that

$$\bigcap_{k=1}^{\infty} \widetilde{S}_{a_1 a_2 a_3 \ldots a_k} = \{A\}$$

from the Cantor intersection theorem. The code representation of A is denoted by $a_1 a_2 a_3 \ldots a_k \ldots$. In the

following proposition, we classify the points on $\widetilde{S}$ according to their code representations:

Proposition 2.1 *The number of the code representations of any point on the added Sierpinski triangle is either* 1, 2, *or* 3.

Proof   First, we show the points that have three code representations as follows: **i)** Let A be the intersection point of any two subtriangles in the same level of S_σ such that

$$\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma a_k} = \{A\},$$

where $\sigma = a_1 a_2 \ldots a_{k-1}$ and $a_k \in \{1, 2, 3\}$.

- If we choose $a_k = 1$, then we have

$$\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma 1} = \widetilde{S}_{\sigma 0} \cap (\widetilde{S}_{\sigma 12} \cap \widetilde{S}_{\sigma 13}) = \{A\}.$$

We now consider three different nested sequences of sets as follows:

$$\widetilde{S}_{\sigma 0}, \widetilde{S}_{\sigma 01}, \widetilde{S}_{\sigma 011}, \ldots, \widetilde{S}_{\sigma 011\ldots 1}, \ldots,$$

$$\widetilde{S}_{\sigma 1}, \widetilde{S}_{\sigma 13}, \widetilde{S}_{\sigma 132}, \ldots, \widetilde{S}_{\sigma 1322\ldots 2}, \ldots,$$

$$\widetilde{S}_{\sigma 1}, \widetilde{S}_{\sigma 12}, \widetilde{S}_{\sigma 123}, \ldots, \widetilde{S}_{\sigma 1233\ldots 3}, \ldots.$$

From the Cantor intersection theorem, the code representations of A are $\sigma 0111 \ldots$, $\sigma 13222 \ldots$, and $\sigma 12333 \ldots$, respectively (see $A = V_\sigma$ in Figure 5).

Figure 5. The points T_σ, V_σ, and W_σ, which have three different code representations on $\tilde{S}_\sigma$.

- Let $a_k = 2$. Since

$$\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma 2} = \widetilde{S}_{\sigma 0} \cap (\widetilde{S}_{\sigma 21} \cap \widetilde{S}_{\sigma 23}) = \{A\},$$

we have three different nested sequences of sets as follows:

$$\widetilde{S}_{\sigma 0}, \widetilde{S}_{\sigma 02}, \widetilde{S}_{\sigma 022}, \dots, \widetilde{S}_{\sigma 022\dots 2}, \dots$$

$$\widetilde{S}_{\sigma 2}, \widetilde{S}_{\sigma 21}, \widetilde{S}_{\sigma 213}, \dots, \widetilde{S}_{\sigma 2133\dots 3}, \dots$$

$$\widetilde{S}_{\sigma 2}, \widetilde{S}_{\sigma 23}, \widetilde{S}_{\sigma 231}, \dots, \widetilde{S}_{\sigma 2311\dots 1}, \dots.$$

Thus, the code representations of A are $\sigma 0222\dots$, $\sigma 21333\dots$, and $\sigma 23111\dots$, respectively (see $A = W_\sigma$ in Figure 5).
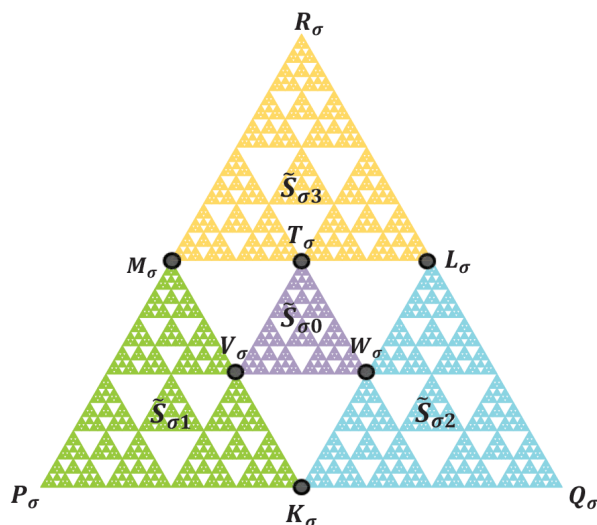
- For $a_k = 3$, we have

$$\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma 3} = \widetilde{S}_{\sigma 0} \cap (\widetilde{S}_{\sigma 31} \cap \widetilde{S}_{\sigma 32}) = \{A\}$$

and three different nested sequences of sets as follows:

$$\widetilde{S}_{\sigma 0}, \widetilde{S}_{\sigma 03}, \widetilde{S}_{\sigma 033}, \dots, \widetilde{S}_{\sigma 033\dots 3}, \dots,$$

$$\widetilde{S}_{\sigma 3}, \widetilde{S}_{\sigma 31}, \widetilde{S}_{\sigma 312}, \dots, \widetilde{S}_{\sigma 3122\dots 2}, \dots,$$

$$\widetilde{S}_{\sigma 3}, \widetilde{S}_{\sigma 32}, \widetilde{S}_{\sigma 321}, \dots, \widetilde{S}_{\sigma 3211\dots 1}, \dots.$$

Then A has code representations such as $\sigma 0333\dots$, $\sigma 31222\dots$, and $\sigma 32111\dots$, respectively (see $A = T_\sigma$ in Figure 5).

Therefore, the vertices points T_σ, V_σ, and W_σ of $\widetilde{S}_{\sigma 0}$ have three different code representations.

ii) Let A be the intersection point of any two subtriangles in the same level of S_σ such that

$$\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma b_k} = \{A\},$$

where $\sigma = a_1 a_2 \dots a_{k-1}$ and $a_k, b_k \in \{1, 2, 3\}$ and $a_k \neq b_k$. Hence, we get the nested sequence of sets such that

$$\widetilde{S}_{\sigma a_k}, \widetilde{S}_{\sigma a_k b_k}, \widetilde{S}_{\sigma a_k b_k b_k}, \dots, \widetilde{S}_{\sigma a_k b_k b_k \dots b_k}, \dots,$$

$$\widetilde{S}_{\sigma b_k}, \widetilde{S}_{\sigma b_k a_k}, \widetilde{S}_{\sigma b_k a_k a_k}, \dots, \widetilde{S}_{\sigma b_k a_k a_k \dots a_k}, \dots.$$

The Cantor intersection theorem states that the code representations of A are $\sigma a_k b_k b_k \dots b_k$ and $\sigma b_k a_k a_k \dots a_k$, respectively, and thus A has two different code representations. For example, the points M_σ, L_σ, and K_σ in Figure 5 have two different code representations.

iii) If A is not the intersection point of any two subtriangles in the same level of S_σ, then A has a unique code representation. That is, it is different from all code representations denoted by cases i and ii above. To exemplify, the vertex points P, Q, and R of $\tilde{S}$ and many points such as $121212\dots$, $01230123\dots$, and nonrepeating forms have unique code representation.

Consequently, the code sets of the subtriangles $\widetilde{S}_\sigma$ are expressed as

$$\widetilde{S}_\sigma = \{\sigma a_k a_{k+1} a_{k+2} \dots \mid a_i \in \{0, 1, 2, 3\}, \ i = k, k+1, k+2, \dots\}.$$

$\square$

3. The construction of the intrinsic metric on the code set of the added Sierpinski triangle

The intrinsic metric on a set K is expressed as follows:

$$d(x,y) = \inf\{\delta \mid \delta \text{ is the length of a rectifiable curve in } K \text{ joining } x \text{ and } y\} \tag{3.1}$$

for $x, y \in K$ (for details see [4]). The intrinsic metric on the code set of the equilateral Sierpinski gasket is also defined in [22]:

Definition 3.1 *Let $a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots$ and $b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots$ be two code representations of the points A and B on the equilateral Sierpinski gasket, respectively, where $a_i = b_i$ for $i = 1, 2, \ldots, k-1$, $a_k \neq b_k$ and $a_i, b_i \in \{0, 1, 2\}$ for $i = 1, 2, 3 \ldots,$. The intrinsic metric between code representations of the points A and B is formulated as follows:*

$$d(A, B) = \min \left\{ \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \ \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i} \right\}, \tag{3.2}$$

where

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \qquad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases},$$

$$\gamma_i = \begin{cases} 0, & a_i \neq a_k \text{ and } a_i \neq b_k \\ 1, & otherwise \end{cases}, \qquad \delta_i = \begin{cases} 0, & b_i \neq b_k \text{ and } b_i \neq a_k \\ 1, & otherwise \end{cases}.$$

We now formulate the intrinsic metric by using the code representations of points on the added Sierpinski triangle. Note that in each step there are middle triangles in the added Sierpinski triangle different from the Sierpinski gasket. To formulate the intrinsic metric on $\tilde{S}$ seems quite complicated due to the different contraction factors and the increase in shortest paths.

Now we express our first observations for this construction. We first need some notations and expressions to define the intrinsic metric by using the code representation of points. Suppose that the code representations of the different points A and B on $\tilde{S}$ are $a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots$ and $b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots$, respectively, where $a_i, \ b_i \in \{0, 1, 2, 3\}$. Let $k = \min\{i \mid a_i \neq b_i\}$ and $\sigma = a_1 a_2 \ldots a_{k-1}$. Assume that the number of elements of the set

$$\kappa = \{i \mid a_i = b_i = 0, i < k\}$$

is t. Moreover, let

$$M = \{i+1 \mid a_i = 0, i > k\} = \{m_1, m_2, m_3, \ldots\},$$
$$L = \{i+1 \mid b_i = 0, i > k\} = \{l_1, l_2, l_3, \ldots\},$$

such that $m_1 < m_2 < m_3 < \ldots$ and $l_1 < l_2 < l_3 < \ldots$.

- Let $a_k \neq 0$ and $b_k \neq 0$. Then the shortest paths between the points A and B must pass through either the point $\tilde{S}_{\sigma a_k} \cap \tilde{S}_{\sigma b_k}$ or the line segment $(\tilde{S}_{\sigma a_k} \cap \tilde{S}_{\sigma c_k})(\tilde{S}_{\sigma b_k} \cap \tilde{S}_{\sigma c_k})$ or the line segment $(\tilde{S}_{\sigma a_k} \cap \tilde{S}_{\sigma 0})(\tilde{S}_{\sigma b_k} \cap \tilde{S}_{\sigma 0})$ (see Case 1, Case 2, and Case 3, respectively). Thus, these three paths should be taken into account and
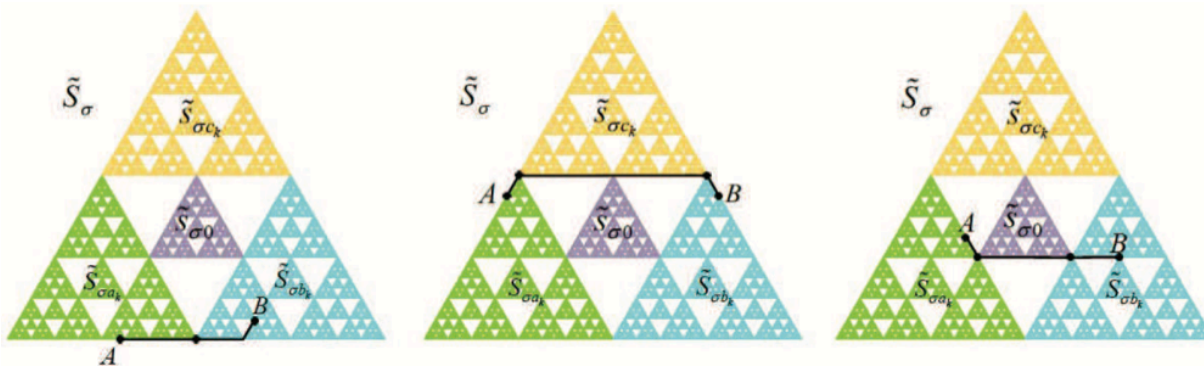
Figure 6. Three different cases for the shortest paths between A and B where $a_k = 1$ and $b_k = 2$.

the minimum of the lengths of them should be taken for the calculation of the length of the shortest paths (see Figure 6).

Case 1. The length of the shortest paths between A and $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma b_k}$ and the length of the shortest paths between B and $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma b_k}$ are obtained by

$$\mathcal{A} = \sum_{i=k+1}^{m_1-1} \frac{\alpha_i}{2^{i+t}} + \frac{1}{2} \sum_{i=m_1}^{m_2-1} \frac{\alpha_i}{2^{i+t}} + \cdots + \frac{1}{2^r} \sum_{i=m_r}^{m_{r+1}-1} \frac{\alpha_i}{2^{i+t}} + \cdots, \tag{3.3}$$

$$\mathcal{B} = \sum_{i=k+1}^{l_1-1} \frac{\beta_i}{2^{i+t}} + \frac{1}{2} \sum_{i=l_1}^{l_2-1} \frac{\beta_i}{2^{i+t}} + \cdots + \frac{1}{2^p} \sum_{i=l_p}^{l_{p+1}-1} \frac{\beta_i}{2^{i+t}} + \cdots, \tag{3.4}$$

respectively, where

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \qquad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases}.$$

Thus, the length of the shortest paths between A and B passing through the point $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma b_k}$ equals $\mathcal{A} + \mathcal{B}$. If $M = \emptyset$ and $L = \emptyset$, then this length is computed by

$$\mathcal{A} + \mathcal{B} = \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^{i+t}}.$$

Case 2. Suppose that $c_k \neq a_k$, $c_k \neq b_k$, and $c_k \in \{1, 2, 3\}$. The length of the shortest paths between point A and point $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma c_k}$ and the length of the shortest paths between point B and point $\widetilde{S}_{\sigma b_k} \cap \widetilde{S}_{\sigma c_k}$ are

$$\mathcal{A}' = \sum_{i=k+1}^{m_1-1} \frac{\gamma_i}{2^{i+t}} + \frac{1}{2} \sum_{i=m_1}^{m_2-1} \frac{\gamma_i}{2^{i+t}} + \cdots + \frac{1}{2^r} \sum_{i=m_r}^{m_{r+1}-1} \frac{\gamma_i}{2^{i+t}} + \cdots, \tag{3.5}$$

$$\mathcal{B}' = \sum_{i=k+1}^{l_1-1} \frac{\delta_i}{2^{i+t}} + \frac{1}{2} \sum_{i=l_1}^{l_2-1} \frac{\delta_i}{2^{i+t}} + \cdots + \frac{1}{2^p} \sum_{i=l_p}^{l_{p+1}-1} \frac{\delta_i}{2^{i+t}} + \cdots, \tag{3.6}$$

respectively, where

$$\gamma_i = \begin{cases} 0, & a_i = c_k \\ 1, & a_i \neq c_k \end{cases}, \qquad \delta_i = \begin{cases} 0, & b_i = c_k \\ 1, & b_i \neq c_k \end{cases}.$$

Moreover, the length of the side $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma c_k})(\widetilde{S}_{\sigma b_k} \cap \widetilde{S}_{\sigma c_k})$ of the subtriangle $\widetilde{S}_{\sigma c_k}$ is $\frac{1}{2^{t+k}}$. Hence, the length of the shortest paths between the points A and B passing through the line segment $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma c_k})(\widetilde{S}_{\sigma b_k} \cap \widetilde{S}_{\sigma c_k})$ equals $\frac{1}{2^{t+k}} + \mathcal{A}' + \mathcal{B}'$. If $M = \emptyset$ and $L = \emptyset$, then this length is obtained as

$$\frac{1}{2^{t+k}} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^{i+t}}.$$

Case 3. To compute the length of the shortest paths between points A and B passing through the line segment $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})(\widetilde{S}_{\sigma b_k} \cap \widetilde{S}_{\sigma 0})$, which is an edge of $\widetilde{S}_{\sigma 0}$, we must take into account the following cases:

i) Let $a_{k+1} \neq a_k$ and $a_{k+1} \neq 0$. For $a_\mu \neq a_{k+1}$, $a_\mu \neq a_k$, and $a_\mu \neq 0$, the length of the shortest paths between points A and point $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0}$ is obtained by

$$\mathcal{A}'' = \sum_{i=k+2}^{m_1-1} \frac{\varphi_i}{2^{i+t}} + \frac{1}{2} \sum_{i=m_1}^{m_2-1} \frac{\varphi_i}{2^{i+t}} + \cdots + \frac{1}{2^r} \sum_{i=m_r}^{m_{r+1}-1} \frac{\varphi_i}{2^{i+t}} + \cdots, \tag{3.7}$$

where

$$\varphi_i = \begin{cases} 0, & a_i = a_\mu \\ 1, & \text{otherwise} \end{cases}.$$

ii) Suppose that $a_{k+1} = 0$. For

$$r = \min\{i \mid a_i \neq 0, \ a_i \neq a_k, \ i \geq k+2\},$$

this length is computed by

$$\mathcal{A}'' = \frac{1}{2^{k+t+2}} + \frac{1}{2} \sum_{i=k+2}^{m_2-1} \frac{\varphi_i}{2^{i+t}} + \frac{1}{2^2} \sum_{i=m_2}^{m_3-1} \frac{\varphi_i}{2^{i+t}} + \cdots + \frac{1}{2^r} \sum_{i=m_r}^{m_{r+1}-1} \frac{\varphi_i}{2^{i+t}} + \cdots, \tag{3.8}$$

where

$$\varphi_i = \begin{cases} 0, & a_i = a_r \\ 1, & \text{otherwise.} \end{cases}$$

Note that we obtain $\varphi_i = 1$ for $i = k+2, k+3, k+4, \ldots$ if

$$\{i \mid a_i \neq 0, \ a_i \neq a_k, \ i \geq k+2\} = \emptyset.$$

iii) There exist three different cases as follows:

a) Let $a_k = a_{k+1} = \ldots = a_{s-1} \neq a_s \neq 0$ $(s > k+1)$. We define

$$r \; = \; \min\{i \mid a_i \neq 0, \; a_i \neq a_k, \; i > s\}.$$

In this case, we obtain

$$\varphi_i = \begin{cases} 0, & a_i = a_r \\ 1, & \text{otherwise} \end{cases}$$

for $i \neq s$ and $\varphi_s = 0$ for $i = s$. If

$$\{i \mid a_i \neq 0, \quad a_i \neq a_k, \quad i > s\} = \emptyset,$$

then we also get $\varphi_i = 1$ for $i \neq s$ and $\varphi_s = 0$ for $i = s$.

b) Let $a_k = a_{k+1} = \ldots = a_{s-1}$ and $a_s = 0$ $(s > k+1)$. In this case, we get

$$\varphi_i = \begin{cases} 0, & a_i = a_r \\ 1, & \text{otherwise} \end{cases}$$

for $i \neq s$ and $\varphi_s = \dfrac{1}{2}$ for $i = s$ where $r = \min\{i \mid a_i \neq 0, \; a_i \neq a_k, \; i \geq k+2\}$. If

$$\{i \mid a_i \neq 0, \; a_i \neq a_k, \; i \geq k+2\} = \emptyset,$$

then we obtain $\varphi_i = 1$ for $i \neq s$ and $\varphi_s = \dfrac{1}{2}$ for $i = s$.

c) If $a_k = a_{k+1} = \cdots = a_i = \cdots$, then $\varphi_i = 1$ for $i = k+2, k+3, k+4, \ldots$

In cases a, b, and c the length of the shortest paths between points A and point $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0}$ is computed by

$$\mathcal{A}'' = \frac{1}{2^{k+t+1}} + \sum_{i=k+2}^{m_1-1} \frac{\varphi_i}{2^{i+t}} + \frac{1}{2} \sum_{i=m_1}^{m_2-1} \frac{\varphi_i}{2^{i+t}} + \cdots + \frac{1}{2^r} \sum_{i=m_r}^{m_{r+1}-1} \frac{\varphi_i}{2^{i+t}} + \cdots. \tag{3.9}$$

Remark 3.2 *Note that similar calculations are valid for the computation of the length of the shortest paths between points B and point $\widetilde{S}_{\sigma b_k} \cap \widetilde{S}_{\sigma 0}$ (in this case, we use θ_i instead of φ_i to avoid confusion). Since the length of the side $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})(\widetilde{S}_{\sigma b_k} \cap \widetilde{S}_{\sigma 0})$ of the subtriangle $\widetilde{S}_{\sigma 0}$ is $\dfrac{1}{2^{t+k+1}}$, we compute that the length of the shortest paths between points A and B passing through the line segment $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})(\widetilde{S}_{\sigma b_k} \cap \widetilde{S}_{\sigma 0})$ equals $\dfrac{1}{2^{t+k+1}} + \mathcal{A}'' + \mathcal{B}''$.*

- Let $a_k \neq 0$ and $b_k = 0$. In this case, the shortest paths between points A and B must pass through one of the vertices of the subadded Sierpinski triangle $\widetilde{S}_{\sigma 0}$. That is, the shortest paths must pass through either point $\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma a_k}$ or $(\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma c_k})$ or $(\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma b'_k})$ (see Case 4, Case 5, and Case 6, respectively). These three paths should thus be taken into account and the minimum of the lengths of them should be taken for the calculation of the length of the shortest paths (see Figure 7).
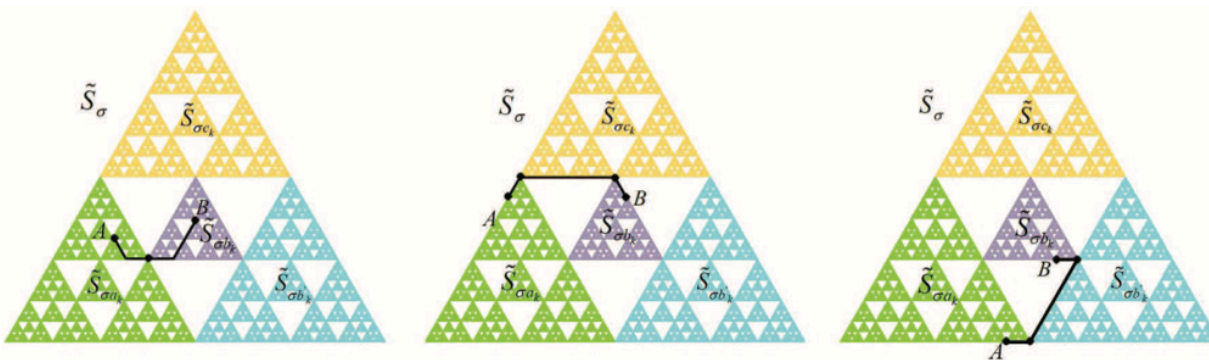
Figure 7. Three different cases for the shortest paths between A and B where $a_k = 1$ and $b_k = 0$.

Case 4. For the computation of the length of the shortest paths between points A and B passing through point $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0}$, we add the length of the shortest paths between points A and $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0}$ and the length of the shortest paths between points B and $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0}$. Note that we use the appropriate value $\mathcal{A}''$ given in Case 3 to compute the length of the shortest paths between points A and $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0}$. Moreover, we obtain that the length of the shortest paths between points B and $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0}$ is

$$\frac{1}{2}\Big(\sum_{i=k+1}^{l_1-1} \frac{\beta_i}{2^{i+t}} + \frac{1}{2}\sum_{i=l_1}^{l_2-1} \frac{\beta_i}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=l_p}^{l_{p+1}-1} \frac{\beta_i}{2^{i+t}} + \cdots \Big) = \frac{1}{2}\mathcal{B}, \tag{3.10}$$

where

$$\beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases}.$$

Thus, the sum of $\mathcal{A}''$ and $\frac{1}{2}\mathcal{B}$ gives us the length of the shortest paths between points A and B passing through the point $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0}$.

Case 5. To compute the length of the shortest paths between points A and B passing through the line segment $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma c_k})(\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma c_k})$, first we obtain the length of the shortest paths between points B and $\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma c_k}$:

$$\frac{1}{2}\Big(\sum_{i=k+1}^{l_1-1} \frac{\delta_i}{2^{i+t}} + \frac{1}{2}\sum_{i=l_1}^{l_2-1} \frac{\delta_i}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=l_p}^{l_{p+1}-1} \frac{\delta_i}{2^{i+t}} + \cdots \Big) = \frac{1}{2}\mathcal{B}', \tag{3.11}$$

where

$$\delta_i = \begin{cases} 0, & b_i = c_k \\ 1, & b_i \neq c_k \end{cases}$$

and $c_k \neq a_k$ for $c_k \in \{1, 2, 3\}$. Also, to compute the length of the shortest paths between points A and $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma c_k}$, we use the appropriate formula given in Case 2. Thus, the sum $\frac{1}{2^{t+k+1}} + \mathcal{A}' + \frac{1}{2}\mathcal{B}'$ gives us the length of the shortest paths between points A and B passing through the line segment $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma c_k})(\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma c_k})$.

Case 6. For the computation of the length of the shortest paths between points A and B passing through the line segment $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma b'_k})(\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma b'_k})$ where $b'_k \neq a_k$, $b'_k \neq c_k$ and $b'_k \in \{1, 2, 3\}$, note that the length of the shortest paths between points A and $\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma b'_k}$ equals the value $\mathcal{A}$ given in Case 1 (use b'_k instead of b_k). Moreover,

$$\mathcal{C} = \frac{1}{2}\Big(\sum_{i=k+1}^{l_1-1} \frac{\beta'_i}{2^{i+t}} + \frac{1}{2} \sum_{i=l_1}^{l_2-1} \frac{\beta'_i}{2^{i+t}} + \cdots + \frac{1}{2^p} \sum_{i=l_p}^{l_{p+1}-1} \frac{\beta'_i}{2^{i+t}} + \cdots \Big) \tag{3.12}$$

denotes the length of the shortest paths between points B and $\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma b'_k}$ where

$$\beta'_i = \begin{cases} 0, & b_i = b'_k \\ 1, & b_i \neq b'_k \end{cases}.$$

Hence, the sum $\dfrac{1}{2^{t+k+1}} + \mathcal{A} + \mathcal{C}$ gives us the length of the shortest paths between points A and B passing through the line segment $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma b'_k})(\widetilde{S}_{\sigma 0} \cap \widetilde{S}_{\sigma b'_k})$.

Theorem 3.3 *Let* $a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots$ *and* $b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots$ *be two representations, respectively, of points A and B on the added Sierpinski triangle such that $a_i = b_i$ for $i = 1, 2, \ldots, k-1$ and $a_k \neq b_k$ and $a_i, b_i \in \{0, 1, 2, 3\}$. If $a_k \neq 0 \neq b_k$, then the intrinsic metric between the code representations of points A and B is formulated as*

$$d(A, B) = \min\left\{ \mathcal{A} + \mathcal{B}, \ \frac{1}{2^{k+t}} + \mathcal{A}' + \mathcal{B}', \ \frac{1}{2^{k+t+1}} + \mathcal{A}'' + \mathcal{B}'' \right\}, \tag{3.13}$$

and if $a_k \neq 0$, $b_k = 0$, *then this formula is obtained as*

$$d(A, B) = \min\left\{ \mathcal{A}'' + \frac{1}{2}\mathcal{B}, \ \frac{1}{2^{t+k+1}} + \mathcal{A}' + \frac{1}{2}\mathcal{B}', \ \frac{1}{2^{t+k+1}} + \mathcal{A} + \mathcal{C} \right\} \tag{3.14}$$

such that $\mathcal{A}, \mathcal{A}', \mathcal{A}'', \mathcal{B}, \mathcal{B}', \mathcal{B}'', \mathcal{C}$ *are defined in Cases 1, 2, 3, 4, 5, and 6.*

Proof We only prove some special cases since the proof of all the cases is extremely long and tedious. Note first that points A and B are in the same subadded Sierpinski triangles $\widetilde{S}_{a_1 a_2 \ldots a_i}$ for $i < k$. Thus, if $a_i \neq 0$ for $i = 1, 2, \ldots, k-1$, then the length of the shortest paths between these points is less than or equal to $\dfrac{1}{2^{k-1}}$. However, an edge length of the subtriangles $\widetilde{S}_1, \widetilde{S}_2$ and $\widetilde{S}_3$ is two times greater than an edge length of $\widetilde{S}_0$. For example, an edge length of the sub-triangle $\widetilde{S}_{123}$ is eight times greater than an edge length of $\widetilde{S}_{000}$. If the element number of the set $\{i \mid a_i = b_i = 0, i < k\}$ is t, then the length of the shortest paths is less than or equal to $\dfrac{1}{2^{k+t-1}}$.

We now begin the proof of Case 3, which involves more complicated situations than Case 1 and Case 2. The proofs of the other cases can also be done in a similar manner.

i) Let $a_{k+1} \neq a_k$ and $a_{k+1} \neq 0$. We first compute the length of the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$. Since $a_i \in \{0, 1, 2, 3\}$ for $i \geq k+2$, there exists a unique number a_μ such that $a_\mu \neq a_k$, $a_\mu \neq a_{k+1}$,

and $a_\mu \neq 0$. If $a_{k+2} \neq a_\mu$, then A is not contained by $\widetilde{S}_{\sigma a_k a_{k+1} a_\mu}$ or $A = \widetilde{S}_{\sigma a_k a_{k+1} a_{k+2}} \cap \widetilde{S}_{\sigma a_k a_{k+1} a_\mu}$. Thus, the shortest paths must pass through $\widetilde{S}_{\sigma a_k a_{k+1} a_{k+2}} \cap \widetilde{S}_{\sigma a_k a_{k+1} a_\mu}$ (that is, the length of the shortest paths is greater than $\dfrac{1}{2^{k+t+2}}$ if A is not contained by $\widetilde{S}_{\sigma a_k a_{k+1} a_\mu}$). If $A = \widetilde{S}_{\sigma a_k a_{k+1} a_{k+2}} \cap \widetilde{S}_{\sigma a_k a_{k+1} a_\mu}$, then the length of the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ equals $\dfrac{1}{2^{k+t+2}}$. If A is contained by $\widetilde{S}_{\sigma a_k a_{k+1} a_\mu}$ (that is, $a_{k+2} = a_\mu$) and $A \neq \widetilde{S}_{\sigma a_k a_{k+1} a_{k+2}} \cap \widetilde{S}_{\sigma a_k a_{k+1} a_\mu}$, then this length is less than $\dfrac{1}{2^{k+t+2}}$. By applying a similar method in the other steps, it is easily seen that if $M = \emptyset$, then the length of the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ is obtained as

$$\mathcal{A}'' = \sum_{i=k+2}^{\infty} \frac{\varphi_i}{2^{i+t}} \tag{3.15}$$

such that

$$\varphi_i = \begin{cases} 0, & a_i = a_\mu \\ 1, & \text{otherwise} \end{cases}.$$

However, if $M \neq \emptyset$, then calculations become a little more complicated. Suppose now that $a_i = 0$ for at least $i \in \{k+2, k+3, k+4, \ldots\}$ and let

$$M = \{i + 1 \mid a_i = 0, i > k + 1\} = \{m_1, m_2, m_3, \ldots\}$$

such that $m_1 < m_2 < m_3 < \ldots$. In this case, point A is the element of the subtriangle $\widetilde{S}_{\sigma a_k \ldots a_{m_1-1} a_{m_1}}$. Thus, the length of the shortest paths between the points $(\widetilde{S}_{\sigma a_k a_{k+1} a_{k+2} \ldots a_{m_1-1}} \cap \widetilde{S}_{\sigma a_k a_{k+1} a_{k+2} \ldots a_\mu})$ and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ is obtained as

$$\sum_{i=k+2}^{m_1-1} \frac{\varphi_i}{2^{i+t}}.$$

However, an edge length of the subtriangle $\widetilde{S}_{\sigma a_k \ldots a_{m_1-2} 0}$ is two times less than an edge length of the subtriangles $\widetilde{S}_{\sigma a_k \ldots a_{m_1-2} 1}$, $\widetilde{S}_{\sigma a_k \ldots a_{m_1-2} 2}$ and $\widetilde{S}_{\sigma a_k \ldots a_{m_1-2} 3}$. The length of the shortest paths between the points $(\widetilde{S}_{\sigma a_k a_{k+1} a_{k+2} \ldots a_{m_1} \ldots a_{m_2-1}} \cap \widetilde{S}_{\sigma a_k a_{k+1} a_{k+2} \ldots a_{m_2-2} a_\mu})$, and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ is obtained as

$$\sum_{i=k+2}^{m_1-1} \frac{\varphi_i}{2^{i+t}} + \frac{1}{2} \sum_{i=m_1}^{m_2-1} \frac{\varphi_i}{2^{i+t}},$$

and if we continue like this, then the length of the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ is computed as in Equation 3.7.

ii) Suppose that $a_{k+1} = 0$. In this case, the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ are determined by the first term (if available), which is different from zero and a_k since there are two different options. That is, if

$$r = \min\{i \mid a_i \neq 0, \ a_i \neq a_k, \ i \geq k + 2\},$$

then one of the shortest paths must pass through the point $(\widetilde{S}_{\sigma a_k 0} \cap \widetilde{S}_{\sigma a_k a_r})$. Note that the length of the shortest path between $(\widetilde{S}_{\sigma a_k 0} \cap \widetilde{S}_{\sigma a_k a_r})$ and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ is $\dfrac{1}{2^{k+t+2}}$. Since $a_{k+1} = 0$, we get $m_1 = k + 2$ and thus M is a nonempty set.

Suppose that A is not contained by $\widetilde{S}_{\sigma a_k 0 a_r}$. That means that $a_{k+2} \neq a_r$. It follows that the shortest paths must pass through $\widetilde{S}_{\sigma a_k 0 a_{k+2}} \cap \widetilde{S}_{\sigma a_k 0 a_r}$ and the length of the shortest paths between $(\widetilde{S}_{\sigma a_k 0} \cap \widetilde{S}_{\sigma a_k a_r})$ and $(\widetilde{S}_{\sigma a_k 0 a_{k+2}} \cap \widetilde{S}_{\sigma a_k 0 a_r})$ is greater than $\dfrac{1}{2^{k+t+3}}$. If $A = \widetilde{S}_{\sigma a_k 0 a_{k+2}} \cap \widetilde{S}_{\sigma a_k 0 a_r}$, then the length of the shortest paths between A and $(\widetilde{S}_{\sigma a_k 0} \cap \widetilde{S}_{\sigma 0 a_r})$ equals $\dfrac{1}{2^{k+t+3}}$. If A is contained by $\widetilde{S}_{\sigma a_k 0 a_r}$ (that is, $a_{k+2} = a_r$), then this length is less than $\dfrac{1}{2^{k+t+3}}$. Following similar steps, the length of the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ is obtained as in Equation 3.8 where

$$\varphi_i = \begin{cases} 0, & a_i = a_r \\ 1, & \text{otherwise} \end{cases}.$$

Note that if

$$\min\{i \mid a_i \neq 0, \ a_i \neq a_k, \ i \geq k+2\} = \emptyset,$$

then for the computation the length of the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ we add in each step the edge lengths of the related subtriangles where the shortest paths pass through. In this case, we get $\varphi_i = 1$ for $i = k+2, k+3, k+4, \ldots$.

iii) In cases a, b, and c, we certainly know that $a_k = a_{k+1}$. Thus, to compute the length of the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$, there are three different options. That is, the shortest paths must pass through one of the points $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k a_l})$ where $a_l \neq a_k$ and $a_l \in \{0, 1, 2, 3\}$. Note that the length of the shortest paths between $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k a_l})$ and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ is $\dfrac{1}{2^{k+t+1}}$. If $a_k = a_{k+1} = a_{k+2}$, then the shortest paths must pass through one of the points $(\widetilde{S}_{\sigma a_k a_k a_k} \cap \widetilde{S}_{\sigma a_k a_k a_l})$. Moreover, the length of the shortest paths between $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k a_l})$ and $(\widetilde{S}_{\sigma a_k a_k a_k} \cap \widetilde{S}_{\sigma a_k a_k a_l})$ is $\dfrac{1}{2^{k+t+2}}$. If $a_{k+2} = a_s \neq a_k$ and $a_s \neq 0$, then there are still two different options and the shortest paths must pass through one of the points $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k a_s})$ or $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k 0})$. Notice that the index s only reduces the number of points that the shortest paths pass through and does not generate an additional length. Moreover, the first term a_r, which is different from a_k and 0 for $i > s$, determines the point where the shortest paths pass through. That means that the shortest paths pass through the point $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k a_s})$ if $a_r = a_s$ or the point $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k 0})$ if $a_r \neq a_s$. Similarly, the index r only determines the point where the shortest paths pass through and does not add an additional length. In the general case $a_k = a_{k+1} = \ldots = a_{s-1} \neq a_s \neq 0$ $(s > k+1)$, a similar way can be followed. Taking into account that M is a nonempty set, we compute the length of the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ as in Equation 3.9 where

$$r \quad = \quad \min\{i \mid a_i \neq 0, \ a_i \neq a_k, \ i > s\},$$

and

$$\varphi_i = \begin{cases} 0, & a_i = a_r \\ 1, & \text{otherwise} \end{cases}$$

for $i \neq s$ and $\varphi_s = 0$ for $i = s$. From the construction, it is clear that $\varphi_i = 1$ for $i \neq s$ and $\varphi_s = 0$ for $i = s$ if

$$\{i \mid a_i \neq 0, \ a_i \neq a_k, \ i \geq k+2\} = \emptyset.$$

Let $a_{k+2} = a_s = 0$. First, the shortest paths between A and $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ must pass through the point $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k 0})$. It is obvious that the length of the shortest paths between $(\widetilde{S}_{\sigma a_k} \cap \widetilde{S}_{\sigma 0})$ and $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k 0})$ is $\dfrac{1}{2^{k+t+1}}$. However, there are still two different options and the shortest paths must pass through one of the points $(\widetilde{S}_{\sigma a_k a_k 0} \cap \widetilde{S}_{\sigma a_k a_k a_l})$ where $a_l \in \{1,2,3\}$ and $a_l \neq a_k$. Note that the length of the shortest paths between $(\widetilde{S}_{\sigma a_k a_k 0} \cap \widetilde{S}_{\sigma a_k a_k a_l})$ and $(\widetilde{S}_{\sigma a_k a_k} \cap \widetilde{S}_{\sigma a_k 0})$ is $\dfrac{1}{2^{k+t+3}}$. In this case, the first term a_r, which is different from a_k and 0 for $i > s$, determines the point where the shortest paths pass through. That means that the shortest path must pass through the points $(\widetilde{S}_{\sigma a_k a_k 0} \cap \widetilde{S}_{\sigma a_k a_k a_r})$. Note that the index r only determines the point where the shortest paths pass through and does not generate an additional length. Hence, generally if $a_k = a_{k+1} = \ldots = a_{s-1}$ and $a_s = 0 \ (s > k+1)$, then we get Equation 3.9 for the computation of the length of the shortest paths where

$$\varphi_i = \begin{cases} 0, & a_i = a_r \\ 1, & \text{otherwise} \end{cases}$$

for $i \neq s$ and $\varphi_s = \dfrac{1}{2}$ for $i = s$. It is clear that if $\{i \mid a_i \neq 0, \ a_i \neq a_k, \ i \geq k+2\} = \emptyset$, then we obtain $\varphi_i = 1$ for $i \neq s$ and $\varphi_s = \dfrac{1}{2}$ for $i = s$.

In the case $a_k = a_{k+1} = a_{k+2} = \ldots$, we add in each step the edge lengths of the related subtriangles where the shortest paths pass. Thus, we obtain $\varphi_i = 1$ for $i = k+2, k+3, k+4, \ldots$. $\qquad\square$

Now we give some geometrical properties of $\widetilde{S}$ by using the intrinsic metric formula given in Theorem 3.3. In the following propositions and lemma, we consider the code representations of A, B, C as $\sigma a_k a_{k+1} a_{k+2} \ldots$, $\sigma b_k b_{k+1} b_{k+2} \ldots$, and $\sigma c_k c_{k+1} c_{k+2} \ldots$ where $a_i, b_i, c_i \in \{0,1,2,3\}$ for $i = 1,2,3\ldots$ and $\sigma = a_1 a_2 \ldots a_{k-1}$. Note that if A is a vertex point of $\widetilde{S}_\sigma$, then this point has the code representation $\sigma a_k a_k a_k \ldots$ where $a_k \in \{1,2,3\}$.

Proposition 3.4 *If A and B are the points on $\widetilde{S}_\sigma$, then $d(A,B) \leq \dfrac{1}{2^{k+t-1}}$. Furthermore, if $a_k = 0$ or $b_k = 0$, then $d(A,B) \leq \dfrac{3}{2^{k+t+1}}$.*

Proof Let A and B be the points on $\widetilde{S}_\sigma$ and let $a_k \neq 0$ and $b_k \neq 0$. To obtain the maximum value of $\mathcal{A} + \mathcal{B}$, it must be $a_k \neq b_k$, $M = L = \emptyset$, and $\alpha_i = \beta_i = 1$ for $i = k+1, k+2, k+3, \ldots$. Therefore, we compute $\mathcal{A} + \mathcal{B} = \dfrac{1}{2^{k+t}} + \dfrac{1}{2^{k+t}} = \dfrac{1}{2^{k+t-1}}$ from formulas (3.3) and (3.4). This shows that $d(A,B) \leq \dfrac{1}{2^{k+t-1}}$ owing to the fact that

$$d(A,B) = \min\left\{ \frac{1}{2^{k+t-1}}, \frac{1}{2^{t+k}} + \mathcal{A}' + \mathcal{B}', \frac{1}{2^{t+k+1}} + \mathcal{A}'' + \mathcal{B}'' \right\}.$$

Suppose now that $a_k \neq 0$ and $b_k = 0$ (the other case is done similarly). For the computation of the maximum value of $\mathcal{A}'' + \frac{1}{2}\mathcal{B}$, we must take into account that $a_k \neq b_k$, $M = L = \emptyset$, and $\varphi_i = \beta_i = 1$ for $i = k+1, k+2, k+3, \ldots$. Hence, we must use the formula given in Case 3-iii-c (see formula (3.9)) for the maximum value of $\mathcal{A}''$. This shows that

$$\mathcal{A}'' = \frac{1}{2^{k+t+1}} + \sum_{i=k+2}^{\infty} \frac{1}{2^{i+t}} = \frac{1}{2^{k+t+1}} + \frac{1}{2^{k+t+1}} = \frac{1}{2^{k+t}}.$$

We also compute

$$\frac{1}{2}\mathcal{B} = \frac{1}{2} \sum_{i=k+1}^{\infty} \frac{1}{2^{i+t}} = \frac{1}{2^{k+t+1}}$$

from (3.10). Thus, we obtain the maximum value as $\mathcal{A}'' + \frac{1}{2}\mathcal{B} = \frac{1}{2^{k+t}} + \frac{1}{2^{k+t+1}} = \frac{3}{2^{k+t+1}}$. Since

$$d(A, B) = \min\left\{ \frac{3}{2^{k+t+1}}, \ \frac{1}{2^{t+k+1}} + \mathcal{A}' + \frac{1}{2}\mathcal{B}', \ \frac{1}{2^{t+k+1}} + \mathcal{A} + \mathcal{C} \right\},$$

we compute $d(A, B) \leq \frac{3}{2^{k+t+1}}$. □

Lemma 3.5 *Let A be a vertex point and B be any point of $\widetilde{S}_\sigma$, and let the code representation of these points be $\sigma a_k a_k a_k \ldots$ and $\sigma b_k b_{k+1} b_{k+2} \ldots$, respectively, where $a_k \in \{1, 2, 3\}$, $b_i \in \{0, 1, 2, 3\}$ and $a_k \neq b_k$.*

a) If $b_k \neq 0$, then $d(A, B) = \mathcal{A} + \mathcal{B}$.

b) If $b_k = 0$, then $d(A, B) = \mathcal{A}'' + \frac{1}{2}\mathcal{B}$.

Proof

a) We first know that $\mathcal{A} + \mathcal{B} \leq \frac{1}{2^{k+t-1}}$ due to Proposition 3.4. We also get $\gamma_i = 1$ since $a_i = a_k$ for $i = k+1, k+2, k+3, \ldots$. In this case, we compute

$$\mathcal{A}' = \sum_{i=k+1}^{\infty} \frac{1}{2^{i+t}} = \frac{1}{2^{k+t}}.$$

This shows that

$$\frac{1}{2^{k+t}} + \mathcal{A}' + \mathcal{B}' = \frac{1}{2^{k+t}} + \frac{1}{2^{k+t}} + \mathcal{B}' = \frac{1}{2^{k+t-1}} + \mathcal{B}' \geq d(A, B).$$

Moreover, it is obvious that

$$\mathcal{A} = \sum_{i=k+1}^{\infty} \frac{1}{2^{i+t}} = \frac{1}{2^{k+t}} = \mathcal{A}'' = \frac{1}{2^{k+t+1}} + \sum_{i=k+2}^{\infty} \frac{1}{2^{i+t}}.$$

We now compute $\mathcal{B}$ and $\mathcal{B}''$, respectively, and then we compare them. First suppose that $b_{k+1} = a_k$. In this case, we get $\beta_{k+1} = 0$. This shows that

$$\mathcal{B} = \sum_{i=k+2}^{l_1-1} \frac{\beta_i}{2^{i+t}} + \frac{1}{2} \sum_{i=l_1}^{l_2-1} \frac{\beta_i}{2^{i+t}} + \cdots + \frac{1}{2^p} \sum_{i=l_p}^{l_{p+1}-1} \frac{\beta_i}{2^{i+t}} + \cdots \leq \frac{1}{2^{k+t+1}}$$

and thus

$$\mathcal{B} \leq \frac{1}{2^{k+t+1}} + \mathcal{B}''.$$

It follows that $\mathcal{A} + \mathcal{B} \leq \frac{1}{2^{k+t+1}} + \mathcal{A}'' + \mathcal{B}''$. Similar cases are also valid for $b_{k+1} = b_k$ and $b_{k+1} = 0$.

However, if $b_{k+1} = c_k$, then $\mathcal{A} + \mathcal{B} = \frac{1}{2^{k+t+1}} + \mathcal{A}'' + \mathcal{B}''$. Since b_μ equals a_k, we have $\beta_i = \theta_i$ $i = k+2, k+3, k+4, \ldots$.

b) It is seen that $\mathcal{A}'' + \frac{1}{2}\mathcal{B} \leq \frac{3}{2^{k+t+1}}$ from Proposition 3.4. We also have $\alpha_i = \gamma_i = 1$ for $i = k+1, k+2, k+3, \ldots$ and thus $\mathcal{A} = \mathcal{A}' = \frac{1}{2^{k+t}}$.

This shows that

$$\frac{1}{2^{k+t+1}} + \mathcal{A}' + \frac{1}{2}\mathcal{B}' \geq \frac{3}{2^{k+t+1}} \quad \text{and} \quad \frac{1}{2^{k+t+1}} + \mathcal{A} + \frac{1}{2}\mathcal{C} \geq \frac{3}{2^{k+t+1}}.$$

This completes the proof. □

Proposition 3.6 *Let A, B, C be the vertices points of $\widetilde{S}_\sigma$. If X is any point of $\widetilde{S}_\sigma$, then*

$$d(A, X) + d(B, X) + d(C, X) = \frac{1}{2^{k+t-2}}.$$

Proof Let the code representation of X be $\sigma x_k x_{k+1} x_{k+2} \ldots$ where $x_i \in \{0, 1, 2, 3\}$ for $i = k, k+1, k+2, \ldots$. There are two cases such that $x_k = 0$ and $x_k \neq 0$. We give the proof of the first case (the other case is similarly done). We use Case b given in Lemma 3.5 since the points A, B, C are the vertices points of $\widetilde{S}_\sigma$. Since $a_i = a_k$, $b_i = b_k$, and $c_i = c_k$ for $i = k+1, k+2, k+3, \ldots$, each of $\mathcal{A}''$s in $d(A, X), d(B, X)$, and $d(C, X)$ equals $\frac{1}{2^{k+t}}$ and thus the sum of them is obtained as $\frac{3}{2^{k+t}}$. We now compute the sum of $\mathcal{B}$s in $d(A, X), d(B, X)$, and $d(C, X)$. Obviously, x_{k+1} equals one of the values a_k, b_k, c_k, or 0. Let $i \geq k+1$. Note that if $x_i = 0$, then we obtain $\beta_i = 1$ for the computation of each $\mathcal{B}$s in $d(A, X), d(B, X)$, and $d(C, X)$. If $x_i = a_k$, then we obtain $\beta_i = 0$ for the computation $\mathcal{B}$ in $d(A, X)$ and $\beta_i = 1$ for the computation of each $\mathcal{B}$ in $d(B, X)$ and $d(C, X)$. That is, if $x_i \neq 0$, one of the β_i equals 0 and the other two equal 1 for the computation of $\mathcal{B}$s in $d(A, X), d(B, X)$, and $d(C, X)$. Therefore, the sum of $\mathcal{B}$s in $d(A, X), d(B, X)$, and $d(C, X)$ turns into the form

$$\frac{1}{2}\left(\frac{1}{2^{k+t}} + \frac{1}{2^{k+t+1}} + \frac{1}{2^{k+t+2}} + \cdots\right) = \frac{1}{2^{k+t}}$$

in each case. Consequently, we obtain

$$d(A, X) + d(B, X) + d(C, X) = \frac{3}{2^{k+t}} + \frac{1}{2^{k+t}} = \frac{1}{2^{k+t-2}}.$$

Therefore, the proof is completed. □

Remark 3.7 *Let the code representation of C be $000\ldots$. We now consider the triangle $\widetilde{T}_0$ given as the initial set in Figure 2. It is easily seen that the point C is the centroid of $\widetilde{T}_0$.*

Proposition 3.8 *The distance between the vertex points of $\widetilde{S}_\sigma$ and the points whose code representations are $\sigma 000\ldots$ equals $\frac{1}{3.2^{k+t-2}}$.*

Proof The code representation of A is $\sigma a_k a_k a_k \ldots$ where $a_k \in \{1, 2, 3\}$ since A is any vertex point of $\widetilde{S}_\sigma$. We use the formula $d(A, B) = \mathcal{A}'' + \frac{1}{2}\mathcal{B}$ to compute the shortest distance between these points from Lemma 3.5. We first get

$$\mathcal{A}'' = \frac{1}{2^{k+t+1}} + \sum_{i=k+2}^{\infty} \frac{1}{2^{i+t}} = \frac{1}{2^{k+t+1}} + \frac{1}{2^{k+t+1}} = \frac{1}{2^{k+t}},$$

due to the fact that $a_k = a_{k+1} = a_{k+2} = \ldots$. Also, we compute

$$
\begin{aligned}
\frac{1}{2}\mathcal{B} &= \frac{1}{2}\Big(\sum_{i=k+1}^{k+1} \frac{1}{2^{i+t}} + \frac{1}{2}\sum_{i=k+2}^{k+2} \frac{1}{2^{i+t}} + \cdots + \frac{1}{2^r}\sum_{i=k+r+1}^{k+r+1} \frac{1}{2^{i+t}} + \cdots \Big) \\
&= \frac{1}{2}\Big(\frac{1}{2^{k+t+1}} + \frac{1}{2}\frac{1}{2^{k+t+2}} + \cdots + \frac{1}{2^r}\frac{1}{2^{k+r+t+1}} + \cdots \Big) \\
&= \frac{1}{2^{k+t+2}}\Big(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots \Big) \\
&= \frac{1}{3}\frac{1}{2^{k+t}},
\end{aligned}
$$

since $l_i = i + 1$ and $\beta_i = 1$ for $i = k+1, k+2, k+3, \ldots$. This shows that

$$d(A, B) = \mathcal{A}'' + \frac{1}{2}\mathcal{B} = \frac{1}{2^{k+t}} + \frac{1}{3}\frac{1}{2^{k+t}} = \frac{1}{3}\frac{1}{2^{k+t-2}}.$$

□

3.1. Some instructive examples

Example 3.9 *Suppose that the code representation of A is $1023111\ldots$ and the code representation of B is $200333\ldots$. We now compute the length of the shortest paths and find one of the shortest paths.*

First, we get $k = 1$ since $k = \min\{i \mid a_i \neq b_i\}$ and $a_1 \neq b_1$. Also, we obtain $t = 0$ owing to the fact that $\{i \mid a_i = b_i = 0, i < k\} = \emptyset$. Moreover, we have

$$
\begin{aligned}
M &= \{i+1 \mid a_i = 0, i > 1\} = \{3\}, \\
L &= \{i+1 \mid b_i = 0, i > 1\} = \{3, 4\},
\end{aligned}
$$

and we thus compute $m_1 = 3$, $l_1 = 3$, and $l_2 = 4$. Note that we consider the related formulas given in Cases 1, 2, and 3, respectively, since $a_1 \neq 0 \neq b_1$. From Equations 3.3 and 3.4, we have

$$
\begin{aligned}
\mathcal{A} &= \frac{1}{2^2} + \frac{1}{2}\left(\frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}\cdots\right) = \frac{1}{2^2} + \frac{1}{2^4} = \frac{5}{16}, \\
\mathcal{B} &= \frac{1}{2^2} + \frac{1}{2}\frac{1}{2^3} + \frac{1}{2^2}\left(\frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots\right) = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^5} = \frac{11}{32}.
\end{aligned}
$$

Thus, the length of the shortest paths passing through $\widetilde{S}_{a_k} \cap \widetilde{S}_{b_k}$ equals

$$
\mathcal{A} + \mathcal{B} = \frac{5}{16} + \frac{11}{32} = \frac{21}{32}
$$

(see blue path in Figure 8). From Equations 3.5 and 3.6, we obtain

$$
\begin{aligned}
\mathcal{A}' &= \frac{1}{2^2} + \frac{1}{2}\left(\frac{1}{2^3} + \frac{0}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7}\cdots\right) = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^5} = \frac{11}{32}, \\
\mathcal{B}' &= \frac{1}{2^2} + \frac{1}{2}\frac{1}{2^3} + \frac{1}{2^2}\left(\frac{0}{2^4} + \frac{0}{2^5} + \frac{0}{2^6} + \cdots\right) = \frac{1}{2^2} + \frac{1}{2^4} = \frac{5}{16},
\end{aligned}
$$

since $c_k = 3$. Therefore, the length of the shortest paths passing through $(\widetilde{S}_{a_k} \cap \widetilde{S}_{c_k})(\widetilde{S}_{b_k} \cap \widetilde{S}_{c_k})$ is computed as

$$
\frac{1}{2} + \mathcal{A}' + \mathcal{B}' = \frac{1}{2} + \frac{11}{32} + \frac{5}{16} = \frac{37}{32}
$$

(see yellow path in Figure 8). For the computation of $\mathcal{A}''$ and $\mathcal{B}''$ we get r as 3 and 4, respectively. That is, we have $a_r = 2$ and $b_r = 3$. Then the following is obtained from Equation 3.8:

$$
\begin{aligned}
\mathcal{A}'' &= \frac{1}{2^3} + \frac{1}{2}\left(\frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots\right) = \frac{1}{2^3} + \frac{1}{2^4} = \frac{3}{16}, \\
\mathcal{B}'' &= \frac{1}{2^3} + \frac{1}{2}\left(\frac{1}{2^3} + \frac{0}{2^4} + \frac{0}{2^5} + \frac{0}{2^6}\cdots\right) = \frac{1}{2^3} + \frac{1}{2^4} = \frac{3}{16}.
\end{aligned}
$$

Therefore, the length of the shortest paths passing through $(\widetilde{S}_{a_k} \cap \widetilde{S}_0)(\widetilde{S}_{b_k} \cap \widetilde{S}_0)$ is computed as

$$
\frac{1}{2^2} + \mathcal{A}'' + \mathcal{B}'' = \frac{1}{2^2} + \frac{3}{16} + \frac{3}{16} = \frac{5}{8}
$$

(see red path in Figure 8). Hence, we compute

$$
d(A, B) = \min\left\{\frac{21}{32}, \frac{37}{32}, \frac{5}{8}\right\} = \frac{5}{8}
$$

and one of the shortest paths is the red path given in Figure 8.

Example 3.10 Let the code representation of A be $0330\overline{1} = 0330111\ldots$ and let the code representation of B be $00\overline{02} = 0020202\ldots$. We must use Formula 3.14 since $a_2 = 3$, $b_2 = 0$, $k = 2$, and $t = 1$. First, we compute
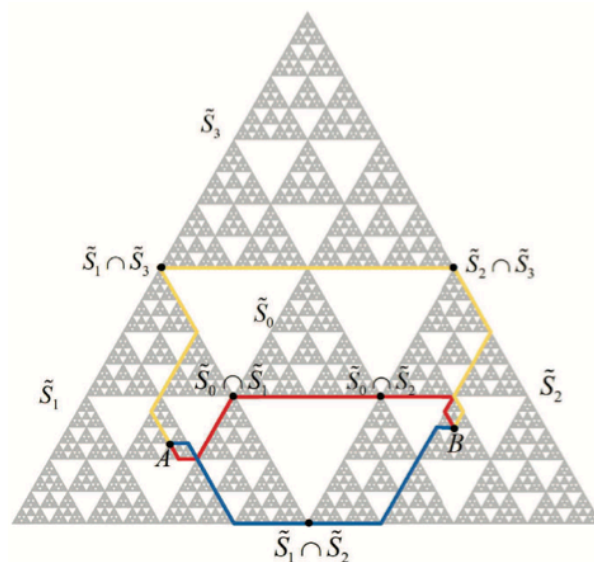
Figure 8. Some of the paths that pass through the intersection points between A and B for $a_1 = 1$ and $b_1 = 2$.

the length of the shortest paths between A and $\widetilde{S}_{03} \cap \widetilde{S}_{00}$. Since $a_2 = a_3 = 3$, we obtain $a_s = 1$ from Case 3-iii-b. Thus, we compute

$$\mathcal{A}'' = \frac{1}{2^{2+1+1}} + \frac{\frac{1}{2}}{2^{4+1}} + \frac{1}{2}\left(\frac{0}{2^{5+1}} + \frac{0}{2^{6+1}} + \frac{0}{2^{7+1}} + \cdots\right) = \frac{1}{2^4} + \frac{1}{2^6} = \frac{5}{64}.$$

Moreover, the length of the shortest paths between the points B and $\widetilde{S}_{03} \cap \widetilde{S}_{00}$ is computed as

$$\frac{1}{2}\mathcal{B} = \frac{1}{2}\left[\frac{1}{2^{3+1}} + \frac{1}{2^{4+1}} + \frac{1}{2}\left(\frac{1}{2^{5+1}} + \frac{1}{2^{6+1}}\right) + \frac{1}{2^2}\left(\frac{1}{2^{7+1}} + \frac{1}{2^{8+1}}\right) + \cdots\right] = \frac{3}{56}$$

(see Case 4). Now let us take $c_2 = 2$. To compute the length of the shortest paths between A and B passing through $(\widetilde{S}_{03} \cap \widetilde{S}_{02})(\widetilde{S}_{00} \cap \widetilde{S}_{02})$, we use Case 5 as follows:

$$\mathcal{A}' = \frac{1}{2^{3+1}} + \frac{1}{2^{4+1}} + \frac{1}{2}\left(\frac{1}{2^{5+1}} + \frac{1}{2^{6+1}} + \frac{1}{2^{7+1}} + \cdots\right) = \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} = \frac{7}{64}.$$

Moreover, the length of the shortest paths between points B and $\widetilde{S}_{03} \cap \widetilde{S}_{00}$ is computed as

$$\frac{1}{2}\mathcal{B}' = \frac{1}{2}\left[\frac{0}{2^{3+1}} + \frac{1}{2^{4+1}} + \frac{1}{2}\left(\frac{0}{2^{5+1}} + \frac{1}{2^{6+1}}\right) + \frac{1}{2^2}\left(\frac{0}{2^{7+1}} + \frac{1}{2^{8+1}}\right) + \cdots\right] = \frac{1}{56}.$$

We also get $b_2' = 1$ since $a_2 = 3$ and $c_2 = 2$. For the computation of the length of the shortest paths between A and B passing through $(\widetilde{S}_{03} \cap \widetilde{S}_{01})(\widetilde{S}_{00} \cap \widetilde{S}_{01})$, the lengths of the shortest paths between points A and $\widetilde{S}_{03} \cap \widetilde{S}_{01}$ and between points B and $\widetilde{S}_{00} \cap \widetilde{S}_{01}$ are obtained:

$$\mathcal{A} = \frac{1}{2^{3+1}} + \frac{1}{2^{4+1}} + \frac{1}{2}\left(\frac{0}{2^{5+1}} + \frac{0}{2^{6+1}} + \frac{0}{2^{7+1}} + \cdots\right) = \frac{1}{2^4} + \frac{1}{2^5} = \frac{3}{32},$$

$$\mathcal{C} \;=\; \frac{1}{2}\left[\frac{1}{2^{3+1}} + \frac{1}{2^{4+1}} + \frac{1}{2}\left(\frac{1}{2^{5+1}} + \frac{1}{2^{6+1}}\right) + \frac{1}{2^2}\left(\frac{1}{2^{7+1}} + \frac{1}{2^{8+1}}\right) + \cdots\right] = \frac{3}{56},$$

respectively (see Case 6). It follows that

$$d(A,B) = \min\left\{\frac{5}{64} + \frac{3}{56}, \frac{1}{16} + \frac{7}{64} + \frac{1}{56}, \frac{1}{16} + \frac{3}{32} + \frac{3}{56}\right\} = \min\left\{\frac{59}{448}, \frac{85}{448}, \frac{47}{224}\right\} = \frac{59}{448},$$

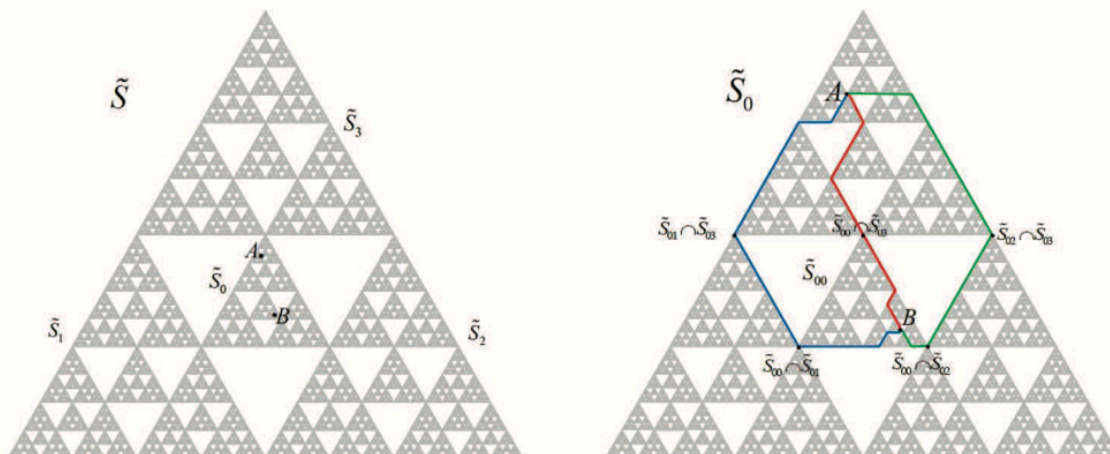*from Formula 3.14. In Figure 9, the red path is one of the shortest paths between A and B.*

Figure 9. Some of the paths that pass through the intersection points between A and B for $a_2 = 3$ and $b_2 = 0$.

4. Conclusion

The intrinsic metric formulas can be defined to examine the geometric properties of different fractals via the code representations of points on them. However, as seen in this model, it is even more difficult to define the intrinsic metric formula on the code set of fractals that have different contraction coefficients. This paper has a different importance from other works given in the literature since it provides the first intrinsic metric formula to be written using the code representations of points on a fractal set that has different contraction coefficients of the related IFS. The formula is also very useful for proving different geometric properties of $\widetilde{S}$. Moreover, this paper will be a guide for different works such as classifications of geodesics.

Acknowledgment

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