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Prolongations of isometric actions to vector bundles

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Abstract: In this paper, we define an isometry on a total space of a vector bundle $\mathbb{E}$ by using a given isometry on the base manifold $\mathbb{M}$. For this definition, we assume that the total space of the bundle is equipped with a special metric which has been introduced in one of our previous papers. We prove that the set of these derived isometries on $\mathbb{E}$ form an imbedded Lie subgroup $\widetilde{G}$ of the isometry group $I(\mathbb{E})$. Using this new subgroup, we construct two different principal bundle structures based one on $\mathbb{E}$ and the other on the orbit space $\mathbb{E}/\widetilde{G}$.

Key words: Fiber bundles, isometry group, vector bundles, principal bundles

1. Introduction

The theory of vector bundles is an important tool used in the field of differential and algebraic geometry. A vector bundle can be thought of as a vector space, varying continuously along a given manifold. Thus, a vector bundle represents "linearization" of nonlinear structure of manifolds. Therefore, they are in many ways much easier to work with than the base manifolds. In algebraic geometry, vector bundles are used to provide functions over algebraic varieties. They are referred as rank 2 stable vector bundles on the complex projective space P^3 [8]. Several other approaches of vector bundles can be found in [3–5, 13, 15]. In this study, we illustrate an application of vector bundles by prolonging isometric actions from a manifold (base manifold) to another by using a vector bundle structure. Since the isometric actions require the detailed study of isometric groups, we would like to mention a few studies in the literature in this regard.

The study of isometry groups is a productive field due to its close relation with symmetric groups and with the proper actions of Riemannian manifolds [2]. There are various studies regarding the theory of isometry groups such as isometry groups of compact manifolds [1, 18], homogeneous manifolds [19], classifications of isometry groups [7, 12, 14], determining the dimension of the isometry group of a finite dimensional Riemannian manifold [9, 16] etc.

The study of isometry groups is often preferred because an isometry group of a Riemannian manifold carries both geometric, and therefore topologic, and algebraic structures. Namely, if $(\mathbb{M}, \mathfrak{g})$ is an n-dimensional Riemannian manifold, then we recall that Myers and Steenrod [17] showed that the isometry group $I(\mathbb{M})$ of a Riemannian manifold $\mathbb{M}$ is a Lie group acting on $\mathbb{M}$ as a Lie transformation group. This group is compact if the manifold is compact. The topology on $I(\mathbb{M})$ is the compact-open topology which was introduced by Ralph

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Fox [6] as follows:

Definition 1.1 *The compact-open topology is a topology that is defined on the set of continuous maps between two topological spaces M and N. Let $C(M,N)$ be the set of all continuous maps between M and N. Given a compact subset $K \subset M$ and an open subset $U \subset N$, let $C(M,N)$ be the set of all functions $f \in C(M,N)$ such that $f(K) \subset U$. Then the collection of all such $V(K,U)$ is a sub-base for the compact open topology on $C(M,N)$.*

Therefore, throughout this paper, we use above topology on the group of isometries of any manifold. Now we would like to mention the special metric (induced metric) on the vector bundle $(\mathbb{E}, \pi, \mathbb{M})$, which is constructed by using the Riemannian structure of the base manifold and the notion of partitions of unity, given by the following [10]:

$$\tilde{\mathfrak{g}}(\bar{V}, \bar{W}) = \mathfrak{g}(\pi_*(\bar{V}), \pi_*(\bar{W})) + \mathfrak{g}_E(\gamma_*(\bar{V}), \gamma_*(\bar{W})) \tag{1.1}$$

where $\mathfrak{g}$ is a Riemannian metric defined on $\mathbb{M}$, $\mathfrak{g}_E$ is the usual metric on the vector space $TE = E \times E$, and $\gamma = pr_2 \circ \Phi$ where Φ represents the local trivialization of the bundle. In Section 2, we redefine this metric considering our newly introduced prolongation. We also define an isometry preserving map between isometry groups of $\mathbb{M}$ to $\mathbb{E}$. Using this map, we construct an imbedded Lie subgroup $\tilde{G} \subset I(\mathbb{E})$. In Section 3, we give two applications of the newly proposed actions of $\tilde{G}$. First we introduce a principal bundle based on $\mathbb{E}$ with the structure group $\tilde{G} \subset I(\mathbb{E})$. Secondly, we obtain another principal bundle based on the orbit space $\mathbb{E}/\widetilde{G}$ by using free isometric actions $I(\mathbb{M})$ on $\mathbb{M}$.

In this work, all manifolds are considered as Hausdorff, second countable, and connected.

2. Metric structure

In this section, we give the metric structure that we plan to use throughout this paper but before that, we start with defining how to fix a trivialization of a given bundle:

2.1. Countable set of trivializations

Suppose that $(\mathbb{E}, \pi, \mathbb{M})$ be a (fiber) bundle. Now consider the following remark:

Remark 2.1 *Since the base manifold $\mathbb{M}$ is second countable, one can always find a countable family $\{U^\alpha\}_{\alpha \in \mathbb{N}}$ of open domains of local coordinates, which are simultaneously domains of the local trivializations. One needs only to observe that the restriction of a local trivialization is again a trivialization and use the Steenrod-Construction of Bundles to create the given bundle $(\mathbb{E}, \pi, \mathbb{M})$. Therefore, without loss of generality, we use a countable set of local trivializations $(\pi^{-1}(U_\alpha), \Phi_\alpha)$ on the given vector bundle $(\mathbb{E}, \pi, \mathbb{M})$.*

Using above remark, one can always fix a countable set of trivializations $\mathbb{B}$ of a given bundle as the following:

$$\mathbb{B} = \{\Phi_\alpha | \;\; \Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times E\}_{\alpha \in I}, \tag{2.1}$$

where I is a subset of $\mathbb{N}$. Using this, we define

$$\tau : \mathbb{E} \to I \tag{2.2}$$

with $\tau(x) = min\{i \in I| \;\; x \in U_i\}$. In the following proposition, we prove that τ is well-defined.

Proposition 2.2 *The relation τ given by the Equation (2.2) is well defined.*

Proof Suppose that $x_1 = x_2$, and $\tau(x_1) = \alpha$ and $\tau(x_2) = \beta$.Then $x_1 \in U_\beta$ and $x_2 \in U_\alpha$.

Since $\tau(x_2) = \beta$ and $x_2 \in U_\alpha$, then from definition of τ, $\beta \leq \alpha$. On the other hand since $\tau(x_1) = \alpha$ and $x_1 \in U_\beta$ then from definition of τ, $\alpha \leq \beta$. Therefore, $\alpha = \beta$. This finishes the proof. □

Now we redefine the induced metric given by Equation (1.1).

Definition 2.3 *Let $\pi : \mathbb{E} \to \mathbb{M}$ be a vector bundle, and $(U_{\tau(x)}, E, \Phi_{\tau(x)})$ be a local fiber bundle trivialization of $\mathbb{E}$ with $h \in \pi^{-1}(U_{\tau(x)})$ and identify TE with $E \times E$. Then by Theorem 3 in [10], there exists an induced Riemannian metric $\widetilde{\mathfrak{g}}$ on $\mathbb{E}$ as follows:*

$$\tilde{\mathfrak{g}}(V_h, W_h) = \mathfrak{g}((\pi)_*(V), (\pi)_*(W)) + Q((pr_2 \circ \Phi_{\tau(x)})_*(V), (pr_2 \circ \Phi_{\tau(x)}))_*(W)) \tag{2.3}$$

where Q is the usual metric on the vector space $TE = E \times E$.

Since $(\mathbb{E}, \pi, \mathbb{M})$ is a vector bundle, then there always exists (global) zero section on this bundle, with

$$\pi \circ \mathfrak{O} = id_M \tag{2.4}$$

where id_M represents the identity map.

Remark 2.4 *Zero section $\mathfrak{O} : (\mathbb{M}, \mathfrak{g}) \to (\mathbb{E}, \tilde{\mathfrak{g}})$ is a distance preserving map.*

Proof

Let $pr_2 \circ \Phi_{\tau(x)} \circ \mathfrak{O} = \mathfrak{O}^2$. Then for all $x \in \mathbb{M}$, $\mathfrak{O}^2(x) = (pr_2 \circ \Phi_{\tau(x)})(0_x)$, where 0_x denotes the zero vector in $\pi^{-1}\{x\}$. Then we have

$$\mathfrak{O}^2(x) = pr_2(\Phi_{\tau(x)}(0_x)) = pr_2(x, 0) = 0,$$

which implies the following:

$$(\mathfrak{O}^2) = 0 \tag{2.5}$$

Now let $v_x, w_x \in T_x\mathbb{M}$. By Equations (2.3) and (2.5) we have

$$\begin{aligned} \tilde{\mathfrak{g}}(\mathfrak{O}_*(v_x), \mathfrak{O}_*(w_x)) &= \mathfrak{g}(\pi_*(\mathfrak{O}_*(v_x)), \pi_*(\mathfrak{O}_*(w_x))) + Q((\mathfrak{O}^2)_*(v_x), (\mathfrak{O}^2)_*(w_x)) \\ &= \mathfrak{g}((\pi \circ \mathfrak{O})_*(v_x), (\pi \circ \mathfrak{O})_*(w_x)) \\ &= \mathfrak{g}(v_x, w_x) \end{aligned}$$

which completes the proof.

□

So far we have seen that for any given bundle $(\mathbb{E}, \pi, \mathbb{M})$, we can fix a countable set of trivializations $\mathbb{B}$. Also, by the above proposition, we show that one can choose a specific trivialization in the Φ_i where a given x belongs to the image of the domains of trivializations under the bundle projection map π. Now we are ready to define a prolonged map F.

Suppose that we fix a countable local trivialization set $\mathbb{B}$ for the bundle $(\mathbb{E}, \pi, \mathbb{M})$. Now we define $F : \mathbb{E} \to \mathbb{E}$ as follows:

$$F(h) = \Phi^{-1}_{\tau(f(x))}(f \circ \pi, pr_2 \circ \Phi_{\tau(x)})(h),$$

where $h \in \pi^{-1}\{x\}$, and f is an isometry on M.

Theorem 2.5 *The relation F is a function.*

Proof

Given $f \in I(\mathbb{M})$, let $h_1 = h_2$, where $h_1, h_2 \in \pi^{-1}(x)$ then

$$(f \circ \pi, pr_2 \circ \Phi_{\tau(x)})(h_1) = (f \circ \pi, pr_2 \circ \Phi_{\tau(x)})(h_2). \tag{2.6}$$

Expressions in both left and right side of the equation are elements of $\{f(x)\} \times E$; therefore, we use the same fixed local trivialization for $f(x)$ and denote it by $\Phi_{\tau(f(x))}$. Using this local trivialization, we have

$$\Phi^{-1}_{\tau(f(x))}(f \circ \pi, pr_2 \circ \Phi_{\tau(x)})(h_1) = \Phi^{-1}_{\tau(f(x))}(f \circ \pi, pr_2 \circ \Phi_{\tau(x)})(h_2) \tag{2.7}$$

Therefore, $\Phi^{-1}_{\tau(f(x))}(f \circ \pi, pr_2 \circ \Phi_{\tau(x)})$ is well defined, and can be expressed as a function from $\mathbb{E}$ to itself. □

Remark 2.6 *F is a fiber preserving map.*

Proof

Since $\Phi^{-1}_{\tau(f(x))}$ is a trivialization, $pr_1 \circ \Phi^{-1}_{\tau(f(x))} = \pi$ where pr_1 stands for the first projection map. Therefore,

$$\pi \circ \Phi^{-1}_{\tau(f(x))} = pr_1. \tag{2.8}$$

On the other hand, we recall that the function F is defined by the following:

$$F = \Phi^{-1}_{\tau(f(x))}(f \circ \pi, pr_2 \circ \Phi_{\tau(x)}).$$

Using definition of F together with Equation (2.8), we have

$$\begin{aligned}(\pi \circ F)(h) &= (\pi \circ \Phi^{-1}_{\tau(f(x))})((f \circ \pi)(h), (pr_2 \circ \Phi_{\tau(x)})(h)) \\ &= pr_1((f \circ \pi)(h), (pr_2 \circ \Phi_{\tau(x)})(h)) \\ &= (f \circ \pi)(h),\end{aligned}$$

which implies

$$\pi \circ F = f \circ \pi. \tag{2.9}$$

This completes the proof. □

Lemma 2.7 *For all $x \in \mathbb{M}$, $(pr_2 \circ \Phi_{\tau(f(x))} \circ F) = pr_2 \circ \Phi_x$.*

Proof

For all $x \in \mathbb{M}$, we have

$$\begin{aligned}(pr_2 \circ \Phi_{\tau(f(x))} \circ F) &= (pr_2 \circ \Phi_{\tau(f(x))})(\Phi^{-1}_{\tau(f(x))}(f \circ \pi, pr_2 \circ \Phi_{\tau(x)})\\ &= pr_2 \circ (f \circ \pi, pr_2 \circ \Phi_{\tau(x)})\\ &= pr_2 \circ \Phi_{\tau(x)}\end{aligned}$$

which finishes the proof.

□

Theorem 2.8 *F is an isometry with respect to $\tilde{\mathfrak{g}}$.*

Proof

We begin the proof by showing that the function F is a bijection:

Suppose that $F(h) = F(h')$ for any $h, h' \in \mathbb{E}$. Then $(\pi \circ F)(h) = (\pi \circ F)(h')$, which implies $(f \circ \pi)(h) = (f \circ \pi)(h')$. Since f is one-to-one, $\pi(h) = \pi(h')$. Thus, h, h' belongs to the same fiber.

Suppose that $h, h' \in \pi^{-1}\{x\}$. Note that we work on the fixed trivialization set $\mathbb{B}$. Therefore, we have $\Phi_{\tau(x)}$ and $\Phi_{\tau(f(x))}$ as in Equation (2.7). Since $\Phi_{\tau(f(x))}$ is a diffeomorphism, then

$$(\Phi_{\tau(f(x))} \circ F)(h) = (\Phi_{\tau(f(x))} \circ F)(h').$$

This implies

$$\begin{aligned}&((f \circ \pi)(h), (pr_2 \circ \Phi_{\tau(x)})(h)) = ((f \circ \pi)(h'), (pr_2 \circ \Phi_{\tau(x)})(h'))\\ &\Rightarrow \pi(h) = \pi(h') \qquad \textit{and} \qquad (pr_2 \circ \Phi_{\tau(x)})(h)) = (pr_2 \circ \Phi_{\tau(x)})(h'))\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\Phi_{\tau(x)}(h) = (\pi(h), (pr_2 \circ \Phi_{\tau(x)})(h)) = (\pi(h'), (pr_2 \circ \Phi_{\tau(x)})(h')) = \Phi_{\tau(x)}(h')\\ &\Rightarrow h = h'.\end{aligned}$$

This implies that F is an injection. We note that we use f being an isometry (f is an injection) on the third line of above equations.

To show that F is a surjective map, suppose that $F(h') = h$ for an arbitrary element $h \in \mathbb{E}$. Then

$$\begin{aligned}&h = \Phi^{-1}_{\tau(f(x))}(f \circ \pi, pr_2 \circ \Phi_{\tau(x)})(h')\\ &\Rightarrow \Phi_{\tau(f(x))}(h) = (f \circ \pi, pr_2 \circ \Phi_{\tau(x)})(h'),\end{aligned}$$

which implies

$$\begin{aligned}&(\pi(h), (pr_2 \circ \Phi_{\tau(f(x))})(h)) = ((f \circ \pi)(h'), (pr_2 \circ \Phi_{\tau(x)})(h'))\\ &\Rightarrow \pi(h) = (f \circ \pi)(h') \qquad \textit{and} \qquad (pr_2 \circ \Phi_{\tau(f(x))})(h) = (pr_2 \circ \Phi_{\tau(x)})(h')\\ &\Rightarrow f^{-1}(\pi(h)) = \pi(h') \qquad \textit{and} \qquad (pr_2 \circ \Phi_{\tau(f(x))})(h) = (pr_2 \circ \Phi_{\tau(x)})(h').\end{aligned}$$

Using the above equations, we have

$$\begin{aligned}\Phi_{\tau(x)}(h') &= (\pi(h'),(pr_2\circ\Phi_{\tau(x)})(h'))\\ &= (f^{-1}(\pi(h)),(pr_2\circ\Phi_{\tau(f(x))})(h)).\end{aligned}$$

Therefore, $h'=\Phi^{-1}_{\tau(x)}(f^{-1}(\pi(h)),(pr_2\circ\Phi_{\tau(f(x))})(h))$. F is a surjective map. Moreover, inverse function is $F^{-1}(h)=\Phi^{-1}_{\tau(x)}(f^{-1}(\pi(h)),(pr_2\circ\Phi_{\tau(f(x))})(h))$, which is a smooth function. Since both F and F^{-1} are smooth functions, then F is a diffeomorphism.

To finish the proof, we need to show that F_* preserves the Riemannian metric $\tilde{\mathfrak{g}}$.

Since $F=\Phi^{-1}_{\tau(f(x))}(f\circ\pi,pr_2\circ\Phi_{\tau(x)})$, by chain rule we have

$$F_*=(\Phi_{\tau(f(x))})^{-1}_*((f\circ\pi)_*,(pr_2\circ\Phi_{\tau(x)})_*).$$

Let $V,W\in T\mathbb{E}_h$, and $h\in\mathbb{E}$.

$$\begin{aligned}&\tilde{\mathfrak{g}}_{\mathbb{E}}(F_{*_h}(V),F_{*_h}(W))\\ =\;&\mathfrak{g}(\pi_*(F_{*_h}(V)),\pi_*(F_{*_h}(W)))+Q((pr_2\circ\Phi_{\tau(x)})_*(F_{*_h}(V)),(pr_2\circ\Phi_{\tau(x)})_*(F_{*_h}(W)))\\ =\;&\mathfrak{g}((\pi\circ F)_{*_h}(V),(\pi\circ F)_{*_h}(W))+Q((pr_2\circ\Phi_{\tau(f(x))}\circ F)_{*_h}(V),(pr_2\circ\Phi_{\tau(f(x))}\circ F)_{*_h}(W))\\ =\;&\mathfrak{g}((f\circ\pi)_{*_h}(V),(f\circ\pi)_{*_h}(W))+Q((pr_2\circ\Phi_{\tau(x)})_*(V),(pr_2\circ\Phi_{\tau(x)})_*(W)) \qquad (2.10)\\ =\;&\mathfrak{g}(\pi_*(V),\pi_*(W))+Q((pr_2\circ\Phi_{\tau(x)})_*(V),(pr_2\circ\Phi_{\tau(x)})_*(W))\\ =\;&\tilde{\mathfrak{g}}_{\mathbb{E}}(V,W)\end{aligned}$$

which finishes the proof. Please note that we use the Lemma (2.7) for Equation (2.10). □

2.2. A map between isometry groups

Definition 2.9 *Now we define a map between isometry groups of $\mathbb{M}$ and $\mathbb{E}$ respectively as follows*

$$\begin{aligned}\Omega: I(\mathbb{M})&\to I(\mathbb{E})\\ f\to F:\mathbb{E}&\to\mathbb{E}\\ h&\to F(h)=\Phi^{-1}_{\tau(f(x))}((f\circ\pi)(h),(pr_2\circ\Phi_{\tau(x)})(h))\end{aligned}$$

such that for each $f\in I(\mathbb{M})$, we choose trivializations $\Phi_{\tau(x)}$ and $\Phi_{\tau(f(x))}$ as in Equation (2.7).

Lemma 2.10 Ω *is a one-to-one group homomorphism*

Proof

For all $x\in\mathbb{M}$, let $h\in\pi^{-1},\{x\}$. We choose trivializations $\Phi_{\tau(x)}$ and $\Phi_{\tau(f(x))}$ as in Equation (2.7). Assume that $\Omega(f_1)=\Omega(f_2)$. Then

$$\begin{aligned}
& (\Omega(f_1))(h) = (\Omega(f_2))(h) \\
\Rightarrow\ & \pi((\Omega(f_1))(h)) = \pi((\Omega(f_2))(h)) \\
\Rightarrow\ & (f_1 \circ \pi)(h) = (f_2 \circ \pi)(h) \\
\Rightarrow\ & f_1(x) = f_2(x) \\
\Rightarrow\ & f_1 = f_2
\end{aligned}$$

which shows that Ω is one-to-one. Now we show that Ω is a homomorphism.

Suppose that $f_1, f_2 \in I(M)$. For all $h \in \pi^{-1}\{x\}$, we use fixed trivializations $\Phi_{\tau(f_1(x))}$ for f_1 and $\Phi_{\tau(f_2(x))}$ for f_2 as in Equation (2.7). Using definition of Ω, we have

$$(\Omega(f_2))(h) = \Phi^{-1}_{\tau(f_2(x))}((f_2 \circ \pi)(h), (pr_2 \circ \Phi_{\tau(x)})(h)).$$

Let $h' = \Omega(f_2)(h)$. Then $\Phi_{\tau(f_2(x))}(h') = ((f_2 \circ \pi)(h), (pr_2 \circ \Phi_{\tau(x)})(h))$. Thus, $\pi(h') = (f_2 \circ \pi)(h)$, and $(pr_2 \circ \Phi_{\tau(f_2(x))})(h') = (pr_2 \circ \Phi_{\tau(x)})(h)$. Then

$$\begin{aligned}
(\Omega(f_1) \circ \Omega(f_2))(h) = (\Omega(f_1))(h') &= \Phi^{-1}_{\tau(f_1(x))}(f_1(\pi(h')), (pr_2 \circ \Phi_{\tau(f_2(x))})(h')) \\
&= \Phi^{-1}_{f_1(x)}((f_1(f_2 \circ \pi)(h), (pr_2 \circ \Phi_{\tau(x)})(h)) \\
&= (\Omega(f_1 \circ f_2))(h)
\end{aligned}$$

which implies that Ω is a homomorphism. □

Theorem 2.11 Ω *is a smooth function.*

Proof

Denote $G = I(\mathbb{M})$ and the image $Im(\Omega) = \widetilde{G}$. Since Ω is a one-to-one group homomorphism, then $\widetilde{G}$ which is defined as

$$\widetilde{G} = \{F \in I(\mathbb{E}) |\ \ F(h) = \Phi^{-1}_{\tau(f(x))}(f \circ \pi, pr_2 \circ \Phi_{\tau(x)})(h), \quad h \in \pi^{-1}\{x\}, \quad f \in G\}$$

is a subgroup of $I(\mathbb{E})$.

Let $\tilde{B}$ be a subbase of $I(\mathbb{E})$ which consists of function sets $V(\tilde{K}, \tilde{U}) \in \tilde{B}$. Since π is both continuous and open function, then $\pi(\tilde{U})$ is open set and $\pi(\tilde{K})$ is compact. Suppose that $\tilde{V}(\tilde{K}, \tilde{U}) \in \tilde{B}$, and let $f \in \Omega^{-1}(\tilde{V}(\tilde{K}, \tilde{U}))$. Then $F = \Omega(f) \in \tilde{V}(\tilde{K}, \tilde{U})$. Therefore, $F(\tilde{K}) \subset \tilde{U}$, which implies $(\pi \circ F)(\tilde{K}) \subset \pi(\tilde{U}) = U$. Since (F, f) is a fiber preserving map, we have $(f \circ \pi)(\tilde{K}) \subset U$, which shows that $f(K) \subset U$, with compact set K, and open set U. This implies that Ω is a continuous function.

Since Ω is continuous, then from (Theorem 3.39 in [20]), Ω is smooth function. □

Theorem 2.12 $\tilde{G}$ *is a Lie subgroup of* $I(\mathbb{E})$.

Proof From Lemma (2.10), $\widetilde{G}$ is a subgroup of $I(\mathbb{E})$. Now, we need to show that $\widetilde{G}$ is a closed subset.

Suppose that (F_k) is a convergent sequence in $\widetilde{G}$ such that $F_k \to F$. By the definition of $\widetilde{G}$, there exists $(f_k) \subset I(\mathbb{M})$ such that $\Omega(f_k) = F_k$. Since π is continuous, then $(\pi \circ F_k) \to (\pi \circ F)$. From Equation (2.9), we have $(f_k \circ \pi) \to (\pi \circ F)$.

Suppose that $\tilde{d}$ and d represent the distance functions corresponding to Riemannian metrics $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ respectively. Then for all $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\forall k \geq N_0$,

$$d((\pi \circ F_k)(h), (\pi \circ F)(h)) < \epsilon \tag{2.11}$$

Therefore, for all $x \in \mathbb{M}$, with $h = \mathfrak{O}(x)$, Equation (2.11) holds, i.e. $\forall \epsilon > 0$, and $\forall k \geq N_0$

$$\begin{aligned} d(f_k(x), (\pi \circ F \circ \mathfrak{O})(x)) &= d((f_k \circ \pi \circ \mathfrak{O})(x), (\pi \circ F \circ \mathfrak{O})(x)) \\ &= d((f_k \circ \pi)(h), (\pi \circ F)(h)) < \epsilon \end{aligned}$$

which implies $f_k \to \pi \circ F \circ \mathfrak{O}$. Let $f = \pi \circ F \circ \mathfrak{O}$. Since $(f_k) \subset I(M)$, then $f \in I(\mathbb{M})$. By Theorem (2.11), Ω is continuous, then $\Omega(f_k) \to \Omega(f)$ thus $F_k \to \Omega(f)$. By the uniqueness of the limit, $\Omega(f) = F$. Then $F \in \tilde{G}$, which finishes the proof. □

Corollary 2.13 *The function $\Omega|_{\Omega^{-1}(\widetilde{G})}$ is a Lie group isomorphism.*

Proof By Lemma (2.10) and Theorem (2.11), the function $\Omega|_{\Omega^{-1}(\widetilde{G})}$ is a smooth bijection. For any section (for example zero section $\mathfrak{O}$),

$$\Omega^{-1}(F) = \pi \circ F \circ \mathfrak{O}. \tag{2.12}$$

Let $\mathfrak{B}$ and $\tilde{\mathfrak{B}}$ be subbases of $I(\mathbb{M})$ and $I(\mathbb{E})$ respectively, which we define in Definition 1.1, and $V(K,U) \in \mathfrak{B}$. Set $\mathfrak{O}(K) = \tilde{K} \subset \mathbb{E}$, and $\pi^{-1}(U) = \tilde{U}$. Since $\mathfrak{O}$ and π are continuous, then $\tilde{K}$ is compact, and $\tilde{U} \subset \mathbb{E}$ is open. Suppose that $f \in V(K,U)$, and $\Omega(f) = F$. Then

$$\begin{aligned} f(K) \subset U \quad &\Leftrightarrow \quad f(\pi(\tilde{K})) \subset U \\ &\Leftrightarrow \quad (\pi \circ F)(\tilde{K}) \subset U \\ &\Leftrightarrow \quad F(\tilde{K}) \subset \tilde{U} \end{aligned}$$

which shows that $\Omega(V(K,U)) = V(\tilde{K},\tilde{U}) \circ \widetilde{G}$, where $V(\tilde{K},\tilde{U}) \in \tilde{\mathfrak{B}}$. From Theorem (2.12), the topology on $\tilde{G}$ is the relative topology such that the family of the sets $V(\tilde{K},\tilde{U}) \cap \widetilde{G}$ forms a subbase for $\tilde{G}$. Thus, Ω^{-1} is continuous (therefore smooth), which concludes that $\Omega : I(\mathbb{M}) \to \widetilde{G}$ is a diffeomorphism. Since Ω is a Lie group homomorphism, then it is a Lie group isomorphism. □

So far we have shown the existence of a Lie subgroup $\widetilde{G}$ of $I(\mathbb{E})$. In the next section, we construct two different principal $\widetilde{G}$-bundle structures.

3. Applications

In this section, we present two principal bundles with the structure group $\widetilde{G}$ as two applications of the prolongation of the isometric actions.

3.1. Induced principal bundle on the total space

Now suppose that $\tau : P \to M$ a principal bundle with structure group $I(\mathbb{M})$, and transition functions $\zeta_{\alpha\beta}$ on $U_\alpha \cap U_\beta$, where α, β belongs to the index set I. Now we define smooth functions $\tilde{\zeta}_{\alpha\beta}$ on $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$ as $\tilde{\zeta}_{\alpha\beta} = \Omega \circ \zeta_{\alpha\beta} \circ \pi$. In the following proposition, we construct a principal bundle based on $\mathbb{E}$.

Lemma 3.1 *$\tilde{\zeta}_{\alpha\beta}$ satisfies Steenrod construction relations.*

Proof Suppose that $h \in \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta) \cap \pi^{-1}(U_\theta)$. Since Ω is a homomorphism, we have

$$\begin{aligned} \tilde{\zeta}_{\alpha\beta}(h)\tilde{\zeta}_{\beta\theta}(h) &= \Omega((\zeta_{\alpha\beta} \circ \pi)(h))\Omega((\zeta_{\beta\theta} \circ \pi)(h)) \\ &= \Omega((\zeta_{\alpha\beta}(\pi(h))(\zeta_{\beta\theta}(\pi(h))) \\ &= \Omega((\zeta_{\alpha\theta}(\pi(h))) \\ &= \tilde{\zeta}_{\alpha\theta}(h). \end{aligned}$$

□

Let us regard the indexing set I for the covering $\{\pi^{-1}(U_\alpha)\}$ as a topological space with the discrete topology. By (Proposition 5.2 in [11]), Lemma (3.1) implies that there exists a principal bundle $(\tilde{P}, \tilde{\tau}, \mathbb{E}, \widetilde{G})$ with the transition functions $\tilde{\zeta}_{\beta\alpha}$ such that the bundle projection $\tilde{\tau}$, and the local trivializations $\tilde{\varphi}_\alpha$ are given by the following:

Let $X_\alpha = \pi^{-1}(U_\alpha) \times \widetilde{G}$ for each index α, and let $X = \bigcup_\alpha X_\alpha$ the topological sum of X_α ; each element of X is a triple $(\alpha, h, F) \in \{\alpha\} \times X_\alpha$. We introduce an equivalence relation $\sim$ on X as

$$(\alpha, h, F) \sim (\beta, h', F') \Leftrightarrow h = h' \in \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta), \quad F' = \zeta_{\beta\alpha}(h) \circ F. \tag{3.1}$$

We define $\tilde{P} = X/\widetilde{G}$ quotient space of X by the relation $\sim$, which makes $\tilde{P}$ a smooth manifold with $\tilde{P}/\widetilde{G} = \mathbb{E}$. The projection $\tilde{\tau} : \tilde{P} \to \mathbb{E}$ as

$$\tilde{\tau}([(\alpha, h, F)]) = h, \tag{3.2}$$

and the local trivializations $\tilde{\varphi}_\alpha : \tilde{\tau}^{-1}(\pi^{-1}(U_\alpha)) \to \pi^{-1}(U_\alpha) \times \widetilde{G}$ as

$$\tilde{\varphi}([(\alpha, h, F)]) = (h, F), \tag{3.3}$$

which make $(\tilde{P}, \tilde{\tau}, \mathbb{E}, \widetilde{G})$ a principal fiber bundle.

3.2. Induced bundle on the orbit space

Let μ be the canonical action of $I(\mathbb{M})$ on M. Then the canonical $\tilde{G}$ action on $\mathbb{E}$ is defined in a usual way as follows:

$$\tilde{\mu}(F, h) = F(h), \tag{3.4}$$

where $F \in \widetilde{G}$. We will call this action as the induced action on $\mathbb{E}$.

Proposition 3.2 *Suppose that $I(\mathbb{M})$ acts freely on $\mathbb{M}$. Then the orbit space $\mathbb{E}/\widetilde{G}$ of the induced action $\tilde{\mu}$ admits a smooth structure in such a way that $(\mathbb{E}, \rho, \mathbb{E}/\widetilde{G}, \widetilde{G})$ is a (smooth) principal fiber bundle, where $\rho : \mathbb{E} \to \mathbb{E}/\widetilde{G}$ is the canonical projection.*

Proof

Suppose that for some $h \in \mathbb{E}$, $F(h) = h$ where $F \in \widetilde{G}$. Suppose that $h \in \pi^{-1}\{x\}$. From Equation (2.2),

$$\Phi^{-1}_{\tau(f(x))}((f \circ \pi)(h), (pr_2 \circ \Phi_{\tau(x)})(h)) = h$$

which implies

$$\pi(h) = (pr_1 \circ \Phi_{\tau(x)})(h) = pr_1((f \circ \pi)(h), (pr_2 \circ \Phi_{\tau(x)})(h)) = (f \circ \pi)(h). \tag{3.5}$$

Since μ is a free action, then $f = id_{I(\mathbb{M})}$. Because Ω is a homomorphism, then it maps the identity of $I(\mathbb{M})$ to the identity of $I(\mathbb{E})$, which implies that $F = id_{\widetilde{G}}$. Since for any $h \in \mathbb{E}$, and $F \in \widetilde{G}$, $F(h) = h$ implies $F = id$, then $\tilde{\mu}$ is a free action.

On the other hand, by Theorem (2.12), $\widetilde{G}$ is an imbedded Lie subgroup of $I(\mathbb{E})$. Then from Proposition 3.62 in [2], the induced action $\tilde{\mu} : \widetilde{G} \times \mathbb{E} \to \mathbb{E}$ is proper. Since $\tilde{\mu}$ is a proper free action on $\mathbb{E}$, by Theorem 3.34 in [2] the orbit space $\mathbb{E}/\widetilde{G}$ admits a smooth structure such that $(\mathbb{E}, \tilde{\rho}, \mathbb{E}/\widetilde{G}, \widetilde{G})$ is a principal bundle, where the bundle projection map $\tilde{\rho} : \mathbb{E} \to \mathbb{E}/\widetilde{G}$ is the quotient map.

$\square$

Corollary 3.3 *By the Proposition (3.2), if the canonical action of the isometry group $I(\mathbb{M})$ is free, then the total space of every vector bundle based on $\mathbb{M}$ can be written as a total space of a principal bundle.*

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