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On orthomorphism elements in ordered algebra

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Abstract: Let C be an ordered algebra with a unit e. The class of orthomorphism elements $Orthe(C)$ of C was introduced and studied by Alekhno in "The order continuity in ordered algebras". If $C = L(G)$, where G is a Dedekind complete Riesz space, this class coincides with the band $Orth(G)$ of all orthomorphism operators on G. In this study, the properties of orthomorphism elements similar to properties of orthomorphism operators are obtained. Firstly, it is shown that if C is an ordered algebra such that C_r, the set of all regular elements of C, is a Riesz space with the principal projection property and $Orthe(C)$ is topologically full with respect to I_e, then $B_e = Orthe(C)$ holds, where B_e is the band generated by e in C_r. Then, under the same hypotheses, it is obtained that $Orthe(C)$ is an f-algebra with a unit e.

Key words: Ordered algebra, orthomorphism elements, orthomorphism, f-algebra

1. Introduction

All vector spaces are considered over the reals only. An ordered vector space (Riesz space) C under an associative multiplication is said to be an *ordered algebra* (*Riesz algebra*) whenever the multiplication makes C an algebra, and in addition it satisfies the following property: $a, b \in C^+$ implies $ab \in C^+$. A Riesz algebra C is called an f-*algebra* if C has the additional property that $a \wedge b = 0$ implies $ac \wedge b = ca \ \wedge b = 0$ for each $c \in C^+$. Throughout the study, we will assume $C \neq \{0\}$ and C has a unit element $e > 0$. An element $a \in C$ is called *a regular element* if $a = b - c$ with b and c positive, the space of all regular elements of C will be denoted by C_r. Obviously, C_r is a real ordered algebra. Let C be an ordered vector space and an element $a \in C^+$, the *order ideal* I_a *generated by* a is the set $I_a = \{b \in C : -\lambda a \leq b \leq \lambda a \text{ for some } \lambda \in \mathbb{R}^+\}$. Under the algebraic operations and the ordering induced by C, I_a is an ordered vector subspace of C. Moreover, I_e is an ordered algebra [1].

An element $q \in C$ is said to be an *order idempotent* whenever $0 \leq q \leq e$ and $q^2 = q$. Under the partial ordering induced by C, the set of all order idempotents $OI(C)$ of C is a Boolean algebra and its lattice operations satisfy the identities $p \wedge q = pq$ and $p \vee q = p + q - pq$ for all $p, q \in OI(C)$. If $c \in C$ and the modulus $|c|$ of c exists, then $q\,|c| = |qc|$ and $|c|\,q = |cq|$ for all $q \in OI(C)$ [2].

Definition 1.1 [1] *Let C be an ordered algebra, an element $a \in C$ is said to be an order idempotent preserving element whenever $(e - q)aq = 0$ for all $q \in OI(C)$. An element a is said to be an orthomorphism element of*

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an ordered algebra C whenever a is an order idempotent preserving element that is also regular.

The collection of all orthomorphism elements of an ordered algebra C will be denoted by $Orthe(C)$. An operator $\pi : G \to G$ on a Riesz space G is said to be *band preserving* whenever $\pi(B) \subseteq B$ holds for each band B of G. π is a band preserving operator if and only if $\pi(x) \perp y$ whenever $x \perp y$ in G. A band preserving and order bounded operator π is called *orthomorphism* of G and the set of all orthomorphisms of G is denoted by $Orth(G)$. If G has the principal projection property, then an operator $\pi : G \to G$ is band preserving if and only if $\pi p = p\pi$ (or $(I-p)\pi p = 0$) for every order projection p on G [3, Theorem 8.3]. If $C = L(G)$ is taken, where G is a Dedekind complete Riesz space, then the set of all order idempotents $OI(C)$ of C is the set of all order projections on G [3, Theorem 3.10] and the band B_e generated by e in C_r is equal to $Orth(G) = Orthe(C)$ [3, Theorem 8.11]. In general, the equality $B_e = Orthe(C)$ does not hold in the case of an arbitrary ordered algebra C. Therefore, the following question might come into mind. Under what condition $Orthe(C)$ could be identified to B_e? In this work, we try to provide an answer to this question. Moreover, we will show that, under the same hypothesis, $Orthe(C)$ has the similar properties of orthomorphisms.

We refer to [3, 5, 7, 9] for definitions and notations which are not explained here. All Riesz spaces in this paper are assumed to be Archimedean.

2. Ortomorphism elements

Proposition 2.1 *Let C be an ordered algebra such that C_r is a Riesz space. Then, $Orthe(C)$ is a band in C_r so that $B_e \subseteq Orthe(C)$ where B_e is the band generated by e in C_r.*

Proof Since $q\,|a| = |qa|$ and $|a|\,q = |aq|$ for all $q \in OI(C)$ and $a \in C_r$, it is easy to show that $Orthe(C)$ is an order ideal. To see that $Orthe(C)$ is a band in C_r, let $0 \le (b_\alpha) \uparrow b$ in C_r with $(b_\alpha) \subseteq Orthe(C)$. Then, for all α we have

$$0 \le (e-q)bq = (e-q)(b-b_\alpha)q + (e-q)b_\alpha q = (e-q)(b-b_\alpha)q \le (b-b_\alpha).$$

Thus, $b - b_\alpha \downarrow 0$ implies $(e-q)bq = 0$ and $b \in Orthe(C)$. $B_e \subseteq Orthe(C)$ is obtained from the definition of B_e. □

Lemma 2.2 *Let C be an ordered algebra such that C_r is a Riesz space with the principal projection property and $b \in C_r$. Then, $b \in Orthe(C)$ if and only if $ba = ab$ for all $a \in I_e$.*

Proof Let $b \in C_r$. If $ba = ab$ for all $a \in I_e$ then $b \in Orthe(C)$ as $OI(C) \subseteq I_e$. Now, let $b \in Orthe(C)$. From Freudenthal's Spectral Theorem [3, Theorem 6.8], there exists a sequence (u_n) of e-step function satisfying

$$0 \le a - u_n \le n^{-1}e \text{ for each } n \text{ and } u_n \uparrow a$$

for every $a \in I_e$. As u_n e-step function, there exist $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{R}$ and $p_1, p_2, ..., p_k \in OI(C)$ such that $u_n = \sum_{i=1}^{k} \lambda_i p_i$. Thus, we have $bu_n = u_n b$ for each n. This yields

$$0 \le |ab - ba| = |ab - u_n b + u_n b - ba| \le |ab - u_n b| + |bu_n - ba| \le n^{-1}b + n^{-1}b$$

for each n. Since C is Archimedean, we have $ab = ba$ for every $a \in I_e$. □

If $C = L(G)$, where G is a Dedekind complete Riesz space, then $Orth(G) = Orthe(C) = B_I$ where B_I is the generated by the identity operator I in C_r. In general, the equality $B_e = Orthe(C)$ does not hold in the case of an ordered algebra C.

Example 2.3 *Let G be the Riesz space of all continuous piecewise linear functions on $[0,1]$, then $Orth(G) = \{\lambda I : \lambda \in \mathbb{R}\}$ by the Problem 7 in [3, p. 124]. If we take $C = L(G)$, then we have $OI(C) = \{\theta, I\}$ as $OI(C) \subseteq Orth(G)$ holds. As a result of these simple observations we obtain that $Orthe(C) = L_r(G) \neq B_I$.*

Now, we will investigate when $B_e = Orthe(C)$ holds.

Definition 2.4 *Let C be an ordered algebra such that C_r is a Riesz space and $Orthe(C)$ has separating order dual. Let $b, c \in Orthe(C)$ be arbitrary and $0 \leq b \leq c$. $Orthe(C)$ is said to be topologically full with respect to I_e if there exists a net $0 \leq a_\alpha \leq e$ with $a_\alpha c \to b$ in $\sigma(Orthe(C), Orthe(C)^\sim)$.*

Example 2.5 *Let G be a Dedekind complete Riesz space with separating order dual. If we take $C = L(G)$, then $Orthe(C) = Orth(G)$ is topologically full with respect to $I_e = Z(G)$ from the Theorem 4.3 in [6].*

Let C be a Riesz algebra such that C_r is a Riesz space. It is easy to see that $(bc)q = q(bc)$ for each $b, c \in Orthe(C)$ and $q \in OI(C)$. Thus, $Orthe(C)$ is a Riesz algebra. For $b \in Orthe(C)$, let us define $L_b : Orthe(C) \to Orthe(C) : L_b(c) = bc$ and $R_b : Orthe(C) \to Orthe(C) : R_b(c) = cb$ for each $c \in Orthe(C)$. L_b, R_b are regular operators and so that the adjoint operators $L_b^\sim$, $R_b^\sim$ are regular operators on $Orthe(C)^\sim$. Let us consider positive linear maps

$$S_h : Orthe(C) \to I_e^\sim,\ b \to S_{b,h} : S_{b,h}(a) = h(ab)$$

$$V_h : Orthe(C) \to I_e^\sim,\ b \to V_{b,h} : V_{b,h}(a) = h(ba)$$

for each $b \in Orthe(C)$, $a \in I_e$ and $h \in Orthe(C)_+^\sim$. If $Orthe(C)$ is topologically full with respect to I_e, then we can say more about the positivity of the maps S_h and V_h. The proof of the following Lemma is the adaptation of the Lemma in [8, p.65].

Lemma 2.6 *If C is an ordered algebra such that C_r is a Riesz space with the principal projection property and $Orthe(C)$ is topologically full with respect to I_e, then $S_h, V_h : Orthe(C) \to I_e^\sim$ are lattice homomorphisms for each $h \in Orthe(C)_+^\sim$.*

Proof Let $0 \leq h \in Orthe(C)^\sim$. To see that S_h is a lattice homomorphism, it is enough to show that $S_{b,h} \wedge S_{c,h} = 0$ for each $b, c \in Orthe(C)$ satisfying $b \wedge c = 0$. Let $d = b + c$ and I_b, I_c, I_d be respectively, the order ideals generated by b, c, and d. Then I_d is actually the order direct sum of I_b and I_c by the Theorem 17.6 [5]. We denote by p the order projection of I_d onto I_b. Let R be the restriction to I_d of order bounded functionals on $Orthe(C)$. Then R is an order ideal in $I_d^\sim$ by the Theorem 2.3 in [3]. The adjoint $p^\sim : I_d^\sim \to I_d^\sim$ of p satisfies $0 \leq p^\sim \leq I$ and as a consequence we obtain $p^\sim(R) \subseteq R$. As a result of these simple observations we obtain that the pair $< I_d, R >$ constitutes a Riesz pair and $p : (I_d, \sigma(I_d, R)) \to (I_d, \sigma(I_d, R))$ is continuous. Since $0 \leq p(d) \leq d$ there exists (a_α) in I_e such that $0 \leq a_\alpha \leq e$ with $a_\alpha d \to p(d) = b$ in $\sigma(Orthe(C), Orthe(C)^\sim)$. As $L_{a_\alpha} \in Z(I_d)$ for each α it is easy to see that $a_\alpha d \to b$ in $\sigma(I_d, R)$ and

$a_\alpha p(d) = p(a_\alpha d)$. By the continuity of p now yields $a_\alpha p(d) = a_\alpha b \to b$ in $\sigma(I_d, R)$. Since $a_\alpha d = a_\alpha b + a_\alpha c$ for each α, we have $a_\alpha c \to 0$ in $\sigma(I_d, R)$. As $(S_{b,h} \wedge S_{c,h})(a) \le h((a - aa_\alpha)b + (aa_\alpha)c)$ for each α, we obtain

$$\begin{aligned} 0 &\le (S_{b,h} \wedge S_{c,h})(a) \le \lim_\alpha h((a - aa_\alpha)b + (aa_\alpha)c) \\ &= \lim_\alpha h(L_a(b - a_\alpha b + a_\alpha c) \\ &= \lim_\alpha L_a^\sim(h)(b - a_\alpha b + a_\alpha c) \\ &= 0 \end{aligned}$$

as $L_a^\sim(Orthe(C)^\sim) \subseteq Orthe(C)^\sim$, which implies that S_h is lattice homomorphism. On the other hand, by the Lemma 2.2 $ba_\alpha \to b$ and $ca_\alpha \to 0$ in $\sigma(I_d, R)$ holds. Similarly, taking V_h instead of S_h and R_a instead of L_a, we get V_h is lattice homomorphism. □

Corollary 2.7 *Let the hypotheses in the Lemma 2.6 hold. If $b, c \in Orthe(C)$ and $b \wedge c = 0$ then $|S_{b,h}| \wedge |S_{c,t}| = 0$ for each $h, t \in Orthe(C)^\sim$.*

Proof Let $b, c \in Orthe(C)$ and $b \wedge c = 0$. From the Lemma 2.6 we have

$$0 \le |S_{b,h}| \wedge |S_{c,t}| \le S_{b,|h|} \wedge S_{c,|t|} \le S_{b,|h|\vee|t|} \wedge S_{c,|h|\vee|t|} = S_{b\wedge c,|h|\vee|t|} = 0.$$

□

Proposition 2.8 *Let C be an ordered algebra such that C_r is a Riesz space with the principal projection property and $Orthe(C)$ is topologically full with respect to I_e. Then, $B_e = Orthe(C)$ holds (where B_e is the band generated by e in $Orthe(C)$).*

Proof Let $b \in Orthe(C)$ with $|b| \wedge e = 0$. Clearly,

$$S_{b,h}(a) = h(ab) = h(ba) = h(L_b(a)) = L_b^\sim(h)(ae) = S_{e,L_b^\sim(h)}(a)$$

holds for each $h \in Orthe(C)_+^\sim$. Then, it follows that

$$0 \le |S_{b,h}| = |S_{b,h}| \wedge |S_{b,h}| \le S_{|b|,h} \wedge S_{e,L_{|b|}^\sim(h)} = 0$$

and so $S_{b,h} = 0$ for each $h \in Orthe(C)^\sim$. Thus, we have $b = 0$ which implies that $B_e = \{e\}^{dd} = Orthe(C)$. □

Corollary 2.9 *Let the hypotheses be as in the Proposition 2.8. Then, the band B_e generated by e in C_r is equal to $Orthe(C)$.*

Proof It is clear that the band generated by e in $Orthe(C)$ is equal to the band generated by e in C_r as $Orthe(C)$ is a band in C_r. □

By the Example 2.5, we have known that if G is a Dedekind complete Riesz space with separating order dual and $C = L(G)$, then $Orthe(C)$ has separating order dual and $Orthe(C) = Orth(G)$ is topologically full with respect to $I_e = Z(G)$. By using this observation and the above result, we can obtain the following Corollary being previously proved as a theorem in a different manner.

Corollary 2.10 *Let G be a Dedekind complete Riesz space and G has separating order dual. Then the band B_I generated by the identity operator in $L_r(G)$ is equal to $Orth(G)$.*

Theorem 2.11 *If C is an ordered algebra such that C_r is a Riesz space with the principal projection property and $Orthe(C)$ is topologically full with respect to I_e, then $Orthe(C)$ is an f-algebra. Moreover, it is a full subalgebra of C.*

Proof Let $b, c, d \in Orthe(C)^+$ and $b \wedge c = 0$. For each $0 \leq h \in Orthe(C)^{\sim}$ and $a \in I_e$

$$\begin{aligned} 0 &\leq S_{db\wedge c,h}(a) = (S_{db,h} \wedge S_{c,h})(a) \\ &\leq S_{db,h}(a) \wedge S_{c,h}(a) \\ &= h(a(db)) \wedge S_{c,h}(a) \\ &= h(d(ab)) \wedge S_{c,h}(a) \\ &= h(L_d(ab)) \wedge S_{c,h}(a) \\ &= L_d^{\sim}(h)(ab) \wedge S_{c,h}(a) \\ &= S_{b,L_d^{\sim}(h)}(a) \wedge S_{c,h}(a) \\ &= 0 \end{aligned}$$

holds, which proves that $db \wedge c = 0$. Similarly, taking V instead of S and R_d instead of L_d, we have $bd \wedge c = 0$. Let $b \in Orthe(C)$ be invertible in C. We will show that $b^{-1} \in Orthe(C)$. As $b \in Orthe(C)$ $bq = qb$ for each $q \in OI(C)$. It is easy to see that $b^{-1}q = qb^{-1}$ for each $q \in OI(C)$. Thus, $Orthe(C)$ is a full subalgebra of C. □

Corollary 2.12 *Let G be a Dedekind complete Riesz space and G has separating order dual. Then, $Orth(G)$ is an f-algebra. Moreover, it is a full subalgebra of $L_r(G)$.*

As each unital f-algebra C with separating order dual is topologically full with respect to I_e [8], we can give the following corollary.

Corollary 2.13 *Let C be an ordered algebra such that C_r is a Riesz space with the principal projection property and $Orthe(C)$ has separating order dual. Then, $Orthe(C)$ is an f-algebra if and only if $Orthe(C)$ is topologically full with respect to I_e.*

As we said before, if G is a Dedekind complete Riesz space with separating order dual and $C = L(G)$ then $Orthe(C) = Orth(G)$ is topologically full with respect to $I_e = Z(G)$. However, even if C is a Dedekind complete ordered algebra, $Orthe(C)$ may not be topologically full with respect to I_e. We now give an example of a Dedekind complete ordered algebra which is not topologically full with respect to I_e.

Example 2.14 *Let f be a multiplicative functional on l_∞ satisfying $f(c_0) = 0$ and C be the linear space $l_\infty \oplus \mathbb{R}$. C is a Dedekind complete ordered Banach algebra with unit $(e, 0)$ under the multiplication*

$$(u_1, \lambda_1) * (u_2, \lambda_2) = (u_1u_2, \lambda_1 f(u_2) + \lambda_2 f(u_1) + \lambda_1\lambda_2),$$

the norm

$$\|(u,\lambda)\| = \|u\|\ + |\lambda|$$

and the order induced by the cone

$$C^{+} = \left\{(u,\lambda) : u \in l_{\infty}^{+} \text{ and } \lambda \in \mathbb{R}\right\}.$$

Furthermore,

$$\begin{aligned} OI(C) &= \{(p,0) : p \in OI(l_{\infty})\} \ \textit{and} \\ Orthe(C) &= \{(u,\lambda) : u \in Orthe(l_{\infty}) \ \textit{and}\ \lambda \in \mathbb{R}\}\,[1]. \end{aligned}$$

Since C is Dedekind complete, C_r is a Riesz space with the principal projection property. As $Orthe(C)$ is order closed, $Orthe(C)$ is norm closed [9, Theorem 100.7]. This implies $Orthe(C)$ Banach lattices, hence $Orthe(C)^{\sim} = Orthe(C)'$ and so $Orthe(C)$ has separating order dual. It is easy that, $(0,1)$, $(e,0) \in Orthe(C)$ and $(0,1) \perp (e,0)$. On the other hand, we have

$$(0,1) * (e,0) = (0e, 1f(e) + 0f(0) + 01) = (0,1) \neq 0$$

so that $Orthe(C)$ is not an f-algebra. By the Corollary 2.13, $Orthe(C)$ is not topologically full with respect to I_e.

Since each f-algebra is commutative, we can give the following corollary.

Corollary 2.15 *Let C be an ordered algebra such that C_r is a Riesz space with the principal projection property and $Orthe(C)$ is topologically full with respect to I_e. Then, $Orthe(C)$ is a commutative algebra.*

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