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Higher-order Sturm–Liouville problems with the same eigenvalues

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Abstract: In this paper, we consider self-adjoint Sturm–Liouville problem (SLP) of higher-order. We define an equivalence relation between second- and higher-order SLP. Using the Darboux lemma and equivalence relation we obtain the closed form of a family of SLP which have the same eigenvalues. Also, some spectral properties of this family of Sturm–Liouville problems are investigated.

Key words: Sturm–Liouville problem, eigenvalue, Darboux lemma, Equivalence relation

1. Introduction

The Sturm–Liouville problem (SLP) arises in many different physical and engineering applications. Usually, self-adjoint SLP appears in quantum mechanics, while nonself-adjoint problems arise in hydrodynamic and magnetohydrodynamic stability theory. The problems in hydrodynamic and magnetohydrodynamic stability are of higher order. Of course, certain quantum mechanic problem can be reduced to a higher-order self-adjoint problem [5, 6]. We consider a $2n$th order Sturm–Liouville equation of the form:

$$(-1)^n y^{(2n)} + (-1)^{n-1}(p_{n-1}(x)y^{(n-1)})^{(n-1)} + \cdots - (p_1(x)y\prime)\prime + p_0(x)y = \lambda y, \quad 0 < x < 1. \tag{1.1}$$

Equation (1.1) together with $2n$ boundary conditions at the end points $x = 0$ and $x = 1$ is called the Sturm–Liouville problem. We suppose that the coefficients $p_i(x)$ are real functions and integrable in $[0, 1]$. The parameter λ is called an eigenvalue, and the corresponding nontrivial solution y is called an eigenfunction. Under the above assumptions equation (1.1) together with $2n$ self-adjoint boundary conditions has real eigenvalues and can be ordered as follows:

$$\lambda_1 \leq \lambda_2 \leq \cdots, \lim_{k\to\infty} \lambda_k = \infty. \tag{1.2}$$

For more details see [5–7]. For Sturm–Liouville problems, we have three types of problems: direct problems, inverse problems, and isospectral problems. In direct problems, the eigenvalues and eigenfunctions are estimated from the known coefficients. Also, some properties of the problem are studied. Different numerical methods for solving direct problem are applied in [5, 7, 10, 11, 13]. In inverse problem, existence, uniqueness, and determination or estimation of the coefficients from the information of eigenvalues or eigenfunctions are being studied. The inverse problems related to the Sturm–Liouville problems are studied in [1, 3, 9, 15–18]. Third

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type of problems related to the Sturm–Liouville problems are isospectral problems. That is problems of the same form, different coefficients, but with the same eigenvalues. In isospectral problems, for a given problem, we want to obtain different problems of the same form, which have the same eigenvalues of the initial problem. Isopectral Sturm–Liouville problems are studied in [2–4, 8, 12, 14, 19].

This paper is organized as follows: In Section 2, we define an equivalence relation between second and higher-order SLP. Also, we obtain the closed form of Sturm–Liouville problems of order 4, 6, 8, 10 and $4n$ which are equivalent to the second-order problem. In section 3, we find a family of Sturm–Liouville problems of order 8, 10, and $4n$ which are isospectral. Finally, two examples are given.

2. Equivalent higher-order Sturm–Liouville problems

In this section, we introduce higher-order SLP equivalent to second-order and obtain some spectral properties of these problems.

Definition 2.1 *Two SLP of order two and $2n$ are said to be equivalent if and only if the following statements are hold:*

(a) *If (λ, y) is an eigenpair of second-order SLP, then (λ^n, y) is an eigenpair of $2n$th order SLP,*

(b) *If (λ^n, y) is an eigenpair of $2n$th order SLP, then (λ, y) or $(-\lambda, y)$ is an eigenpair of second-order problem.*

In the following Theorem, we show that for every $n \in \mathbb{N}$ there exists a SLP of order $2n$ such that if (λ, y) is an eigenpair of second order SLP then (λ^n, y) is an eigenpair of $2n$th order SLP.

Theorem 2.2 *Suppose that $q(x)$ is an analytic real function in $[0,1]$. There exists an SLP of order $2n$ such that if (λ, y) is an arbitrary eigenpair of the problem*

$$\begin{aligned} &y''(x) + (\lambda - q(x))y(x) = 0, \quad x \in (0,1), \\ &y(0) = 0 = y(1), \end{aligned} \tag{2.1}$$

then, (λ^n, y) is an eigenpair of $2n$th order SLP.

Proof Suppose that (λ, y) is an eigenpair of the problem (2.1). Differentiating twice from Equation (2.1) and substituting $y'' = (q - \lambda)y$ we find, the following SLP of order 4 such that (λ^2, y) is an eigenpair of it.

$$\begin{aligned} &y^{(4)}(x) - 2(q(x)y'(x))' + (q^2(x) - q''(x))y(x) = \lambda^2 y(x), \\ &y(0) = y''(0) = 0, \quad y(1) = y''(1) = 0. \end{aligned} \tag{2.2}$$

Again, differentiating twice from Equation (2.2) and using (2.1) and (2.2), we obtain a sixth-order SLP as follows:

$$\begin{aligned} &-y^{(6)}(x) + 3[q(x)y''(x)]'' - [(3q^2(x) - 4q''(x))y'(x)]' \\ &+[q^{(4)} + q^3 - 2q'^2 - 3qq'']y(x) = \lambda^3 y(x), \\ &y(0) = y''(0) = 0, \quad y(1) = y''(1) = 0, \\ &y^{(4)}(0) - 2q'(0)y'(0) = 0, \quad y^{(4)}(1) - 2q'(1)y'(1) = 0. \end{aligned} \tag{2.3}$$

Since $q(x)$ is an analytic function, one can continue this process to obtain SLP of order $2n$. □

For $n=4$ and $n=5$, the SLP of Theorem 2.2 are as follows:

$$\begin{array}{l} y^{(8)}(x)-4[qy'''(x)]''' \\ +[(6q^2-10q'')y''(x)]''+[(-6q^{(4)}-4q^3+8q'^2+16qq'')y'(x)]' \\ +[-8qq'^2-6q^2q''+4qq^{(4)}+q^4-q^{(6)}+7q''^2+10q'q''']y(x)=\lambda^4y(x), \\ y(0)=y''(0)=0, \quad y(1)=y''(1)=0, \\ y^{(4)}(0)-2q'(0)y'(0)=0, \quad y^{(4)}(1)-2q'(1)y'(1)=0, \\ y^{(6)}(0)-6q'(0)y'''(0)-4q'''(0)y'(0)=0, \quad y^{(6)}(1)-6q'(1)y'''(1)-4q'''(1)y'(1)=0. \end{array} \tag{2.4}$$

$$\begin{array}{l} -y^{(10)}(x)+5[qy^{(4)}(x)]^{(4)}-[10q^2-20q'')y'''(x)]''' \\ -[(-21q^{(4)}10q^3+20q'^2+50qq'')y''(x)]'' \\ +[(49q''^2+62q'q'''-40qq'^2+30qq^{(4)}-8q^{(6)}+40q^2q''+5q^4)y'(x)]'+[q^{(8)}-24q'''^2-38q''q^{(4)} \\ -18q'q^{(5)}-5qq^{(6)}+52q''q'^2+35qq''^2-50qq'q'''+10q^2q^{(4)}+10q''q^3+20q^2q'^2]y(x)=\lambda^5y(x), \\ y(0)=y''(0)=0, \quad y(1)=y''(1)=0, \\ y^{(4)}(0)-2q'(0)y'(0)=0, \quad y^{(4)}(1)-2q'(1)y'(1)=0, \\ y^{(6)}(0)-6q'(0)y'''(0)-4q'''(0)y'(0)=0, \quad y^{(6)}(1)-6q'(1)y'''(1)-4q'''(1)y'(1)=0, \\ y^{(8)}(0)-12q'(0)y^{(5)}(0)-24q'''(0)y'''(0)-12q'(0)q''(0)y'(0)-6q^{(5)}(1)y'(0)=0, \\ y^{(8)}(1)-12q'(1)y^{(5)}(1)-24q'''(1)y'''(1)-12q'(1)q''(1)y'(1)-6q^{(5)}(1)y'(1)=0. \end{array} \tag{2.5}$$

In this paper, we obtained the closed form of SLP (satisfy in Theorem 2.2) up to order 10. Thus, here after we will focus on these orders of SLPs. We denote the $2n$th order Sturm–Liouville equations obtained in Theorem 2.2 with $L^{2n,q}y=\lambda^n y,\ n=1,2,\cdots$. The general form of boundary conditions for these problems are as follows:

$$L^{2k,q}y(x)|_{x=0,1}=0,\ k=0,1,\cdots,n-1. \tag{2.6}$$

Lemma 2.3 *The SLPs obtained in Theorem 2.2 with inner product* $\langle u,v\rangle=\int_0^1\overline{v}udx$ *are self-adjoint.*

Proof Let u and v be two functions and satisfy in the boundary conditions (2.6). Using integrating by parts and applying the boundary conditions, we obtain that $\langle Lu,v\rangle=\langle u,Lv\rangle$ for $L=L^{2n,q},\ n=1,2,3,4,5$. □

Lemma 2.3 shows that the eigenvalues of the operators $L^{2n,q}$ with the boundary conditions (2.6) are real. In [13, 14], some spectral properties of the operators $L^{4,q}$ and $L^{6,q}$ are studied. Here, we study some spectral properties of the operators $L^{8,q}, L^{10,q}$, and $L^{4n,q}$.

2.1. 8th order Sturm–Liouville problem

Theorem 2.4 *For 8th order SLP (2.4), the following statements are satisfy:*

(a) The eigenvalues are nonnegative,

(b) If (λ^4,y) is an eigenpair of $L^{8,q}$, then only one of the eigenpairs (λ,y) or $(-\lambda,y)$ is an eigenpair of $L^{2,q}$,

(c) The multiplicity of the eigenvalues are at most two,

(d) If $\lambda=0$ is an eigenvalue of the operator $L^{8,q}$, then it is a simple eigenvalue.

Proof Part (a). We prove by contradiction. Let $\lambda \neq 0$ and $(-\lambda^4, y)$ is an eigenpair of the problem (2.4). Thus, we have $L^{8,q}y = -\lambda^4 y$. We can factorize this equation as follows

$$(L^{4,q} + \lambda^2 i)(L^{4,q} - \lambda^2 i)y(x) = 0.$$

Suppose that $\Phi(x) := (L^{4,q} - \lambda^2 i)y(x)$. The function $\Phi(x)$ satisfies in the boundary conditions (2.2). We have two cases: $\Phi(x) \neq 0$ and $\Phi(x) \equiv 0$. If $\Phi(x) \equiv 0$, then $(\lambda^2 i, y(x))$ is an eigenpair of fourth order problem (2.2). If $\Phi(x) \neq 0$, then $(-\lambda^2 i, y(x))$ is an eigenpair of fourth order problem (2.2). In two cases we have a contradiction, since the eigenvalues of $L^{4,q}$ are real [14].

Part (b). Suppose that for $\lambda \neq 0$, $(\lambda^4, y(x))$ is an eigenpair of the problem (2.4). We can write this equation as follows:

$$(L^{4,q} + \lambda^2)(L^{4,q} - \lambda^2)y(x) = 0.$$

We define $\Phi(x) = (L^{4,q} - \lambda^2)y(x)$. The function $\Phi(x)$ is zero, otherwise $(-\lambda^2, \Phi(x))$ is an eigenpair of $L^{4,q}$ and this is a contradiction with the nonnegativity of the eigenvalues of problem (2.2), see [14]. From $\Phi(x) \equiv 0$, we conclude that $(\lambda^2, y(x))$ is an eigenpair of the problem (2.2) and by Theorem 1 in [12] (λ, y) or $(-\lambda, y)$ is an eigenpair of second order SLP (2.1).

Part (c). Let $\lambda \neq 0$ and (λ, y_1) and $(-\lambda, y_2)$ be the eigenpairs of the second order SLP* (2.1), then by Theorem 2.2, (λ^4, y_1) and (λ^4, y_2) are the eigenpairs of 8th order problem (2.4). Thus, some eigenvalues of the problem (2.4) can be of the multiplicity two. We prove that the multiplicity of nonzero eigenvalues are at most two. Suppose that problem (2.4) has an eigenvalue λ^4 with multiplicity three. Thus, there exist three independent functions y_1, y_2, and y_3 such that

$$L^{8,q}y_1 = \lambda^4 y_1, \quad L^{8,q}y_2 = \lambda^4 y_2, \quad L^{8,q}y_3 = \lambda^4 y_3.$$

By part (b), λ or $-\lambda$ is an eigenvalue of second order problem (2.1) with multiplicity at least two. This is a contradiction, the eigenvalues of (2.1) are simple. The proof of part (d), is similar to those of part (b) and part (c). □

Theorems 2.2 and 2.4 show that the second-order problem (2.1) and 8th order problem (2.4) are equivalent.

2.2. 10th order Sturm–Liouville problem

In the following Theorem, we prove some properties of 10th order SLP (2.5).

Theorem 2.5 *The 10th order SLP (2.5) have the following properties:*

(a) If (λ^5, y) is an eigenpair of the problem (2.5), then (λ, y) is an eigenpair of second order problem (2.1),

(b) The eigenvalues are simple.

Proof Let (λ^5, y) be an eigenpair of the problem (2.5). We can factorize the equation $L^{10,q}y = \lambda^5 y$ as follows

$$\{D^8 - (\lambda + 4q(x))D^6 - 12q\prime(x)D^5 + AD^4 + BD^3 + CD^2 + ED + F\}\{D^2 + (\lambda - q(x))\}y(x) = 0, \tag{2.7}$$

*For example $(-4, sin(x))$ and $(4, sin(3x))$ are the eigenpairs of the second order problem $y\prime\prime + (\lambda + 5)y = 0,\ y(0) = y(\pi) = 0$.

where D is the differential operator and

$$
\begin{aligned}
A &= 6q^2 - 22q'' + 3\lambda q + \lambda^2, \\
B &= -24q''' + 24qq' + 6\lambda q', \\
C &= 20q'^2 - 16q^{(4)} + 28qq'' - 4q^3 + 7\lambda q'' - 3\lambda q^2 - 2\lambda^2 q - \lambda^3, \\
E &= -6q^{(5)} + 32q'q'' + 16qq''' - 12q'q^2 + 4\lambda q''' - 2\lambda^2 q' - 6\lambda qq', \\
F &= -q^{(6)} + 4qq^{(4)} - 6q^2q'' + 7q''^2 + 10q'q''' - 8qq'^2 + q^4 + \lambda q^{(4)} \\
&\quad -\lambda^2 q'' - 3\lambda qq'' - 2\lambda q'^2 + \lambda q^3 + \lambda^2 q^2 + \lambda^3 q + \lambda^4.
\end{aligned}
$$

We define $\Phi(x) = (D^2 + (\lambda - q(x)))y(x)$. It is easy to verify that $\Phi(x)$ satisfies in the boundary conditions of the problem (2.4). We claim that $\Phi(x) \equiv 0$. If $\Phi(x) \neq 0$, then equation (2.7) can be written as follows

$$L^{8,q+\frac{\lambda}{4}}\Phi(x) + \frac{5}{8}\lambda^2 L^{4,q+\frac{3}{4}\lambda}\Phi(x) = -\frac{3}{2}\lambda^4\Phi(x). \tag{2.8}$$

It is proved that the operators $L^{4,q}$ and $L^{8,q}$ for all functions $q(x) \in C^6[0,1]$ have nonnegative eigenvalues. Thus, the eigenvalues of the operator $L^{8,q+\frac{\lambda}{4}} + \frac{5}{8}\lambda^2 L^{4,q+\frac{3}{4}\lambda}$ must be nonnegative, but equation (2.8) shows that $(-\frac{3}{2}\lambda^4, \Phi(x))$ is an eigenpair of this operator. This is a contradiction; thus, $\Phi(x) \equiv 0$ and (λ, y) is an eigenpair of second order problem (2.1).

Part (b) concludes from part (a) and simplicity of the eigenvalues of second-order problem. □

Theorems 2.2 and 2.5 state that 10th order problem (2.5) and second order problem (2.1) are equivalent by means of definition 2.1.

2.3. Other higher-order problems

Having SLP of order $2n$, we can obtain a SLP of order $4n$ as follows.

$$L^{4n,q} := L^{2n,q}L^{2n,q}. \tag{2.9}$$

We prove some properties of the operator $L^{4n,q}$ in the following Theorem:

Theorem 2.6 *(i) If (λ, y) is an eigenpair of second-order problem, then (λ^{2n}, y) is an eigenpair of the operator $L^{4n,q}$,*

(ii) The eigenvalues are nonnegative,

(iii) If λ^{2n} is an eigenvalue of the operator $L^{4n,q}$, then λ or $-\lambda$ or both are the eigenvalues of second order problem,

(iv) The multiplicity of the eigenvalues are at most two.

Proof Let (λ, y) be an eigenpair of second-order problem; thus, by Theorem 2.2, (λ^n, y) is an eigenpair of the operator $L^{2n,q}$,i.e.,

$$L^{2n,q} = \lambda^n y \Longrightarrow L^{4n,q}y = \lambda^{2n}y.$$

Part (ii). Suppose that $(-\lambda^{2n}, y)$ is an eigenpair of the operator $L^{4n,q}$; thus, we have

$$L^{4n,q} = -\lambda^{2n}y \Longrightarrow (L^{2n,q} + \lambda^n i)(L^{2n,q} - \lambda^n i)y = 0,$$

the rest of proof is similar to Part (a) of Theorem 2.4.

Part (iii). Suppose that $L^{4n,q}y = \lambda^{2n}y$. We can write this equation as $(L^{2n,q} + \lambda^n)(L^{2n,q} - \lambda^n)y = 0$. Define $\Phi(x) = (L^{2n,q} - \lambda^n)y$, if $\Phi(x) \equiv 0$, then λ^n is an eigenvalue of $L^{2n,q}$. If $\Phi(x) \neq 0$, then $-\lambda^n$ is an eigenvalue of $L^{2n,q}$. According to the equivalence relation between the operators $L^{2n,q}$ and $L^{2,q}$, we conclude that λ or $-\lambda$ or both are the eigenvalues of second order problem.

The proof of Part (iv) is similar to Part (d) of Theorem 2.4. □

By Definition 2.1 and Theorem 2.6 the operators $L^{4n,q}$ and $L^{2,q}$ are equivalent.

3. Family of SLP with the same eigenvalues

If A and B are two operators, then the operators AB and BA have the same eigenvalues. One can apply this idea for finding different operators with the same eigenvalues. For this purpose, the given operator must be factorized as product of two operators, then by reversing the factors we obtain the new operator. This idea is applied for second-order problem and known as Darboux Lemma:

Lemma 3.1 *(Darboux Lemma)[3] Suppose that* (λ_m, y_m) *is an arbitrary eigenpair of the problem:*

$$\begin{aligned} &y'' + (\lambda - \hat{q})y = 0, \\ &y(0) = 0, \quad y(1) = 0. \end{aligned} \tag{3.1}$$

Then problem (3.1) *and problems*

$$\begin{aligned} &w'' + (\lambda - q_{m,\alpha}(x))w = 0, \\ &w(0) = 0, \quad w(1) = 0. \end{aligned} \tag{3.2}$$

have the same eigenvalues, where $q_{m,\alpha}(x) = \hat{q}(x) - 2(ln(1 + \alpha \int_0^x y_m^2(t)dt))''$, $m = 1, 2, \cdots,$ $\int_0^1 y_m^2(t)dt = 1$ *and* $\alpha > -1$ *is an arbitrary real number. In problem (3.2), the corresponding orthogonal eigenfunctions to the eigenvalues* λ_k *are* $w_k = y_k(x) - \frac{\alpha y_m(x) \int_0^x y_m(s) y_k(s) ds}{1 + \alpha \int_0^x y_m^2(s) ds}$, *for* $k = 1, 2, .$

Using Darboux lemma and equivalence relations obtained in the previous section, we find that, if we replace the function $q(x)$ in problems (2.4) and (2.5) with $q_{m,\alpha}(x)$, then the eigenvalues do not change. Thus, we can construct family of isospectral SLP. We apply the mentioned method in the following examples.

Example 3.2 *Consider the* $8th$ *order SLP of the form* (2.4) *corresponding to* $q(x) = 0$ *as follows:*

$$\begin{aligned} &y^{(8)}(x) = \lambda^4 y(x), \\ &y^{(2k)}(0) = 0, \quad y^{(2k)}(1) = 0, \quad k = 0, 1, 2, 3. \end{aligned} \tag{3.3}$$

Using Theorems 2.2 and 2.4, the problem (3.3) *is equivalent to the problem:*

$$\begin{aligned} &y''(x) = \lambda y(x), \\ &y(0) = 0, \quad y(1) = 0. \end{aligned} \tag{3.4}$$

The eigenvalues of the problem (3.4) are $\lambda_m = m^2\pi^2$ *and the corresponding orthogonal eigenfunctions are* $y_m(x) = \sqrt{2}\sin(m\pi x)$. *By Darboux Lemma, the problem (3.4) is isospectral to the problem*

$$\begin{array}{l} w''(x) + (\lambda - q_{m,\alpha}(x))w(x) = 0, \qquad x \in (0,1), \\ w(0) = 0, \quad w(1) = 0, \end{array} \tag{3.5}$$

where

$$q_{m,\alpha} = 4\alpha \frac{\alpha - \alpha cos(2m\pi x) - m\pi(1+\alpha x)sin(2m\pi x)}{(1 + \alpha x - \frac{\alpha}{2m\pi} sin(2m\pi x))^2}. \tag{3.6}$$

Again, applying the equivalence relation we find that problem (3.5) and the following problem are equivalent.

$$\begin{array}{l} y^{(8)} - 4[q_{m,\alpha}y''']''' + [(6q^2_{m,\alpha} - 10q_{m,\alpha}'')y'']'' \\ +[(-6q^{(4)}_{m,\alpha} - 4q^3_{m,\alpha} + 8q_{m,\alpha}'^2 + 16q_{m,\alpha}q_{m,\alpha}'')y'(x)]' \\ +[-8q_{m,\alpha}q_{m,\alpha}'^2 - 6q^2_{m,\alpha}q_{m,\alpha}'' + 4q_{m,\alpha}q^{(4)}_{m,\alpha} + q^4_{m,\alpha} - q^{(6)}_{m,\alpha} + 7q''^2_{m,\alpha} + 10q_{m,\alpha}'q_{m,\alpha}''']y = \lambda^4 y, \\ y(0) = y''(0) = 0, \quad y(1) = y''(1) = 0, \\ y^{(4)}(0) - 2q_{m,\alpha}'(0)y'(0) = 0, \quad y^{(4)}(1) - 2q_{m,\alpha}'(1)y'(1) = 0, \\ y^{(6)}(0) - 6q_{m,\alpha}'(0)y'''(0) - 4q_{m,\alpha}'''(0)y'(0) = 0, \quad y^{(6)}(1) - 6q_{m,\alpha}'(1)y'''(1) - 4q_{m,\alpha}'''(1)y'(1) = 0. \end{array} \tag{3.7}$$

Thus, problems (3.3) and (3.7) have the same eigenvalues. According to the equivalence relation the eigenvalues of the problems (3.3) and (3.7) are $\lambda_r = r^8\pi^8$, *for* $r = 1, 2, \cdots$.
By a similar method, we obtain that the $10th$ *order problems*

$$\begin{array}{l} -y^{(10)}(x) = \lambda^5 y(x), \\ y^{(2k)}(0) = 0, \quad y^{(2k)}(1) = 0, \quad k = 0,1,2,3,4. \end{array} \tag{3.8}$$

and

$$\begin{array}{l} -y^{(10)} + 5[q_{m,\alpha}y^{(4)}]^{(4)} - [10q^2_{m,\alpha} - 20q_{m,\alpha}'')y''']''' \\ -[(-21q^{(4)}_{m,\alpha}10q^3_{m,\alpha} + 20q_{m,\alpha}'^2 + 50q_{m,\alpha}q_{m,\alpha}'')y'']'' \\ +[(49q''^2_{m,\alpha} + 62q_{m,\alpha}'q_{m,\alpha}''' - 40q_{m,\alpha}q_{m,\alpha}'^2 + 30q_{m,\alpha}q^{(4)}_{m,\alpha} - 8q^{(6)}_{m,\alpha} + 40q^2_{m,\alpha}q_{m,\alpha}'' + 5q^4_{m,\alpha})y']' \\ +[q^{(8)}_{m,\alpha} - 24q'''^2_{m,\alpha} - 38q_{m,\alpha}''q^{(4)}_{m,\alpha} - 18q_{m,\alpha}'q^{(5)}_{m,\alpha} - 5q_{m,\alpha}q^{(6)}_{m,\alpha} + 52q_{m,\alpha}''q_{m,\alpha}'^2 + 35q_{m,\alpha}q''^2_{m,\alpha} \\ -50q_{m,\alpha}q_{m,\alpha}'q_{m,\alpha}''' + 10q^2_{m,\alpha}q^{(4)}_{m,\alpha} + 10q_{m,\alpha}''q^3_{m,\alpha} + 20q^2_{m,\alpha}q_{m,\alpha}'^2]y = \lambda^5 y, \\ \\ y(0) = y''(0) = 0, \quad y(1) = y''(1) = 0, \\ y^{(4)}(0) - 2q_{m,\alpha}'(0)y'(0) = 0, \quad y^{(4)}(1) - 2q_{m,\alpha}'(1)y'(1) = 0, \\ y^{(6)}(0) - 6q_{m,\alpha}'(0)y'''(0) - 4q_{m,\alpha}'''(0)y'(0) = 0, \quad y^{(6)}(1) - 6q_{m,\alpha}'(1)y'''(1) - 4q_{m,\alpha}'''(1)y'(1) = 0, \\ y^{(8)}(0) - 12q_{m,\alpha}'(0)y^{(5)}(0) - 24q_{m,\alpha}'''(0)y'''(0) - 12q_{m,\alpha}'(0)q_{m,\alpha}''(0)y'(0) - 6q^{(5)}_{m,\alpha}(1)y'(0) = 0, \\ y^{(8)}(1) - 12q_{m,\alpha}'(1)y^{(5)}(1) - 24q_{m,\alpha}'''(1)y'''(1) - 12q_{m,\alpha}'(1)q_{m,\alpha}''(1)y'(1) - 6q^{(5)}_{m,\alpha}(1)y'(1) = 0, \end{array} \tag{3.9}$$

have the same eigenvalues $\lambda_l = l^{10}\pi^{10}$, *for* $l = 1, 2, \ldots$. *the corresponding eigenfunctions for problems (3.8) and (3.9) are* $y_l(x) = sin(l\pi x)$ *and* $w_l(x) = \sqrt{2}sin(l\pi x) - \frac{2\sqrt{2}\alpha sin(m\pi x)\int_0^x sin(m\pi s)sin(l\pi s)ds}{1+2\alpha\int_0^x sin^2(m\pi s)ds}$, *respectively.*
Also, for every $n, m \in \mathbb{N}$ *and* $\alpha > -1$, *the operators* $L^{4n,0}$ *and* $L^{4n,q_{m,\alpha}}$ *have the same eigenvalues* $\lambda_k = k^{4n}\pi^{4n}$, $k = 1, 2, \cdots$.

Example 3.3 *[10] Consider the one dimentional quantum harmonic oscillator as follows*

$$y''(x) + (\lambda - x^2)y(x) = 0, \quad x \in (-\infty, +\infty), \tag{3.10}$$

with the boundary conditions $\lim_{x\to\pm\infty} y(x) = 0$. *This problem has the eigenvalues* $\lambda_m = 2m+1$ *and eigenfunctions* $y_m(x) = \frac{(-1)^m \pi^{\frac{-1}{4}}}{\sqrt{2^m m!}} e^{\frac{x^2}{2}} \frac{d^m}{dx^m}(e^{-x^2})$. *Using this problem and the results of the previous sections, we can construct a family of Sturm–Liouville problems of order* $2n$ *with the same eigenvalues* $\lambda_m = (2m+1)^n, m = 0, 1, 2, \ldots$. *In other words, if in problems (2.1)–(2.5) and the operator* $L^{4n,q}$, *we replace the function* $q(x)$ *with the new function*

$$q_{m,\alpha}(x) = x^2 - 2(ln(1+\alpha\int_{-\infty}^{x} y_m^2(s)ds))'',$$

then we obtain a family of isospectral Sturm–Liouville problems of order $2n$ *and* $4n$.

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