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The existence and compactness of the set of solutions for a nonlinear integrodifferential equation in N variables in a Banach space

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Abstract: The paper is devoted to the study of a nonlinear integrodifferential equation in N variables with values in a general Banach space. By applying fixed point theorems in a suitable Banach space under appropriate conditions for subsets to be relatively compact, we prove the existence and the compactness of the set of solutions. In order to illustrate the results, we give two examples.

Key words: Nonlinear integrodifferential equation in N variables, the Banach fixed point theorem, Schauder fixed point theorem

1. Introduction

In this paper, we consider the following nonlinear integrodifferential equation in N variables

$$u(x) = g(x) + \int_{\Omega} K(x, y; u(y), D_1 u(y))dy, \quad (1.1)$$

where $(x_1, \cdots, x_N) \in \Omega = [0,1]^N$ and $g : \Omega \to E,\ K : \Omega \times \Omega \times E^2 \to E$ are given functions, E is a Banach space with norm $\|\cdot\|_E$. Denote by $D_1 u = \frac{\partial u}{\partial x_1}$ the partial derivative of a function $u(x)$ defined on Ω with respect to the first variable.

It is well known that integral and integrodifferential equations have attracted interest of many scientists due to a large number of applications in different branches of science and engineering. These equations arise naturally in various models in mechanics, physics, population dynamics, economics, and other fields of science, for example, see the books written by Corduneanu [4], Deimling [5].

Some interesting kinds of equations similar to (1.1) are also studied; the fixed point theorems are often applied in these equations, see [1–17] and the references therein.

In the case where E is an arbitrary Banach space, Bica et al. [3] presented a new approach for the following neutral Fredholm integro-differential equation

$$x(t) = g(t) + \int_a^b f(t, s, x(s), x'(s))ds,\ t \in [a, b], \quad (1.2)$$

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where $f:[a,b]\times[a,b]\times E\times E\to E$ is continuous and $g\in C^1([a,b];E)$. Here, the authors used Perov's fixed point theorem to obtain the existence, the uniqueness, and the global approximation of the solution of (1.2).

In the case where $E=\mathbb{R}^d$, motivated by the results in [3], based on the application of the well-known Banach fixed point theorem coupled with a Bielecki-type norm and a certain integral inequality with explicit estimates, Pachpatte [14] proved the uniqueness and other properties of solutions of the following Fredholm-type integrodifferential equation

$$x(t)=g(t)+\int_a^b f(t,s,x(s),x'(s),\cdots,x^{(n-1)}(s))ds,\ t\in[a,b],$$

where $x,\ g,\ f$ are real valued functions and $n\geq 2$ is an integer. By the same methods, Pachpatte [15] studied the existence, the uniqueness, and some basic properties of solutions of the Fredholm-type integral equation in two variables as the following

$$u(x,y)=f(x,y)+\int_0^a\int_0^b g\left(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)\right)dtds.$$

In [1], Abdou et al. also considered the existence of an integrable solution of a nonlinear integral equation of Hammerstein-Volterra type of the second kind by using the technique of measure of weak noncompactness and the Schauder fixed point theorem. In [2], Aghajani et al. proved some results on the existence, the uniqueness, and the estimation of the solutions of Fredholm-type integro-differential equations in two variables by using Perov's fixed point theorem.

In [9–11], by using tools of functional analysis and a fixed point theorem of Krasnosel'skii type, the solvability and the asymptotic stability of nonlinear functional integral equations in one variable, two variables, and N variables were investigated.

Recently, in [6, 12], the authors continued to prove that the Banach fixed point theorem, Schauder fixed point theorem, the fixed point theorem of Krasnosel'skii type associated with tools of functional analysis can be applied in order to obtain the existence result and some properties of solutions, such as the uniqueness of a solution or the compactness of the set of solutions or furthermore, the set of solutions is a continuum that is nonempty, compact, and connected. Such a structure of solutions set for differential equations and integral equations have been studied by many authors; for examples, we refer to [5, 7, 8, 13] and references therein. In [8], solution sets of abstract, Volterra, functional and functional differential equations in appropriate Fréchet spaces were discussed and applications to integral and integrodifferential equations and initial value problems were examined. Here, the authors have determined a set of conditions in order that the solution set of each considering equation is a continuum. In particular, the authors proved that if for each $n\in N$, the solutions set of an abstract Volterra operator is a continuum in the Banach space $C[0,n]$, then it has a continuum of solutions in the Fréchet space $C[0,\infty)$.

Based on the above works, we consider (1.1). This paper is organized as follows. Section 2 is devoted to preliminaries, where we present the definition of a suitable Banach space and a sufficient condition for relatively compact subsets. In Section 3, by applying the Banach theorem and the Schauder theorem, we prove two existence theorems. Furthermore, the compactness of the solution set is also proved. In order to illustrate the results obtained here, two examples are given.

2. Preliminaries

First, we construct an appropriate Banach space for (1.1) as follows. Let $X = C(\Omega; E)$ be the space of all continuous functions from Ω into E equipped with the following norm

$$\|u\|_X = \sup_{x\in\Omega} \|u(x)\|_E\,,\ u \in X. \tag{2.1}$$

Put

$$X_1 = \{u \in X : D_1 u \in X\}. \tag{2.2}$$

We remark that $C^1(\Omega; E) \subsetneqq X_1 \subsetneqq X$.

Indeed, let $e_1 \in E,\ e_1 \neq 0,$

(i) Consider $u(x) = u(x_1, \cdots, x_N) = \left(\left|x_1 - \frac{1}{2}\right| + \sum_{i=2}^N \left|x_i - \frac{1}{i+1}\right|\right) e_1,$ we have $u \in X,$ but $u \notin X_1;$

(ii) Consider $v = v(x) = \left(x_1^2 + \sum_{i=2}^N \left|x_i - \frac{1}{i+1}\right|\right) e_1,$ we have $v \in X_1,$ but $v \notin C^1(\Omega; E).$

Lemma 2.1. *X_1 is a Banach space with the norm defined by*

$$\|u\|_{X_1} = \|u\|_X + \|D_1 u\|_X\,,\quad u \in X_1. \tag{2.3}$$

Proof. Let $\{u_p\} \subset X_1$ be a Cauchy sequence in X_1, it means that

$$\|u_p - u_q\|_{X_1} = \|u_p - u_q\|_X + \|D_1 u_p - D_1 u_q\|_X \to 0 \text{ as } p,\ q \to \infty.$$

Then $\{u_p\}$ and $\{D_1 u_p\}$ are also Cauchy sequences in X. Since X is complete, $\{u_p\}$ converges to u and $\{D_1 u_p\}$ converges to v in X, i.e.,

$$\|u_p - u\|_X \to 0, \|D_1 u_p - v\|_X \to 0 \text{ as } p \to \infty. \tag{2.4}$$

We shall show that $D_1 u = v$.

We have

$$u_p(x) - u_p(0, x') = \int_0^{x_1} D_1 u_p(s, x') ds\ \forall x = (x_1, x') \in \Omega. \tag{2.5}$$

By $\|u_p - u\|_X \to 0,$ we get

$$u_p(x) - u_p(0, x') \to u(x) - u(0, x') \text{ in } E,\ \forall x = (x_1, x') \in \Omega. \tag{2.6}$$

On the other hand, it follows from $\|D_1 u_p - v\|_X \to 0$ that

$$\int_0^{x_1} D_1 u_p(s, x') ds \to \int_0^{x_1} v(s, x') ds\ \forall x = (x_1, x') \in \Omega \tag{2.7}$$

since

$$\left\|\int_0^{x_1} D_1 u_p(s, x') ds - \int_0^{x_1} v(s, x') ds\right\|_E \leq \int_0^{x_1} \|D_1 u_p(s, x') - v(s, x')\|_E\, ds \leq \|D_1 u_p - v\|_X \to 0.$$

Combining (2.5)–(2.7) leads to

$$u(x) - u(0, x') = \int_0^{x_1} v(s, x')ds \ \forall x = (x_1, x') \in \Omega. \tag{2.8}$$

It implies that $D_1 u = v \in X$. Therefore, $u \in X_1$ and $u_p \to u$ in X_1. Lemma 2.1 is proved. □

Next, we give a sufficient condition for relatively compact subsets of X_1.

Lemma 2.2. *Let $\mathcal{F} \subset X_1$. Then $\mathcal{F}$ is relatively compact in X_1 if and only if the following conditions are satisfied*

$$\begin{aligned} &\text{(i) } \forall x \in \Omega,\ \mathcal{F}(x) = \{u(x) : u \in \mathcal{F}\} \text{ and} \\ &\qquad\qquad D_1\mathcal{F}(x) = \{D_1 u(x) : u \in \mathcal{F}\} \text{ are relatively compact subsets of } E; \\ &\text{(ii) } \forall \varepsilon > 0, \exists \delta > 0 : \forall x, \bar{x} \in \Omega,\ |x - \bar{x}| < \delta \Longrightarrow \sup_{u \in \mathcal{F}} [u(x) - u(\bar{x})]_E < \varepsilon, \end{aligned} \tag{2.9}$$

where $[u(x) - u(\bar{x})]_E = \|u(x) - u(\bar{x})\|_E + \|D_1 u(x) - D_1 u(\bar{x})\|_E$.

Proof.

(a) Let $\mathcal{F}$ be relatively compact in X_1.

First, we show that (2.9) (i) is true.

Proof $\mathcal{F}(x) = \{u(x) : u \in \mathcal{F}\}$ is a relatively compact subset of E.

To prove that $\mathcal{F}(x)$ is relatively compact in E, let $\{u_p(x)\}$ be a sequence in $\mathcal{F}(x)$, we show that $\{u_p(x)\}$ contains a convergent subsequence in E. Because $\overline{\mathcal{F}}$ is compact in X_1, we have $\{u_p\} \subset \mathcal{F}$ contains a convergent subsequence $\{u_{p_k}\}$ in X_1. So there exists $u \in X_1$ such that

$$\|u_{p_k} - u\|_{X_1} \to 0 \text{ as } k \to \infty.$$

By $\|u_{p_k}(x) - u(x)\|_E \le \|u_{p_k} - u\|_X \le \|u_{p_k} - u\|_{X_1} \to 0$. Hence, $u_{p_k}(x) \to u(x)$ in E. Thus, $\mathcal{F}(x)$ is relatively compact in E.

Similarly, by $\|D_1 u_{p_k}(x) - D_1 u(x)\|_E \le \|D_1 u_{p_k} - D_1 u\|_X \le \|u_{p_k} - u\|_{X_1} \to 0$, we have $D_1\mathcal{F}(x)$ is also relatively compact in E. It implies that (2.9) (i) is true.

Next, we show that (2.9) (ii) is also true.

For every $\varepsilon > 0$, considering a collection of open balls in X_1 centered at $u \in \mathcal{F}$ with radius $\frac{\varepsilon}{4}$, as the following

$$B(u, \frac{\varepsilon}{4}) = \{\bar{u} \in X_1 : \|u - \bar{u}\|_{X_1} < \frac{\varepsilon}{4}\},\ u \in \mathcal{F}.$$

It is clear that $\overline{\mathcal{F}} \subset \bigcup_{u \in \mathcal{F}} B(u, \frac{\varepsilon}{4})$. Because $\overline{\mathcal{F}}$ is compact in X_1, the open cover $\bigcup_{u \in \mathcal{F}} B(u, \frac{\varepsilon}{4})$ of $\overline{\mathcal{F}}$ contains a finite subcover and there are $u_1, \cdots, u_q \in \mathcal{F}$ such that $\overline{\mathcal{F}} \subset \bigcup_{j=1}^q B(u_j, \frac{\varepsilon}{4})$.

By the functions $u_j,\ D_1 u_j,\ j = \overline{1, q}$ are uniformly continuous on Ω, there exists $\delta > 0$ such that

$$\forall x, \bar{x} \in \Omega, |x - \bar{x}| < \delta \Longrightarrow [u_j(x) - u_j(\bar{x})]_E < \frac{\varepsilon}{2}\ \forall j = \overline{1, q}.$$

For all $u \in \mathcal{F}$, $u \in B(u_{j_0}, \frac{\varepsilon}{4})$ for some $j_0 = \overline{1,q}$. Thus, for all x, $\bar{x} \in \Omega$, if $|x - \bar{x}| < \delta$ then we obtain

$$\begin{aligned} [u(x) - u(\bar{x})]_E &\leq [u(x) - u_{j_0}(x)]_E + [u_{j_0}(x) - u_{j_0}(\bar{x})]_E + [u_{j_0}(\bar{x}) - u(\bar{x})]_E \\ &\leq 2\left\|u - u_{j_0}\right\|_{X_1} + [u_{j_0}(x) - u_{j_0}(\bar{x})]_E \\ &< \frac{2\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

It implies that (2.9) (ii) is true.

(b) Conversely, let (2.9) be correct.

To prove that $\mathcal{F}$ is relatively compact in X_1, let $\{u_p\}$ be a sequence in $\mathcal{F}$, we show that $\{u_p\}$ contains a convergent subsequence.

Put $\mathcal{F}_1 = \{u_p : p \in \mathbb{N}\}$. By (2.9), $\mathcal{F}_1(x) = \{u_p(x) : p \in \mathbb{N}\}$ is a relatively compact subset of E, for all $x \in \Omega$ and $\mathcal{F}_1$ is equicontinuous in X. Applying the Ascoli-Arzela theorem to $\mathcal{F}_1$, it is relatively compact in X, so there exists a subsequence $\{u_{p_k}\}$ of $\{u_p\}$ and $u \in X$ such that

$$\left\|u_{p_k} - u\right\|_X \to 0 \text{ as } k \to \infty.$$

Similarly, $\mathcal{F}_2 = \{D_1 u_{p_k} : k \in \mathbb{N}\}$ is also relatively compact in X. We obtain the existence of a subsequence of $\{D_1 u_{p_k}\}$, denoted by the same symbol, and $w \in X$ such that

$$\left\|D_1 u_{p_k} - w\right\|_X \to 0 \text{ as } k \to \infty.$$

Since

$$u_{p_k}(x) - u_{p_k}(0, x') = \int_0^{x_1} D_1 u_{p_k}(s, x')ds \;\forall x = (x_1, x') \in \Omega.$$

Furthermore, since $\left\|u_{p_k} - u\right\|_X \to 0$ and $\left\|D_1 u_{p_k} - w\right\|_X \to 0$, we obtain

$$u(x) - u(0, x') = \int_0^{x_1} w(s, x')ds \;\forall x = (x_1, x') \in \Omega.$$

It gives $D_1 u = w \in X$. Therefore, $u \in X_1$ and $u_{p_k} \to u$ in X_1. Lemma 2.2 is proved. □

3. The existence theorems

We make the following assumptions.

(A_1) $\quad g \in X_1$,

(A_2) $\quad K \in C(\Omega \times \Omega \times E^2; E)$ such that $\frac{\partial K}{\partial x_1} \in C(\Omega \times \Omega \times E^2; E)$,

and there exist nonnegative functions k_0, $k_1 : \Omega \times \Omega \to \mathbb{R}$ satisfying

(i) $\quad \beta = \sup\limits_{x\in\Omega} \int_\Omega k_0(x, y)dy + \sup\limits_{x\in\Omega} \int_\Omega k_1(x, y)dy < 1$,

(ii) $\quad \left\|K(x, y; u, v) - K(x, y; \bar{u}, \bar{v})\right\|_E \leq k_0(x, y)\left(\left\|u - \bar{u}\right\|_E + \left\|v - \bar{v}\right\|_E\right)$
$\forall (x, y) \in \Omega \times \Omega$, $\forall (u, v)$, $(\bar{u}, \bar{v}) \in E^2$,

(iii) $\left\| \frac{\partial K}{\partial x_1}(x,y;u,v) - \frac{\partial K}{\partial x_1}(x,y;\bar{u},\bar{v}) \right\|_E \le k_1(x,y)\left(\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right)$
$\forall (x,y) \in \Omega \times \Omega,\ \forall (u,v),\ (\bar{u},\bar{v}) \in E^2.$

Theorem 3.1. *Let the functions $g,\ K$ in (1.1) satisfy the assumptions (A_1), (A_2). Then the equation (1.1) has a unique solution in X_1.*

Proof. For every $u \in X_1$, we put

$$(Au)(x) = g(x) + \int_\Omega K(x,y;u(y),D_1u(y))dy,\ x \in \Omega. \tag{3.1}$$

It is obvious that $Au \in X_1\ \ \forall u \in X_1$. We shall show that $A : X_1 \to X_1$ is a contraction mapping, by proving

$$\|Au - A\bar{u}\|_{X_1} \le \beta \|u - \bar{u}\|_{X_1}\ \ \forall u, \bar{u} \in X_1. \tag{3.2}$$

For every $u,\ \bar{u} \in X_1$, for all $(x,y) \in \Omega$, by (A_2, ii), (3.1) leads to

$$\begin{aligned} \|(Au)(x) - (A\bar{u})(x)\|_E &\le \int_\Omega \|K(x,y;u(y),D_1u(y)) - K(x,y;\bar{u}(y),D_1\bar{u}(y))\|_E\, dy \\ &\le \int_\Omega k_0(x,y)\left[\|u(y)-\bar{u}(y)\|_E + \|D_1u(y) - D_1\bar{u}(y)\|_E\right] dy \\ &\le \left(\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy\right)\|u-\bar{u}\|_{X_1}. \end{aligned}$$

Hence,

$$\|Au - A\bar{u}\|_X \le \left(\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy\right)\|u-\bar{u}\|_{X_1}. \tag{3.3}$$

Similarly, by

$$D_1(Au)(x) = D_1g(x) + \int_\Omega \frac{\partial K}{\partial x_1}(x,y;u(y),D_1u(y))dy,\ x \in \Omega,$$

and (A_2, ii), we obtain

$$\begin{aligned} &\|D_1(Au)(x) - D_1(A\bar{u})(x)\|_E \\ &\le \int_\Omega \left\| \frac{\partial K}{\partial x_1}(x,y;u(y),D_1u(y)) - \frac{\partial K}{\partial x_1}(x,y;\bar{u}(y),D_1\bar{u}(y)) \right\|_E dy \\ &\le \int_\Omega k_1(x,y)\left[\|u(y)-\bar{u}(y)\|_E + \|D_1u(y)-D_1\bar{u}(y)\|_E\right] dy \\ &\le \left(\sup_{x\in\Omega}\int_\Omega k_1(x,y)dy\right)\|u-\bar{u}\|_{X_1}. \end{aligned}$$

Thus,

$$\|D_1(Au) - D_1(A\bar{u})\|_X \le \left(\sup_{x\in\Omega}\int_\Omega k_1(x,y)dy\right)\|u-\bar{u}\|_{X_1}. \tag{3.4}$$

From (3.3) and (3.4), we have (3.2). Applying the Banach fixed point theorem, Theorem 3.1 is proved. □

We also obtain the existence of solutions of (1.1) in X_1 via the Schauder fixed point theorem by making the following assumptions.

(A_1) $\quad g \in X_1$;

$(\bar{A}_2)$ $\quad K \in C(\Omega \times \Omega \times E^2; E)$ such that $\frac{\partial K}{\partial x_1} \in C(\Omega \times \Omega \times E^2; E)$

and there exist nonnegative functions $\bar{k}_0,\ \bar{k}_1 : \Omega \times \Omega \to \mathbb{R}$ satisfying

(i) $\quad \bar{\beta} = \sup_{x\in\Omega} \int_\Omega \bar{k}_0(x,y)dy + \sup_{x\in\Omega} \int_\Omega \bar{k}_1(x,y)dy < 1$,

(ii) $\|K(x,y;u,v)\|_E \le \bar{k}_0(x,y)\left(1 + \|u\|_E + \|v\|_E\right)$ $\forall (x,y) \in \Omega \times \Omega$, $\forall (u,v) \in E^2$,

(iii) $\left\|\frac{\partial K}{\partial x_1}(x,y;u,v)\right\|_E \le \bar{k}_1(x,y)\left(1 + \|u\|_E + \|v\|_E\right)$ $\forall (x,y) \in \Omega \times \Omega$, $\forall (u,v) \in E^2$;

$(\bar{A}_3)$ $\ K,\ \frac{\partial K}{\partial x_1} : \Omega \times \Omega \times E^2 \to E$ are completely continuous such that for any bounded subset J of E^2, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}\forall x,\, \bar{x} \in \Omega,\, |x - \bar{x}| < \delta \Longrightarrow & \|K(x,y;u,v) - K(\bar{x},y;u,v)\|_E \\ & + \left\|\frac{\partial K}{\partial x_1}(x,y;u,v) - \frac{\partial K}{\partial x_1}(\bar{x},y;u,v)\right\|_E < \varepsilon \ \forall (y,u,v) \in \Omega \times J.\end{aligned}$$

Theorem 3.2. *Let the functions g, K in (1.1) satisfy the assumptions (A_1), $(\bar{A}_2)$, $(\bar{A}_3)$. Then the equation (1.1) has a solution in X_1. Furthermore, the set of solutions is compact.*

Proof. With the operator A as in (3.1), it is clear that $A : X_1 \to X_1$. For $M > 0$, we define a closed ball in X_1 as the following

$$B_M = \{u \in X_1 : \|u\|_{X_1} \le M\}.$$

We shall show that there exists $M > 0$ such that $A : B_M \to B_M$. For every $u \in B_M$, for all $(x,y) \in \Omega$, we have

$$\begin{aligned}\|(Au)(x)\|_E &\le \|g(x)\|_E + \int_\Omega \|K(x,y;u(y),D_1u(y))\|_E\, dy \\ &\le \|g\|_X + \int_\Omega \bar{k}_0(x,y)\left(1 + \|u(y)\|_E + \|D_1u(y)\|_E\right) dy \\ &\le \|g\|_X + \left(1 + \|u\|_{X_1}\right)\left(\sup_{x\in\Omega}\int_\Omega \bar{k}_0(x,y)dy\right),\end{aligned}$$

it gives

$$\|Au\|_X \le \|g\|_X + (1 + M)\left(\sup_{x\in\Omega}\int_\Omega \bar{k}_0(x,y)dy\right). \tag{3.5}$$

Similarly,

$$\begin{aligned}\|D_1(Au)(x)\|_E &\le \|D_1g(x)\|_E + \int_\Omega \left\|\frac{\partial K}{\partial x_1}(x,y;u(y),D_1u(y))\right\|_E dy \\ &\le \|D_1g\|_X + \left(1 + \|u\|_{X_1}\right)\left(\sup_{x\in\Omega}\int_\Omega \bar{k}_1(x,y)dy\right).\end{aligned}$$

Thus,

$$\|D_1(Au)\|_X \leq \|D_1 g\|_X + (1+M)\left(\sup_{x\in\Omega}\int_\Omega \bar{k}_1(x,y)dy\right). \tag{3.6}$$

This gives

$$\|Au\|_{X_1} \leq \|g\|_{X_1} + (1+M)\,\bar{\beta}. \tag{3.7}$$

Choosing $M \geq \|g\|_{X_1} + (1+M)\,\bar{\beta}$, i.e., $M \geq \frac{\|g\|_{X_1}+\bar{\beta}}{1-\bar{\beta}}$. Therefore, $A : B_M \to B_M$.

Now we show that two conditions as below are satisfied.

(a) $A : B_M \to B_M$ is continuous.

(b) $\mathcal{F} = A(B_M)$ is relatively compact in X_1.

To have (a), let $\{u_p\} \subset B_M$, $\|u_p - u_0\|_{X_1} \to 0$, as $p \to \infty$, we need to show that

$$\|Au_p - Au_0\|_X \to 0 \text{ and } \|D_1(Au_p) - D_1(Au_0)\|_X \to 0 \text{ as } p \to \infty. \tag{3.8}$$

Remark that

$$\|(Au_p)(x) - (Au_0)(x)\|_E \leq \int_\Omega \|K(x,y;u_p(y),D_1u_p(y)) - K(x,y;u_0(y),D_1u_0(y))\|_E\,dy. \tag{3.9}$$

Put

$$\begin{aligned} S_1 &= \{u_p(y) : y \in \Omega,\ p \in \mathbb{Z}_+\}, \\ S_2 &= \{D_1u_p(y) : y \in \Omega,\ p \in \mathbb{Z}_+\}. \end{aligned} \tag{3.10}$$

We have S_1, S_2 are compact in E since $\|u_p - u_0\|_{X_1} \to 0$.

(j) S_1 is compact in E.

Indeed, let $\{u_{p_j}(y_j)\}_j$ be a sequence in S_1. We can assume that $\lim_{j\to\infty} y_j = y_0$ and $\lim_{j\to\infty} \left\|u_{p_j} - u_0\right\|_{X_1} = 0$. We have

$$\begin{aligned} \left\|u_{p_j}(y_j) - u_0(y_0)\right\|_E &\leq \left\|u_{p_j}(y_j) - u_0(y_j)\right\|_E + \|u_0(y_j) - u_0(y_0)\|_E \\ &\leq \left\|u_{p_j} - u_0\right\|_{X_1} + \|u_0(y_j) - u_0(y_0)\|_E \to 0 \text{ as } j \to \infty, \end{aligned} \tag{3.11}$$

which shows that $\lim_{j\to\infty} u_{p_j}(y_j) = u_0(y_0)$ in E. This means that S_1 is compact in E.

(jj) Similarly S_2 is also compact in E.

Give $\varepsilon > 0$. By K is uniformly continuous on $\Omega\times\Omega\times S_1\times S_2$, there exists $\delta > 0$ such that

$$\forall (u,v),\ (\bar{u},\bar{v}) \in S_1\times S_2,\ \ \|u-\bar{u}\|_E + \|v-\bar{v}\|_E < \delta \Longrightarrow \|K(x,y;u,v) - K(x,y;\bar{u},\bar{v})\|_E < \varepsilon$$

$\forall (x,y) \in \Omega\times\Omega$.

Since $\|u_p - u_0\|_X \to 0$ and $\|D_1u_p - D_1u_0\|_X \to 0$, there is $p_0 \in \mathbb{N}$ such that

$$\forall p \in \mathbb{N},\ p \geq p_0 \Longrightarrow \|u_p - u_0\|_X + \|D_1u_p - D_1u_0\|_X < \delta.$$

It implies that $\forall p \in \mathbb{N},\ p \geq p_0$

$$\Longrightarrow \|K(x, y; u_p(y), D_1 u_p(y)) - K(x, y; u_0(y), D_1 u_0(y))\|_E < \varepsilon$$

$\forall (x, y) \in \Omega \times \Omega$. Consequently,

$$\|(Au_p)(x) - (Au_0)(x)\|_E < \varepsilon\ \forall x \in \Omega,\ \forall p \geq p_0.$$

It means that

$$\|Au_p - Au_0\|_X < \varepsilon\ \forall p \geq p_0, \tag{3.12}$$

i.e., $\|Au_p - Au_0\|_X \to 0$ as $p \to \infty$.

By the same argument, we obtain that $\|D_1(Au_p) - D_1(Au_0)\|_X \to 0$ as $p \to \infty$. The continuity of A is proved.

To have (b), we use Lemma 2.2.

The condition (2.9) (i) holds, i.e., $A(B_M)(x) = \{Au(x) : u \in B_M\}$ and $D_1 A(B_M)(x) = \{D_1(Au)(x) : u \in B_M\}$ are relatively compact in E.

Indeed, put

$$\begin{aligned} R_1 &= \{u(y) : y \in \Omega,\ u \in B_M\}, \\ R_2 &= \{D_1 u(y) : y \in \Omega,\ u \in B_M\}. \end{aligned} \tag{3.13}$$

Then $R_1,\ R_2$ are bounded in E.

Since K is completely continuous, $K(\Omega \times \Omega \times R_1 \times R_2)$ is relatively compact in E. It implies that $\overline{K(\Omega \times \Omega \times R_1 \times R_2)}$ is compact in E. So is $\overline{conv}\left(K(\Omega \times \Omega \times R_1 \times R_2)\right)$, where $\overline{conv}\left(K(\Omega \times \Omega \times R_1 \times R_2)\right)$ is the convex closure of $K(\Omega \times \Omega \times R_1 \times R_2)$.

For every $x \in \Omega$, for all $u \in B_M$, it follows from

$$\begin{aligned} K(x, y; u(y), D_1 u(y)) &\in K(\Omega \times \Omega \times R_1 \times R_2)\ \forall y \in \Omega, \\ (Au)(x) &= g(x) + \int_\Omega K(x, y; u(y), D_1 u(y)) dy \end{aligned} \tag{3.14}$$

that

$$\begin{aligned} \overline{A(B_M)(x)} &\subset g(x) + |\Omega|\, \overline{conv}\left(K(\Omega \times \Omega \times R_1 \times R_2)\right) \\ &= g(x) + \overline{conv}\left(K(\Omega \times \Omega \times R_1 \times R_2)\right). \end{aligned} \tag{3.15}$$

Hence, the set $A(B_M)(x)$ is relatively compact in E.

Similarly, $\overline{D_1 A(B_M)(x)} \subset D_1 g(x) + \overline{conv}\left(\frac{\partial K}{\partial x}(\Omega \times \Omega \times R_1 \times R_2)\right)$. Hence, the set $D_1 A(B_M)(x)$ is relatively compact in E.

The condition (2.9) (ii) also holds. Indeed, give $\varepsilon > 0$. By $(\bar{A}_3)$, there exists $\delta_1 > 0$ such that

$$\begin{aligned} &\forall x,\ \bar{x} \in \Omega,\ |x - \bar{x}| < \delta_1 \Longrightarrow [K(x, y; u, v) - K(\bar{x}, y; u, v)]_E \\ &\qquad = \|K(x, y; u, v) - K(\bar{x}, y; u, v)\|_E + \left\|\tfrac{\partial K}{\partial x_1}(x, y; u, v) - \tfrac{\partial K}{\partial x_1}(\bar{x}, y; u, v)\right\|_E < \tfrac{\varepsilon}{2} \\ &\forall y \in \Omega,\ \forall (u, v) \in R_1 \times R_2. \end{aligned}$$

Since $g,\ D_1g$ are uniformly continuous on Ω, there is $\delta_2>0$ such that

$$\forall x,\bar{x}\in\Omega,\ \ |x-\bar{x}|<\delta_2 \Longrightarrow [g(x)-g(\bar{x})]_E<\frac{\varepsilon}{2}.$$

Choose $\delta=\min\{\delta_1,\delta_2\}$, it gives, $\forall x,\ \bar{x}\in\Omega,\ |x-\bar{x}|<\delta$,

$$\begin{aligned}[(Au)(x)-(Au)(\bar{x})]_E &\leq [g(x)-g(\bar{x})]_E \\ &\quad +\int_\Omega [K(x,y;u(y),D_1u(y))-K(\bar{x},y;u(y),D_1u(y))]_E dy \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\ \forall u\in B_M.\end{aligned} \tag{3.16}$$

Using Lemma 2.2, $\mathcal{F}=A(B_M)$ is relatively compact in X_1. And applying the Schauder fixed point theorem, the existence of a solution is proved.

Next, we show that the set of solutions, $S=\{u\in B_M: u=Au\}$, is compact in X_1. By the compactness of the operator $A:B_M\to B_M$ and $S=A(S)$, we only prove that S is closed. Let $\{u_p\}\subset S,\ \|u_p-u\|_{X_1}\to 0.$ The continuity of A leads to

$$\|u-Au\|_{X_1}\leq\|u-u_p\|_{X_1}+\|u_p-Au\|_{X_1}=\|u-u_p\|_{X_1}+\|Au_p-Au\|_{X_1}\to 0.$$

Thus, $u=Au\in S$. Theorem 3.2 is proved. □

4. Examples

In this section, we illustrate the results obtained in Section 3 by two examples.

Let $E=C([0,1];\mathbb{R})$ be the Banach space of all continuous functions $v:[0,1]\to\mathbb{R}$ equipped with the norm

$$\|v\|_E=\sup_{0\leq\eta\leq 1}|v(\eta)|\,,\ v\in E. \tag{4.1}$$

Let $X=C(\Omega;E)$ be the space of all continuous functions from Ω into E equipped with the following norm

$$\|u\|_X=\sup_{x\in\Omega}\|u(x)\|_E\,,\ u\in X. \tag{4.2}$$

Put

$$X_1=\{u\in X: D_1u\in X\}. \tag{4.3}$$

Then, for all $u\in X_1$ and $x\in\Omega,\ u(x)$ is an element of E and we denote

$$u(x)(\eta)=u(x;\eta),\ 0\leq\eta\leq 1. \tag{4.4}$$

Example 4.1. We consider (1.1) with the functions $g:\Omega\to E,\ K:\Omega\times\Omega\times E^2\to E$ as the following

(i) Function K :

$$\begin{aligned}&K:\Omega\times\Omega\times E^2\to E\\ &\quad (x,y;u,v)\longmapsto K(x,y;u,v),\\ &K(x,y;u,v)(\eta)=k(x;\eta)\left[(y_1\cdots y_N)^{\alpha_0}\sin\left(\tfrac{\pi u(\eta)}{2w_0(y;\eta)}\right)+(y_1\cdots y_N)^{\alpha_1}\cos\left(\tfrac{2\pi v(\eta)}{D_1w_0(y;\eta)}\right)\right],\end{aligned} \tag{4.5}$$

$0 \le \eta \le 1,\ (x,y;u,v) \in \Omega \times \Omega \times E^2,$ with

$$\begin{cases} k,\, w_0 : \Omega \to E, \\ k(x;\eta) = \frac{1}{1+\eta} x_1^{\tilde{\gamma}_1} \sum_{i=2}^N |x_i - \tilde{\alpha}|^{\tilde{\gamma}_2}, \\ w_0(x;\eta) = \frac{1}{1+\eta}\left[e^{x_1+\cdots+x_N} + x_1^{\gamma_1}\sum_{i=2}^N |x_i-\alpha|^{\gamma_2}\right],\, 0 \le \eta \le 1,\, x \in \Omega. \end{cases} \tag{4.6}$$

(ii) Function $g : \Omega \to E,$

$$g(x;\eta) = w_0(x;\eta) - \left[\frac{1}{(1+\alpha_0)^N} + \frac{1}{(1+\alpha_1)^N}\right] k(x;\eta),\ 0 \le \eta \le 1,\ x \in \Omega; \tag{4.7}$$

where $\alpha,\ \gamma_1,\ \gamma_2,\ \tilde{\alpha},\ \tilde{\gamma}_1,\ \tilde{\gamma}_2,\ \alpha_0,\ \alpha_1$ are positive constants satisfying

$$\begin{cases} 0 < \alpha < 1,\, 0 < \gamma_2 \le 1,\, \gamma_1 > 1, \\ 0 < \tilde{\alpha} < 1,\, 0 < \tilde{\gamma}_2 \le 1,\, \tilde{\gamma}_1 > 1, \\ \alpha_0,\, \alpha_1 > 0, \\ \pi(1+\tilde{\gamma}_1)(N-1)\left[\frac{1}{(1+\alpha_0)^N} + \frac{4}{(1+\alpha_1)^N}\right] \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} < 1. \end{cases} \tag{4.8}$$

We now prove that $(A_1),\ (A_2)$ hold. It is obvious that (A_1) holds by $w_0,\ k \in X_1$.

Assumption (A_2) holds, it is proved below.

First, we show that $K : \Omega \times \Omega \times E^2 \to E$ is continuous. For all $(x,y;u,v),\ (\bar{x},\bar{y};\bar{u},\bar{v}) \in \Omega \times \Omega \times E^2$, $0 \le \eta \le 1$,

$$\begin{aligned} & K(x,y;u,v)(\eta) - K(\bar{x},\bar{y};\bar{u},\bar{v})(\eta) \\ = \;& [k(x;\eta) - k(\bar{x};\eta)]\left[(y_1\cdots y_N)^{\alpha_0}\sin\left(\frac{\pi u(\eta)}{2w_0(y;\eta)}\right) + (y_1\cdots y_N)^{\alpha_1}\cos\left(\frac{2\pi v(\eta)}{D_1 w_0(y;\eta)}\right)\right] \\ & + k(\bar{x};\eta)\left[(y_1\cdots y_N)^{\alpha_0} - (\bar{y}_1\cdots\bar{y}_N)^{\alpha_0}\right]\sin\left(\frac{\pi u(\eta)}{2w_0(y;\eta)}\right) \\ & + k(\bar{x};\eta)\left[(y_1\cdots y_N)^{\alpha_1} - (\bar{y}_1\cdots\bar{y}_N)^{\alpha_1}\right]\cos\left(\frac{2\pi v(\eta)}{D_1 w_0(y;\eta)}\right) \\ & + k(\bar{x};\eta)(\bar{y}_1\cdots\bar{y}_N)^{\alpha_0}\left[\sin\left(\frac{\pi u(\eta)}{2w_0(y;\eta)}\right) - \sin\left(\frac{\pi\bar{u}(\eta)}{2w_0(\bar{y};\eta)}\right)\right] \\ & + k(\bar{x};\eta)(\bar{y}_1\cdots\bar{y}_N)^{\alpha_1}\left[\cos\left(\frac{2\pi v(\eta)}{D_1 w_0(y;\eta)}\right) - \cos\left(\frac{2\pi\bar{v}(\eta)}{D_1 w_0(\bar{y};\eta)}\right)\right]. \end{aligned} \tag{4.9}$$

We have

$$\begin{aligned} w_0(x;\eta) &= \frac{1}{1+\eta}\left[e^{x_1+\cdots+x_N} + x_1^{\gamma_1}\sum_{i=2}^N |x_i-\alpha|^{\gamma_2}\right], \\ D_1 w_0(x;\eta) &= \frac{1}{1+\eta}\left[e^{x_1+\cdots+x_N} + \gamma_1 x^{\gamma_1-1}\sum_{i=2}^N |x_i-\alpha|^{\gamma_2}\right],\ 0 \le \eta \le 1,\ x \in \Omega, \end{aligned} \tag{4.10}$$

so w_0, $D_1 w_0 \in X$ and $w_0(x;\eta) \geq \frac{1}{2}$, $D_1^i w_0(x;\eta) \geq \frac{1}{2}$, it follows that

$$\begin{aligned} & |K(x,y;u,v)(\eta) - K(\bar{x},\bar{y};\bar{u},\bar{v})(\eta)| \\ \leq\ & 2\|k(x)-k(\bar{x})\|_E + \|k(\bar{x})\|_E \left|(y_1\cdots y_N)^{\alpha_0} - (\bar{y}_1\cdots\bar{y}_N)^{\alpha_0}\right| \\ & + \|k(\bar{x})\|_E \left|(y_1\cdots y_N)^{\alpha_1} - (\bar{y}_1\cdots\bar{y}_N)^{\alpha_1}\right| + \|k(\bar{x})\|_E \left|\frac{\pi u(\eta)}{2w_0(y;\eta)} - \frac{\pi\bar{u}(\eta)}{2w_0(\bar{y};\eta)}\right| \\ & + \|k(\bar{x})\|_E \left|\frac{2\pi v(\eta)}{D_1 w_0(y;\eta)} - \frac{2\pi\bar{v}(\eta)}{D_1 w_0(\bar{y};\eta)}\right|. \end{aligned} \tag{4.11}$$

We have

$$\begin{aligned} \left|\frac{u(\zeta)}{w_0(y;\zeta)} - \frac{\bar{u}(\zeta)}{w_0(\bar{y};\zeta)}\right| &= \left|\frac{[w_0(\bar{y};\zeta) - w_0(y;\zeta)]\,u(\zeta) + w_0(y;\zeta)\,[u(\zeta)-\bar{u}(\zeta)]}{w_0(y;\zeta)w_0(\bar{y};\zeta)}\right| \\ &\leq 4\left(\|w_0(\bar{y}) - w_0(y)\|_E \|u\|_E + \|w_0(y)\|_E \|u-\bar{u}\|_E\right). \end{aligned} \tag{4.12}$$

Similarly

$$\left|\frac{v(\zeta)}{D_1 w_0(y;\zeta)} - \frac{\bar{v}(\zeta)}{D_1 w_0(\bar{y};\zeta)}\right| \leq 4\left[\|D_1 w_0(\bar{y}) - D_1 w_0(y)\|_E \|v\|_E + \|D_1 w_0(y)\|_E \|v-\bar{v}\|_E\right]. \tag{4.13}$$

This gives

$$\begin{aligned} & \|K(x,y;u,v) - K(\bar{x},\bar{y};\bar{u},\bar{v})\|_E \\ \leq\ & 2\|k(x)-k(\bar{x})\|_E + \|k(\bar{x})\|_E \left|(y_1\cdots y_N)^{\alpha_0} - (\bar{y}_1\cdots\bar{y}_N)^{\alpha_0}\right| \\ & + \|k(\bar{x})\|_E \left|(y_1\cdots y_N)^{\alpha_1} - (\bar{y}_1\cdots\bar{y}_N)^{\alpha_1}\right| \\ & + 2\pi\|k(\bar{x})\|_E \left(\|w_0(\bar{y}) - w_0(y)\|_E \|u\|_E + \|w_0(y)\|_E \|u-\bar{u}\|_E\right) \\ & + 8\pi\|k(\bar{x})\|_E \left[\|D_1 w_0(\bar{y}) - D_1 w_0(y)\|_E \|v\|_E + \|D_1 w_0(y)\|_E \|v-\bar{v}\|_E\right] \end{aligned} \tag{4.14}$$

and the continuity of K is proved.

Similarly, we also have $\frac{\partial K}{\partial x_1} : \Omega \times \Omega \times E^2 \to E$ is continuous.

Next, the assumption $(A_2, (i), (ii), (iii))$ is true by the following.

For all $(x,y;u,v)$, $(x,y;\bar{u},\bar{v}) \in \Omega \times \Omega \times E^2$, $0 \leq \eta \leq 1$,

$$\begin{aligned} & |K(x,y;u,v)(\eta) - K(x,y;\bar{u},\bar{v})(\eta)| \\ \leq\ & k(x;\eta)\left[(y_1\cdots y_N)^{\alpha_0} \frac{\pi\,|u(\eta)-\bar{u}(\eta)|}{2w_0(y;\eta)} + (y_1\cdots y_N)^{\alpha_1} \frac{2\pi\,|v(\eta)-\bar{v}(\eta)|}{D_1 w_0(y;\eta)}\right] \\ \leq\ & \pi k(x;\eta)\left[(y_1\cdots y_N)^{\alpha_0}\,|u(\eta)-\bar{u}(\eta)| + 4(y_1\cdots y_N)^{\alpha_1}\,|v(\eta)-\bar{v}(\eta)|\right] \\ \leq\ & \pi k(x;\eta)\left[(y_1\cdots y_N)^{\alpha_0} + 4(y_1\cdots y_N)^{\alpha_1}\right]\left[\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right]. \end{aligned} \tag{4.15}$$

Hence

$$\|K(x,y;u,v) - K(x,y;\bar{u},\bar{v})\|_E \leq k_0(x,y)\left[\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right], \tag{4.16}$$

in which

$$\begin{aligned} k_0(x,y) &= \pi\left\|k(x)\right\|_E\left[(y_1\cdots y_N)^{\alpha_0}+4(y_1\cdots y_N)^{\alpha_1}\right] \\ &= \pi x_1^{\tilde{\gamma}_1}\sum_{i=2}^{N}|x_i-\tilde{\alpha}|^{\tilde{\gamma}_2}\left[(y_1\cdots y_N)^{\alpha_0}+4(y_1\cdots y_N)^{\alpha_1}\right]. \end{aligned} \tag{4.17}$$

Similarly, since

$$\frac{\partial K}{\partial x_1}(x,y;u,v)(\eta)=D_1k(x;\eta)\left[(y_1\cdots y_N)^{\alpha_0}\sin\left(\frac{\pi u(\eta)}{2w_0(y;\eta)}\right)+(y_1\cdots y_N)^{\alpha_1}\cos\left(\frac{2\pi v(\eta)}{D_1w_0(y;\eta)}\right)\right], \tag{4.18}$$

we have

$$\left\|\frac{\partial K}{\partial x}(x,y;u,v)-\frac{\partial K}{\partial x}(x,y;\bar{u},\bar{v})\right\|_E\le k_1(x,y)\left[\|u-\bar{u}\|_E+\|v-\bar{v}\|_E\right] \tag{4.19}$$

with

$$\begin{aligned} k_1(x,y) &= \pi\left\|D_1k(x)\right\|_E\left[(y_1\cdots y_N)^{\alpha_0}+4(y_1\cdots y_N)^{\alpha_1}\right] \\ &= \pi\tilde{\gamma}_1 x^{\tilde{\gamma}_1-1}\sum_{i=2}^{N}|x_i-\tilde{\alpha}|^{\tilde{\gamma}_2}\left[(y_1\cdots y_N)^{\alpha_0}+4(y_1\cdots y_N)^{\alpha_1}\right]. \end{aligned} \tag{4.20}$$

We also have the following lemma. We omit its proof.

Lemma 4.2. *Let positive constants* $\alpha,\ \gamma_2,\ \gamma_1$ *satisfy* $0<\alpha<1,\ 0<\gamma_2\le 1<\gamma_1$. *Then*

$$\begin{aligned} 0 &\le x_1^{\gamma_1}\sum_{i=2}^{N}|x_i-\alpha|^{\gamma_2}\le(N-1)\max\{\alpha^{\gamma_2},(1-\alpha)^{\gamma_2}\}\ \forall x\in\Omega, \\ 0 &\le x_1^{\gamma_1-1}\sum_{i=2}^{N}|x_i-\alpha|^{\gamma_2}\le(N-1)\max\{\alpha^{\gamma_2},(1-\alpha)^{\gamma_2}\}\ \forall x\in\Omega. \end{aligned}$$

Using Lemma 4.2, we get

$$\begin{aligned} \int_\Omega k_0(x,y)dy &= \pi x_1^{\tilde{\gamma}_1}\sum_{i=2}^{N}|x_i-\tilde{\alpha}|^{\tilde{\gamma}_2}\int_\Omega\left[(y_1\cdots y_N)^{\alpha_0}+4(y_1\cdots y_N)^{\alpha_1}\right]dy \\ &= \pi x_1^{\tilde{\gamma}_1}\sum_{i=2}^{N}|x_i-\tilde{\alpha}|^{\tilde{\gamma}_2}\left[\frac{1}{(1+\alpha_0)^N}+\frac{4}{(1+\alpha_1)^N}\right] \\ &\le \pi(N-1)\left[\frac{1}{(1+\alpha_0)^N}+\frac{4}{(1+\alpha_1)^N}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}; \\ \int_\Omega k_1(x,y)dy &= \pi\tilde{\gamma}_1 x^{\tilde{\gamma}_1-1}\sum_{i=2}^{N}|x_i-\tilde{\alpha}|^{\tilde{\gamma}_2}\int_\Omega\left[(y_1\cdots y_N)^{\alpha_0}+4(y_1\cdots y_N)^{\alpha_1}\right]dy \\ &\le \pi\tilde{\gamma}_1(N-1)\left[\frac{1}{(1+\alpha_0)^N}+\frac{4}{(1+\alpha_1)^N}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}. \end{aligned} \tag{4.21}$$

Therefore,

$$\begin{aligned} \sup_{x\in\Omega}\int_\Omega k_0(x,y)dy &\leq \pi(N-1)\left[\frac{1}{(1+\alpha_0)^N}+\frac{4}{(1+\alpha_1)^N}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}, \\ \sup_{x\in\Omega}\int_\Omega k_1(x,y)dy &\leq \pi\tilde{\gamma}_1(N-1)\left[\frac{1}{(1+\alpha_0)^N}+\frac{4}{(1+\alpha_1)^N}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}. \end{aligned} \tag{4.22}$$

Consequently,

$$\begin{aligned} \beta &= \sup_{x\in\Omega}\int_\Omega k_0(x,y)dy+\sup_{x\in\Omega}\int_\Omega k_1(x,y)dy \\ &\leq \pi(1+\tilde{\gamma}_1)(N-1)\left[\frac{1}{(1+\alpha_0)^N}+\frac{4}{(1+\alpha_1)^N}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}<1. \end{aligned} \tag{4.23}$$

Then, Theorem 3.1 is fulfilled and it is obvious that $w_0 \in X_1$ is also a unique solution of (1.1). □

Example 4.3. We consider (1.1) with the functions $g:\Omega\to E,\ K:\Omega\times\Omega\times E^2\to E$ defined as below

(i) Function K :

$$\begin{aligned} &K:\Omega\times\Omega\times E^2\to E \\ &\quad (x,y;u,v)\longmapsto K(x,y;u,v), \\ &K(x,y;u,v)(\eta)=k(x;\eta)\left[(y_1\cdots y_N)^{\alpha_0}\int_0^1\left|\frac{u(\zeta)}{w_0(y;\zeta)}\right|^{1/2}d\zeta+(y_1\cdots y_N)^{\alpha_1}\int_0^1\left(\frac{v(\zeta)}{D_1w_0(y;\zeta)}\right)^{1/5}d\zeta\right], \end{aligned} \tag{4.24}$$

$0\leq\eta\leq 1,\ (x,y;u,v)\in\Omega\times\Omega\times E^2,$ with

$$\begin{cases} k,\ w_0:\Omega\to E, \\ k(x;\eta)=\frac{1}{1+\eta}x_1^{\tilde{\gamma}_1}\sum_{i=2}^N|x_i-\tilde{\alpha}|^{\tilde{\gamma}_2}, \\ w_0(x;\eta)=\frac{1}{1+\eta}\left[e^{x_1+\cdots+x_N}+x_1^{\gamma_1}\sum_{i=2}^N|x_i-\alpha|^{\gamma_2}\right],\ 0\leq\eta\leq 1,\ x\in\Omega. \end{cases} \tag{4.25}$$

(ii) Function $g:\Omega\to E$,

$$g(x;\eta)=w_0(x;\eta)-\left[\frac{1}{(1+\alpha_0)^N}+\frac{1}{(1+\alpha_1)^N}\right]k(x;\eta),\ 0\leq\eta\leq 1,\ x\in\Omega, \tag{4.26}$$

where $\alpha,\ \gamma_1,\ \gamma_2,\ \tilde{\alpha},\ \tilde{\gamma}_1,\ \tilde{\gamma}_2,\ \alpha_0,\ \alpha_1$ are positive constants satisfying

$$\begin{cases} 0<\alpha<1,\ 0<\gamma_2\leq 1,\ \gamma_1>1, \\ 0<\tilde{\alpha}<1,\ 0<\tilde{\gamma}_2\leq 1,\ \tilde{\gamma}_1>1, \\ \alpha_0,\ \alpha_1>0, \\ 2(1+\tilde{\gamma}_1)(N-1)\left[\frac{1}{(1+\alpha_0)^N}+\frac{1}{(1+\alpha_1)^N}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}<1. \end{cases} \tag{4.27}$$

We can prove that $(A_1),\ (\bar{A}_2),\ (\bar{A}_3)$ hold by the following.

We have $w_0,\ k\in X_1$. Hence, (A_1) holds.

Assumption $(\bar{A}_2)$ holds. Indeed, we first show $K : \Omega \times \Omega \times E^2 \to E$ is continuous. For all $(x, y; u, v)$, $(\bar{x}, \bar{y}; \bar{u}, \bar{v}) \in \Omega \times \Omega \times E^2$, $0 \le \eta \le 1$,

$$\begin{aligned} & K(x,y;u,v)(\eta) - K(\bar{x},\bar{y};\bar{u},\bar{v})(\eta) \qquad (4.28) \\ = \; & (k(x;\eta) - k(\bar{x};\eta)) \left[(y_1 \cdots y_N)^{\alpha_0} \int_0^1 \left| \frac{u(\zeta)}{w_0(y;\zeta)} \right|^{1/2} d\zeta \right. \\ & \left. + (y_1 \cdots y_N)^{\alpha_1} \int_0^1 \left(\frac{v(\zeta)}{D_1 w_0(y;\zeta)} \right)^{1/5} d\zeta \right] \\ & + k(\bar{x};\eta) \left[(y_1 \cdots y_N)^{\alpha_0} - (\bar{y}_1 \cdots \bar{y}_N)^{\alpha_0} \right] \int_0^1 \left| \frac{u(\zeta)}{w_0(y;\zeta)} \right|^{1/2} d\zeta \\ & + k(\bar{x};\eta) \left[(y_1 \cdots y_N)^{\alpha_1} - (\bar{y}_1 \cdots \bar{y}_N)^{\alpha_1} \right] \int_0^1 \left(\frac{v(\zeta)}{D_1 w_0(y;\zeta)} \right)^{1/5} d\zeta \\ & + k(\bar{x};\eta)(\bar{y}_1 \cdots \bar{y}_N)^{\alpha_0} \int_0^1 \left(\left| \frac{u(\zeta)}{w_0(y;\zeta)} \right|^{1/2} - \left| \frac{\bar{u}(\zeta)}{w_0(\bar{y};\zeta)} \right|^{1/2} \right) d\zeta \\ & + k(\bar{x};\eta)(\bar{y}_1 \cdots \bar{y}_N)^{\alpha_1} \int_0^1 \left[\left(\frac{v(\zeta)}{D_1 w_0(y;\zeta)} \right)^{1/5} - \left(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{y};\zeta)} \right)^{1/5} \right] d\zeta. \end{aligned}$$

By $w_0(x;\eta) \ge \frac{1}{2}$, $D_1 w_0(x;\eta) \ge \frac{1}{2}$, it follows that

$$\begin{aligned} & \| K(x,y;u,v) - K(\bar{x},\bar{y};\bar{u},\bar{v}) \|_E \qquad (4.29) \\ \le \; & 2 \| k(x) - k(\bar{x}) \|_E \left[\|u\|_E^{1/2} + \|v\|_E^{1/5} \right] \\ & + 2 \| k(\bar{x}) \|_E \left| (y_1 \cdots y_N)^{\alpha_0} - (\bar{y}_1 \cdots \bar{y}_N)^{\alpha_0} \right| \|u\|_E^{1/2} \\ & + 2 \| k(\bar{x}) \|_E \left| (y_1 \cdots y_N)^{\alpha_1} - (\bar{y}_1 \cdots \bar{y}_N)^{\alpha_1} \right| \|v\|_E^{1/5} \\ & + \| k(\bar{x}) \|_E \int_0^1 \left| \left| \frac{u(\zeta)}{w_0(y;\zeta)} \right|^{1/2} - \left| \frac{\bar{u}(\zeta)}{w_0(\bar{y};\zeta)} \right|^{1/2} \right| d\zeta \\ & + \| k(\bar{x}) \|_E \int_0^1 \left| \left(\frac{v(\zeta)}{D_1 w_0(y;\zeta)} \right)^{1/5} - \left(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{y};\zeta)} \right)^{1/5} \right| d\zeta \\ \equiv \; & R_1 + R_2 + R_3 + R_4 + R_5. \end{aligned}$$

We estimate the terms on the right-hand side of (4.29) as follows.

Estimating $R_1 + R_2 + R_3$. It is easy to see that

$$\begin{aligned} R_1 &= 2 \| k(x) - k(\bar{x}) \|_E \left[\|u\|_E^{1/2} + \|v\|_E^{1/5} \right] \to 0, \qquad (4.30) \\ R_2 &= 2 \| k(\bar{x}) \|_E \left| (y_1 \cdots y_N)^{\alpha_0} - (\bar{y}_1 \cdots \bar{y}_N)^{\alpha_0} \right| \|u\|_E^{1/2} \to 0, \\ R_3 &= 2 \| k(\bar{x}) \|_E \left| (y_1 \cdots y_N)^{\alpha_1} - (\bar{y}_1 \cdots \bar{y}_N)^{\alpha_1} \right| \|v\|_E^{1/5} \to 0, \end{aligned}$$

as $|x-\bar{x}|+|\bar{y}-y|\to 0$.

Estimating R_4. We have

$$\begin{aligned}\left|\frac{u(\zeta)}{w_0(y;\zeta)}-\frac{\bar{u}(\zeta)}{w_0(\bar{y};\zeta)}\right| &= \left|\frac{[w_0(\bar{y};\zeta)-w_0(y;\zeta)]\,u(\zeta)+w_0(y;\zeta)\,[u(\zeta)-\bar{u}(\zeta)]}{w_0(y;\zeta)w_0(\bar{y};\zeta)}\right| \\ &\leq 4\left(\|w_0(\bar{y})-w_0(y)\|_E\,\|u\|_E+\|w_0(y)\|_E\,\|u-\bar{u}\|_E\right).\end{aligned} \tag{4.31}$$

Applying the following inequalities

$$\begin{aligned}\left||a|^q-|b|^q\right| &\leq |a-b|^q \;\;\forall a,b\in\mathbb{R},\;\;\forall q\in(0,1], \\ (a+b)^q &\leq a^q+b^q \;\forall a,b\geq 0,\;\;\forall q\in(0,1],\end{aligned} \tag{4.32}$$

we obtain

$$\begin{aligned}\left|\left|\frac{u(\zeta)}{w_0(y;\zeta)}\right|^{1/2}-\left|\frac{\bar{u}(\zeta)}{w_0(\bar{y};\zeta)}\right|^{1/2}\right| &\leq \left|\frac{u(\zeta)}{w_0(y;\zeta)}-\frac{\bar{u}(\zeta)}{w_0(\bar{y};\zeta)}\right|^{1/2} \\ &\leq 2\left[\|w_0(\bar{y})-w_0(y)\|_E\,\|u\|_E+\|w_0(y)\|_E\,\|u-\bar{u}\|_E\right]^{1/6} \\ &\leq 2\left[\|w_0(\bar{y})-w_0(y)\|_E^{1/2}\,\|u\|_E^{1/2}+\|w_0(y)\|_E^{1/2}\,\|u-\bar{u}\|_E^{1/2}\right].\end{aligned} \tag{4.33}$$

Thus,

$$\begin{aligned}R_4 &= \|k(\bar{x})\|_E\int_0^1\left|\left|\frac{u(\zeta)}{w_0(y;\zeta)}\right|^{1/2}-\left|\frac{\bar{u}(\zeta)}{w_0(\bar{y};\zeta)}\right|^{1/2}\right|d\zeta \\ &\leq 2\,\|k(\bar{x})\|_E\left[\|w_0(\bar{y})-w_0(y)\|_E^{1/2}\,\|u\|_E^{1/2}+\|w_0(y)\|_E^{1/2}\,\|u-\bar{u}\|_E^{1/2}\right].\end{aligned} \tag{4.34}$$

Hence,

$$R_4\leq 2\,\|k(\bar{x})\|_E\left[\|w_0(\bar{y})-w_0(y)\|_E^{1/2}\,\|u\|_E^{1/2}+\|w_0(y)\|_E^{1/2}\,\|u-\bar{u}\|_E^{1/2}\right]\to 0 \tag{4.35}$$

as $|\bar{y}-y|+\|u-\bar{u}\|_E\to 0$.

Estimating R_5. Similarly

$$\left|\frac{v(\zeta)}{D_1w_0(y;\zeta)}-\frac{\bar{v}(\zeta)}{D_1w_0(\bar{y};\zeta)}\right|\leq 4\left[\|D_1w_0(\bar{y})-D_1w_0(y)\|_E\,\|v\|_E+\|D_1w_0(y)\|_E\,\|v-\bar{v}\|_E\right]. \tag{4.36}$$

Applying the following inequalities

$$\begin{aligned}\left||a|^{q-1}\,a-|b|^{q-1}\,b\right| &\leq 2^{1-q}\,|a-b|^q \;\;\forall a,b\in\mathbb{R},\;\;\forall q\in(0,1], \\ (a+b)^q &\leq a^q+b^q \;\forall a,b\geq 0,\;\;\forall q\in(0,1],\end{aligned} \tag{4.37}$$

we obtain

$$\left|\left(\frac{v(\zeta)}{D_1 w_0(y;\zeta)}\right)^{1/5} - \left(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{y};\zeta)}\right)^{1/5}\right| \tag{4.38}$$

$$= \left|\left|\frac{v(\zeta)}{D_1 w_0(y;\zeta)}\right|^{-4/5}\left(\frac{v(\zeta)}{D_1 w_0(y;\zeta)}\right) - \left|\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{y};\zeta)}\right|^{-4/5}\left(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{y};\zeta)}\right)\right|$$

$$\leq 2^{4/5}\left|\frac{v(\zeta)}{D_1 w_0(y;\zeta)} - \frac{\bar{v}(\zeta)}{D_1 w_0(\bar{y};\zeta)}\right|^{1/5}$$

$$\leq 2^{4/5}4^{1/5}\left[\|D_1 w_0(\bar{y}) - D_1 w_0(y)\|_E \|v\|_E + \|D_1 w_0(y)\|_E \|v-\bar{v}\|_E\right]^{1/5}$$

$$\leq 2^{6/5}\left[\|D_1 w_0(\bar{y}) - D_1 w_0(y)\|_E^{1/5} \|v\|_E^{1/5} + \|D_1 w_0(y)\|_E^{1/5} \|v-\bar{v}\|_E^{1/5}\right].$$

It implies that

$$R_5 = \|k(\bar{x})\|_E \int_0^1 \left|\left(\frac{v(\zeta)}{D_1 w_0(y;\zeta)}\right)^{1/5} - \left(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{y};\zeta)}\right)^{1/5}\right| d\zeta \tag{4.39}$$

$$\leq 2^{6/5}\|k(\bar{x})\|_E\left[\|D_1 w_0(\bar{y}) - D_1 w_0(y)\|_E^{1/5} \|v\|_E^{1/5} + \|D_1 w_0(y)\|_E^{1/5} \|v-\bar{v}\|_E^{1/5}\right].$$

Hence,

$$R_5 \leq 2^{6/5}\|k(\bar{x})\|_E\left[\|D_1 w_0(\bar{y}) - D_1 w_0(y)\|_E^{1/5} \|v\|_E^{1/5} + \|D_1 w_0(y)\|_E^{1/5} \|v-\bar{v}\|_E^{1/5}\right] \to 0, \tag{4.40}$$

as $|\bar{y}-y| + \|v-\bar{v}\|_E \to 0.$

It follows from (4.29), (4.30), (4.35), (4.40) that

$$\|K(x,y;u,v) - K(\bar{x},\bar{y};\bar{u},\bar{v})\|_E \leq \sum_{i=1}^{5} R_i \to 0, \tag{4.41}$$

as $|x-\bar{x}| + |\bar{y}-y| + \|u-\bar{u}\|_E + \|v-\bar{v}\|_E \to 0,$ and the continuity of K is proved.

Similarly, we also have $\frac{\partial K}{\partial x_1} : \Omega\times\Omega\times E^2 \to E$ is continuous.

Now, $(\bar{A}_2, (i), (ii), (iii))$ holds by the following.

Applying the inequality

$$a \leq 1 + a^q \ \forall a \geq 0,\ \forall q \geq 1, \tag{4.42}$$

we obtain

$$|K(x,y;u,v)(\eta)| \leq 2|k(x;\eta)|\left[(y_1\cdots y_N)^{\alpha_0}\int_0^1 |u(\zeta)|^{1/2} d\zeta + (y_1\cdots y_N)^{\alpha_1}\int_0^1 |v(\zeta)|^{1/5} d\zeta\right] \tag{4.43}$$

$$\leq 2\|k(x)\|_E\left[(y_1\cdots y_N)^{\alpha_0}(1+\|u\|_E) + (y_1\cdots y_N)^{\alpha_1}(1+\|v\|_E)\right]$$

$$\leq 2\|k(x)\|_E\left[(y_1\cdots y_N)^{\alpha_0} + (y_1\cdots y_N)^{\alpha_1}\right](1+\|u\|_E+\|v\|_E).$$

It leads to

$$\|K(x,y;u,v)\|_E \leq \bar{k}_0(x,y)\left(1+\|u\|_E+\|v\|_E\right), \tag{4.44}$$

in which

$$\begin{aligned}\bar{k}_0(x,y) &= 2\left\|k(x)\right\|_E\left[(y_1\cdots y_N)^{\alpha_0}+(y_1\cdots y_N)^{\alpha_1}\right] \\ &= 2x_1^{\tilde{\gamma}_1}\sum_{i=2}^{N}\left|x_i-\tilde{\alpha}\right|^{\tilde{\gamma}_2}\left[(y_1\cdots y_N)^{\alpha_0}+(y_1\cdots y_N)^{\alpha_1}\right].\end{aligned} \tag{4.45}$$

Similarly,

$$\left\|\frac{\partial K}{\partial x_1}(x,y;u,v)\right\|_E \le \bar{k}_1(x,y)\left(1+\|u\|_E+\|v\|_E\right), \tag{4.46}$$

where

$$\begin{aligned}\bar{k}_1(x,y) &= 2\left\|D_1k(x)\right\|_E\left[(y_1\cdots y_N)^{\alpha_0}+(y_1\cdots y_N)^{\alpha_1}\right] \\ &= 2\tilde{\gamma}_1x^{\tilde{\gamma}_1-1}\sum_{i=2}^{N}\left|x_i-\tilde{\alpha}\right|^{\tilde{\gamma}_2}\left[(y_1\cdots y_N)^{\alpha_0}+(y_1\cdots y_N)^{\alpha_1}\right].\end{aligned} \tag{4.47}$$

We have

$$\begin{aligned}\int_\Omega\bar{k}_0(x,y)dy &= 2x_1^{\tilde{\gamma}_1}\sum_{i=2}^{N}\left|x_i-\tilde{\alpha}\right|^{\tilde{\gamma}_2}\int_\Omega\left[(y_1\cdots y_N)^{\alpha_0}+(y_1\cdots y_N)^{\alpha_1}\right]dy \\ &\le 2(N-1)\left[\frac{1}{(1+\alpha_0)^N}+\frac{1}{(1+\alpha_1)^N}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}; \\ \int_\Omega\bar{k}_1(x,y)dy &= 2\tilde{\gamma}_1x^{\tilde{\gamma}_1-1}\left|y-\tilde{\alpha}\right|^{\tilde{\gamma}_2}\int_\Omega\left[(y_1\cdots y_N)^{\alpha_0}+(y_1\cdots y_N)^{\alpha_1}\right]dy \\ &\le 2\tilde{\gamma}_1(N-1)\left[\frac{1}{(1+\alpha_0)^N}+\frac{1}{(1+\alpha_1)^N}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}.\end{aligned} \tag{4.48}$$

Thus,

$$\begin{aligned}\bar{\beta} &= \sup_{x\in\Omega}\int_\Omega\bar{k}_0(x,y)dy+\sup_{x\in\Omega}\int_\Omega\bar{k}_1(x,y)dy \\ &= 2(1+\tilde{\gamma}_1)(N-1)\left[\frac{1}{(1+\alpha_0)^N}+\frac{1}{(1+\alpha_1)^N}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}<1.\end{aligned} \tag{4.49}$$

Thus, assumption $(\bar{A}_2,(i))$ holds.

Assumption $(\bar{A}_3)$ also holds, the proof is as below.

(a) Prove $K:\Omega\times\Omega\times E^2\to E$ is completely continuous.

By $K,\ \frac{\partial K}{\partial x_1}\in C(\Omega\times\Omega\times E^2;E)$, we have to prove that $K,\ \frac{\partial K}{\partial x_1}:\Omega\times\Omega\times E^2\to E$ are compact.

Let B be bounded in $\Omega\times\Omega\times E^2$. We have

$$\begin{aligned}\|K(x,y;u,v)\|_E &\le \bar{k}_0(x,y)\left(1+\|u\|_E+\|v\|_E\right) \\ &\le \sup_{(x,y;u,v)\in B}\bar{k}_0(x,y)\left(1+\|u\|_E+\|v\|_E\right)\equiv M_1\end{aligned} \tag{4.50}$$

for all $(x,y;u,v) \in B,$ which implies that $K(B)$ is uniformly bounded in $E.$

For all $\eta,\ \bar{\eta} \in [0,1],$ for all $(x,y;u,v) \in B,$

$$\begin{aligned} & |K(x,y;u,v)(\eta) - K(x,y;u,v)(\bar{\eta})| \\ = \ & |k(x;\eta) - k(x;\bar{\eta})| \left| (y_1 \cdots y_N)^{\alpha_0} \int_0^1 \left| \frac{u(\zeta)}{w_0(y;\zeta)} \right|^{1/2} d\zeta + (y_1 \cdots y_N)^{\alpha_1} \int_0^1 \left(\frac{v(\zeta)}{D_1 w_0(y;\zeta)} \right)^{1/5} d\zeta \right| \\ \leq \ & 2 \left| k(x;\eta) - k(x;\bar{\eta}) \right| \left(\|u\|_E^{1/2} + \|v\|_E^{1/5} \right). \end{aligned} \tag{4.51}$$

On the other hand

$$\begin{aligned} |k(x;\eta) - k(x;\bar{\eta})| &= \left| \left(\frac{1}{1+\eta} - \frac{1}{1+\bar{\eta}} \right) \right| x_1^{\tilde{\gamma}_1} \sum_{i=2}^{N} |x_i - \tilde{\alpha}|^{\tilde{\gamma}_2} \\ &= \frac{|\bar{\eta} - \eta|}{(1+\eta)(1+\bar{\eta})} x_1^{\tilde{\gamma}_1} \sum_{i=2}^{N} |x_i - \tilde{\alpha}|^{\tilde{\gamma}_2} \\ &\leq (N-1) \left| \bar{\eta} - \eta \right|. \end{aligned} \tag{4.52}$$

Thus,

$$\begin{aligned} |K(x,y;u,v)(\eta) - K(x,y;u,v)(\bar{\eta})| &\leq 2(N-1) \left(\|u\|_E^{1/2} + \|v\|_E^{1/5} \right) |\bar{\eta} - \eta| \\ &\leq C \left| \bar{\eta} - \eta \right| \text{ for all } (x,y;u,v) \in B \text{ and } \eta, \bar{\eta} \in [0,1]. \end{aligned} \tag{4.53}$$

Consequently, $K(B)$ is equicontinuous.

(b) Similarly, we also have $\frac{\partial K}{\partial x_1} : \Omega \times \Omega \times E^2 \to E$ is compact.

(c) Finally, for all bounded subset J of $E^2,$ for all $\varepsilon > 0,$ there exists $\delta > 0$ such that

$$\begin{aligned} \forall x, \bar{x} \in \Omega,\ \ |x - \bar{x}| < \delta \Longrightarrow & \|K(x,y;u,v) - K(\bar{x},y;u,v)\|_E \\ & + \left\| \tfrac{\partial K}{\partial x_1}(x,y;u,v) - \tfrac{\partial K}{\partial x_1}(\bar{x},y;u,v) \right\|_E < \varepsilon \ \forall (y,u,v) \in \Omega \times J. \end{aligned} \tag{4.54}$$

Indeed, we get the above property since

$$\begin{aligned} & \|K(x,y;u,v) - K(\bar{x},y;u,v)\|_E + \left\| \frac{\partial K}{\partial x_1}(x,y;u,v) - \frac{\partial K}{\partial x_1}(\bar{x},y;u,v) \right\|_E \\ \leq \ & 2 \left(\|u\|_E^{1/2} + \|v\|_E^{1/5} \right) \left[\|k(x) - k(\bar{x})\|_E + \left\| \frac{\partial k}{\partial x_1}(x) - \frac{\partial k}{\partial x_1}(\bar{x}) \right\|_E \right] \\ \leq \ & C \left[\|k(x) - k(\bar{x})\|_E + \left\| \frac{\partial k}{\partial x_1}(x) - \frac{\partial k}{\partial x_1}(\bar{x}) \right\|_E \right] \end{aligned} \tag{4.55}$$

$\forall (y,u,v) \in \Omega \times J,\ \forall x,\ \bar{x} \in \Omega,$ where $k,\ \frac{\partial k}{\partial x_1} : \Omega \to E$ are uniformly continuous on $\Omega.$

Theorem 3.2 is true. Furthermore, $w_0 \in X_1$ is also a solution of (1.1). $\square$

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