Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 438 – 445

doi:10.3906/mat-1911-45

Regular sequences in the subrings of $C(X)$

Fariborz AZARPANAH*, **Delavar ESMAEILVANDI**
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

Received: 13.11.2019 • **Accepted/Published Online:** 17.01.2020 • **Final Version:** 17.03.2020

Abstract: We show that the intermediate subalgebras between $C^*(X)$ and $C(X)$ do not contain regular sequences with length ≥ 2. This shows that $\text{depth}(A(X)) \leq 1$ for each intermediate subalgebra $A(X)$ between $C^*(X)$ and $C(X)$. Whenever an intermediate subalgebra $A(X)$ is proper, i.e. $A(X) \neq C(X)$, we observe that the depth of $A(X)$ is exactly 1. Using this, it turns out that $\text{depth}(C^*(X)) = 0$ if and only if X is a pseudocompact almost P-space. The regular sequences in the subrings of the form $I + \mathbb{R}$ of $C(X)$, where I is a z-ideal of $C(X)$, are also investigated and we have shown that the length of regular sequences in such rings is at most 1. In contrast to the depth of intermediate subalgebras, we see that the depth of a proper subring of the form $I + \mathbb{R}$ may be zero. Finally, regular sequences of extension rings of $C(X)$ are also studied and some examples of subrings of $C(X)$ are given with depths different from the depth of $C(X)$.

Key words: Regular sequence, depth, almost P-space, z-ideal

1. Introduction

Throughout this article, topological spaces are assumed to be completely regular Hausdorff (Tychonoff) spaces. We denote by $C(X)$ the ring of all real-valued continuous functions on a space X. The subring $C^*(X)$ of $C(X)$ is the set of bounded elements of $C(X)$, and whenever $C(X) = C^*(X)$, we say that X is *pseudocompact*. Recall that for each $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$, the *zero-set* of f, while $\operatorname{coz} f = X \setminus Z(f)$, the *cozero-set* of f. For every ideal I of $C(X)$ the set of zero-sets $\{Z(f) : f \in I\}$ is denoted by $Z[I]$, and $\bigcap Z[I] := \bigcap_{f \in I} Z(f)$. It is well known that a Hausdorff space X is completely regular if and only if the set of all zero-sets is a base for closed subsets of X, or equivalently the set of all cozero-sets is a base for open subsets of X; see Theorem 3.2 in [5].

An ideal I in $C(X)$ is called a *z-ideal* if whenever $f \in I$, $g \in C(X)$, and $Z(f) \subseteq Z(g)$, then $g \in I$. Every maximal ideal of $C(X)$ is precisely of the form $M^p = \{f \in C(X) : p \in \text{cl}_{\beta X} Z(f)\}$, for some $p \in \beta X$, where βX is the *Stone-Čech compactification* of X. Whenever $p \in X$, then M^p is denoted by M_p and in this case $M_p = \{f \in C(X) : p \in Z(f)\}$. Hence, every maximal ideal itself is a z-ideal. It is easy to see that the intersection of all maximal ideals containing $f \in C(X)$ coincides with $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$ and it is the smallest z-ideal containing f. Using Problem 4A in [5], an ideal I is a z-ideal if and only if $M_f \subseteq I$ for each $f \in I$.

*Correspondence: azarpanah@ipm.ir
2010 *AMS Mathematics Subject Classification:* 23584, 13A15, 54C40

Every invertible element of a ring R is called a *unit* and an element of R is said to be *regular* if it is not a zero divisor. The following lemma, which gives the well-known topological characterizations of unit elements and regular elements of $C(X)$, is an immediate consequence of the definitions.

Lemma 1.1 *The following statements hold:*

1. *An element $f \in C(X)$ is a unit if and only if $Z(f) = \emptyset$.*
2. *An element $f \in C(X)$ is a regular element (non-zero divisor) if and only if $\text{int}_X Z(f) = \emptyset$, or equivalently $cozf$ is dense in X.*

Recall that a point $x \in X$ an *almost P-point* if for every $f \in C(X)$, $x \in Z(f)$ implies that $\text{int}_X Z(f) \neq \emptyset$. Hence, using Lemma 1.1, $x \in X$ is not an almost P-point if and only if there exists a regular element $r \in C(X)$ such that $x \in Z(r)$. A space X is called an *almost P-space* if every point of X is an almost P-point. By Lemma 1.1, it is easy to see that a space X is an *almost P-space* if and only if every nonempty G_δ-set or equivalently every nonempty zero-set in X has a nonempty interior. Using this and the above lemma, a space X is an almost P-space if and only if the set of unit elements and the set of regular elements of $C(X)$ coincide; see [7] for more details about almost P-spaces. In case every zero-set or every G_δ-set in X is open, X is called a P-space. Clearly every P-space is an almost P-space but not conversely. For instance, a one-point compactification of an uncountable discrete space is an almost P-space that is not a P-space. We refer the reader to 4J in [5] for more details and properties of P-spaces.

Whenever R is a ring, a sequence $a_1, \cdots, a_n$ of elements of R is said to be a regular sequence of length n if a_1 is regular in R, a_2 (in fact $a_2 + a_1R$) is regular in R/a_1R, a_3 is regular in $R/(a_1R + a_2R)$, ... such that $R \neq \sum_{i=1}^{n} a_iR$. The maximum length of all regular sequences in R, if it exists, is called the *depth* of R and it is denoted by $\text{depth}(R)$. The concept of regular sequences of a ring was first introduced in [10]. Regular sequences and the concept of depth are usually studied in the context of local rings and algebraic geometry. Nevertheless, these concepts are defined and studied in general rings, modules, and recently in rings of continuous functions; see [2]. We also refer interested readers to [6] and Auslander's works [1]. In the present paper we generalize Theorem 2.5 in [2] and its corollaries and we compute the depth of some important subrings of $C(X)$.

2. Regular sequences in the intermediate algebras between $C^*(X)$ and $C(X)$

Since $C^*(X) \cong C(\beta X)$ and $C(Y)$ for each space Y does not contain a regular sequence of length more than 1 by Corollary 2.7 in [2], the lengths of regular sequences in $C^*(X)$ are at most 1. More generally, we observe in this section that the length of any regular sequence in the intermediate algebras between $C^*(X)$ and $C(X)$ is also at most 1 and then we obtain the depth of such algebras. First we cite some known results. The following lemma is Proposition 3.3 in [4]. We recall that a subring A of $C(X)$ absolutely convex if whenever $f \in C(X)$, $g \in A$, and $|f| \leq |g|$, then $f \in A$.

Lemma 2.1 *If $A(X)$ is an intermediate algebra between $C^*(X)$ and $C(X)$, then $A(X)$ is an absolutely convex subalgebra of $C(X)$.*

Using the above lemma, for each intermediate algebra $A(X)$ between $C^*(X)$ and $C(X)$, we have $f \in A(X)$ if and only if $|f| \in A(X)$. The following result also seems to be known, but we could not find the related reference.

Lemma 2.2 *If $A(X)$ is an intermediate algebra between $C^*(X)$ and $C(X)$, then $f \in A(X)$ if and only if $|f|^{1/2} \in A(X)$. More generally, $f \in A(X)$ if and only if $|f|^{1/2n} \in A(X)$.*

Proof Since $\frac{|f|^{1/2}}{1+|f|} \leq \frac{1}{2}$, we have $\frac{|f|^{1/2}}{1+|f|} \in C^*(X) \subseteq A(X)$. Hence, there exists $h \in A(X)$ such that $|f|^{1/2} = h(1+|f|)$, and since $|f| \in A(X)$ by Lemma 2.1, we have $|f|^{1/2} \in A(X)$. Conversely, $|f|^{1/2} \in A(X)$ implies $|f| = |f|^{1/2}|f|^{1/2} \in A(X)$. By induction, we may prove that $f \in A(X)$ if and only if $|f|^{1/2n} \in A(X)$.
□

Theorem 2.3 *Let $A(X)$ be an intermediate algebra between $C^*(X)$ and $C(X)$, r be a nonunit regular element of $A(X)$, and $f \in A(X)$. Let (r) denote the principal ideal in $A(X)$ generated by r. If $f+(r)$ is a nonunit in $A(X)/(r)$, then $f+(r)$ is a zero divisor in $A(X)/(r)$.*

Proof Using Lemma 2.1, $|f|+|r| \in A(X)$. First we show that $|f|+|r|$ is not a unit in $A(X)$. Suppose, on the contrary, that $|f|+|r|$ is a unit in $A(X)$ and let $h \in A(X)$ such that $(|f|+|r|)h = 1$. Then $1-h^2f^2-r^2h^2 = 2|r||f|h^2$. Again by squaring both sides, we get $f(h^4f^3-2h^2f-2h^4fr^2)+1 = r(2h^2r-h^4r^3)$, and this means that $f+(r)$ is a unit in $A(X)/(r)$, a contradiction. Hence, $|f|+|r|$ is not a unit in $A(X)$. Next we consider two cases:

Case 1. $Z(r) \cap Z(f) = \emptyset$. Define $h = \frac{r}{|f|+|r|}$ and $k = \frac{f}{|f|+|r|}$. Clearly $h, k \in C^*(X) \subseteq A(X)$ and $fh = rk$, i.e. $(f+(r))(k+(r)) = (r) = 0$. To see that $f+(r)$ is a zero divisor in $A(X)/(r)$, it is enough to show that $h \notin (r)$. Suppose, on the contrary, that $h \in (r)$. Then $h = rt$ for some $t \in A(X)$, so we have $\frac{r}{|f|+|r|} = rt$ or $r(1-t(|f|+|r|)) = 0$. This implies that $t(|f|+|r|) = 1$, since r is a regular element of $A(X)$, but $|f|+|r|$ is not a unit in $A(X)$, a contradiction. Therefore, $f+(r)$ is indeed a zero divisor in $A(X)/(r)$ and we are through.

Case 2. $Z(r) \cap Z(f) \neq \emptyset$. First, whenever $\mathrm{int}_X Z(r) \cap \mathrm{int}_X Z(f) \neq \emptyset$, then we take $x \in \mathrm{int}_X Z(r) \cap \mathrm{int}_X Z(f)$ and define $g \in C^*(X) \subseteq A(X)$ such that $g(x) = 1$ and $g(X \setminus \mathrm{int}_X Z(r) \cap \mathrm{int}_X Z(f)) = 0$. Clearly $gf = 0$ and $g \notin (r)$ because $g(x) = 1 \neq 0 = r(x)$. This implies that $f+(r)$ is a zero divisor in $A(X)/(r)$. Next, suppose that $\mathrm{int}_X Z(r) \cap \mathrm{int}_X Z(f) = \emptyset$. Thus, $|f|+|r|$ is a nonunit regular element of $A(X)$. Define

$$h(x) = \begin{cases} \frac{r}{|f|^{1/2}+|r|^{1/2}}(x) & , \quad x \notin Z(r) \cap Z(f) \\ 0 & , \quad x \in Z(r) \cap Z(f) \end{cases}$$

$$k(x) = \begin{cases} \frac{f}{|f|^{1/2}+|r|^{1/2}}(x) & , \quad x \notin Z(r) \cap Z(f) \\ 0 & , \quad x \in Z(r) \cap Z(f). \end{cases}$$

Clearly $h, k \in C(X)$ (in fact $|h| \leq |r|^{1/2}$ and $|k| \leq |f|^{1/2}$). Furthermore, $|r|^{1/2}, |f|^{1/2} \in A(X)$ by Lemma 2.2 and $|h| \leq |r|^{1/2}$, $|k| \leq |f|^{1/2}$ imply that $h, k \in A(X)$ by Lemma 2.1. Moreover, we have $fh = rk$, i.e. $(f+(r))(h+(r)) = 0$ in $A(X)/(r)$. We show that $h \notin (r)$. Indeed, if $h = rt$ for some $t \in A(X)$, then $r(x)[t(x)(|f|^{1/2}(x)+|r|^{1/2}(x))-1] = 0$ for all $x \notin Z(r) \cap Z(f)$. Clearly, this equality is also valid for each $x \in Z(r) \cap Z(f)$, so $r(t(|f|^{1/2}+|r|^{1/2})-1) = 0$. Since $t(|r|^{1/2}+|f|^{1/2})-1 \in A(X)$ and r is regular in $A(X)$, we must have $t(|r|^{1/2}+|f|^{1/2})-1 = 0$. Therefore, $t(x) = \frac{1}{|r|^{1/2}+|f|^{1/2}}(x)$ for each $x \notin Z(r) \cap Z(f)$. But $\mathrm{int}_X Z(r) \cap \mathrm{int}_X Z(f) = \emptyset$, so $\emptyset \neq Z(r) \cap Z(f) = \partial(Z(r) \cap Z(f)) = \partial Z(|f|^{1/2}+|r|^{1/2})$.

Now if we take $x_0 \in Z(r) \cap Z(f)$, there exists a net (x_λ) in $X \setminus (Z(r) \cap Z(f))$ such that $x_\lambda \to x_0$. Therefore, $|r|^{1/2}(x_\lambda) + |f|^{1/2}(x_\lambda) \to 0$, whence $t(x_\lambda) \to \infty$, i.e. t is not continuous at x_0, a contradiction. This implies that $h \notin (r)$; hence, $f + (r)$ is a zero divisor in $A(X)/(r)$ and we are done. □

Corollary 2.4 *Whenever $A(X)$ is an intermediate algebra between $C^*(X)$ and $C(X)$, then $A(X)$ does not contain a regular sequence of length ≥ 2. In other words, depth$(A(X)) \leq 1$.*

Corollary 2.5 . *If $A(X)$ is a proper intermediate algebra between $C^*(X)$ and $C(X)$, i.e. $A(X) \neq C(X)$, then depth$(A(X))$ is exactly 1. In particular, if X is not pseudocompact, then depth$(C^*(X)) = 1$.*

Proof Suppose that $f \in C(X) \setminus A(X)$. Using Lemma 2.1 we also have $1 + |f| \notin A(X)$. Now $\frac{1}{1+|f|} \in C^*(X) \subseteq A(X)$ implies that $\frac{1}{1+|f|}$ is a nonunit element of $A(X)$. On the other hand, $\frac{1}{1+|f|}$ is not a zero divisor in $A(X)$, so it is a nonunit regular element of $A(X)$. Moreover, $\frac{1}{1+|f|}A(X) \neq A(X)$, for if $1 = \frac{a}{1+|f|}$ for some $a \in A(X)$, then we have $1 + |f| = a \in A(X)$, which is impossible. Therefore, depth($A(X)) \geq 1$, and using Corollary 2.4, depth($A(X)) = 1$ □

Using Proposition 2.2 in [7], βX is an almost P-space if and only if X is a pseudocompact almost P-space. On the other hand we know that X is an almost P-space if and only if every regular element of $C(X)$ is a unit. Therefore, depth$(C(X)) = 0$ if and only if X is an almost P-space. Now, using these facts and $C^*(X) \cong C(\beta X)$, the following result is evident.

Proposition 2.6 *Depth$(C^*(X)) = 0$ if and only if X is a pseudocompact almost P-space.*

Remark 2.7 *Let $A(X)$ be a proper intermediate algebra between $C^*(X)$ and $C(X)$, i.e. $A(X) \neq C(X)$. Then, by the above results, whenever X is not an almost P-space, we have depth$(C^*(X))$ = depth$(A(X))$ = depth$(C(X)) = 1$, and whenever X is an almost P-space, then depth$(C^*(X))$ = depth$(A(X)) = 1$ but depth$(C(X)) = 0$.*

3. Regular sequences in the subrings of the form $I + \mathbb{R}$ of $C(X)$

The class of subrings of $C(X)$ of the form $I + \mathbb{R}$, where I is a z-ideal of $C(X)$, is considered in this section. Some properties of these subrings of $C(X)$ were studied in [8], [9], and [3]. In this section we show that the lengths of regular sequences in such subrings are also at most 1.

Whenever I is a z-ideal of $C(X)$, then the subring $I + \mathbb{R}$ has a useful representation as follows:

$$I + \mathbb{R} = \{f \in C(X) : f \text{ is constant on an element of } Z[I]\}.$$

To see this, if $f \in I + \mathbb{R}$, then $f = i + r$ fo some $i \in I$ and $r \in \mathbb{R}$. This implies that $f(Z(i)) = r$, i.e. f is constant on $Z(i) \in Z[I]$. Conversely, if f is constant on a zero-set $Z(i)$, where $i \in I$, say $f(Z(i)) = c$, then $Z(i) \subseteq Z(f - c)$. Since I is a z-ideal, $f - c \in I$ and therefore $f = (f - c) + c \in I + \mathbb{R}$.

This representation helps us to see that whenever $f \in I + \mathbb{R}$ and $f \geq 0$, then $f^r \in I + \mathbb{R}$ for each nonnegative $r \in \mathbb{R}$. In fact, if $f \in I + \mathbb{R}$, then f is constant on some $Z \in Z[I]$, say $f(Z) = c$, and this implies that $f^r(Z) = c^r$, i.e. $f^r \in I + \mathbb{R}$. In particular, whenever $f \in I + \mathbb{R}$, then $f^{\frac{1}{n}} \in I + \mathbb{R}$ for each odd integer n. Also, $f \in I + \mathbb{R}$ if and only if $|f| \in I + \mathbb{R}$, because f is constant on a zero-set if and only if $|f|$ is.

To prove the next proposition we need the following well-known fact, which topologically characterizes unit elements of $I+\mathbb{R}$. For its simple proof, see [3] Proposition 1.1.

Lemma 3.1 *Let I be a z-ideal of $C(X)$, $i \in I$ and $r \in \mathbb{R}$. Then $i+r$ is a unit in $I+\mathbb{R}$ if and only if $Z(i+r) = i^{-1}(\{-r\}) = \emptyset$.*

Proposition 3.2 *Let I be a z-ideal of $C(X)$, $i, j \in I$, and $r, s \in \mathbb{R}$. Let $i+r$ be a nonunit regular element in $I+\mathbb{R}$ and $(i+r)$ be the principal ideal in $I+\mathbb{R}$ generated by $i+r$. Then every nonunit element of $(I+\mathbb{R})/(i+r)$ is a zero divisor.*

Proof Let $j+s+(i+r)$ be a nonunit element of $(I+\mathbb{R})/(i+r)$. First we show that $Z(i+r) \cap Z(j+s) \neq \emptyset$, i.e. $(i+r)^2+(j+s)^2$ is a nonunit in $I+\mathbb{R}$ by Lemma 3.1. In fact, if $(i+r)^2+(j+s)^2$ is a unit, then $t(i+r)^2+t(j+s)^2 = 1$ for some $t \in I+\mathbb{R}$, but this means that $j+s+(i+r)$ is a unit element of $(I+\mathbb{R})/(i+r)$, which contradicts our hypothesis. Next define

$$h(x) = \begin{cases} \frac{i+r}{|i+r|^{1/2}+|j+s|^{1/2}}(x) & , \quad x \notin Z(i+r) \cap Z(j+s) \\ 0 & , \quad x \in Z(i+r) \cap Z(j+s) \end{cases}$$

$$k(x) = \begin{cases} \frac{j+s}{|i+r|^{1/2}+|j+s|^{1/2}}(x) & , \quad x \notin Z(i+r) \cap Z(j+s) \\ 0 & , \quad x \in Z(i+r) \cap Z(j+s). \end{cases}$$

Clearly, $h, k \in C(X)$. Furthermore, the value of h on $Z(i) \cap Z(j)$ is the constant real number $\frac{r}{|r|^{1/2}+|s|^{1/2}}$ if $|r|+|s| \neq 0$. Similarly, the value of k on $Z(i) \cap Z(j)$ is the constant real number $\frac{s}{|r|^{1/2}+|s|^{1/2}}$ if $|r|+|s| \neq 0$. Whenever $r = s = 0$, then both h and k are zero on $Z(i) \cap Z(j)$ by definitions of h and k. In any case, h and k are constant on $Z(i) \cap Z(j) \in Z[I]$; therefore, $h, k \in I+\mathbb{R}$. Moreover, we have $(j+s)h = (i+r)k$. If we show that $h \notin (i+r)(I+\mathbb{R})$, then $j+s+(i+r)$ will be a zero divisor element of $(I+\mathbb{R})/(i+r)$ and we are done.

Suppose, on the contrary, that $h = (i+r)t$ for some $t \in I+\mathbb{R}$. Then on $X \setminus (Z(i+r) \cap Z(j+s))$ we have

$$\frac{i+r}{|i+r|^{1/2}+|j+s|^{1/2}} = (i+r)t.$$

Thus, on the outside of $Z(i+r) \cap Z(j+s)$ we have $i+r = (|i+r|^{1/2}+|j+s|^{1/2})(i+r)t$. It is clear that this equality also holds on $Z(i+r)$ and hence on $Z(i+r) \cap Z(j+s)$. Therefore, $(i+r)(1-(|i+r|^{1/2}+|j+s|^{1/2})t) = 0$ on X. However, $1-(|i+r|^{1/2}+|j+s|^{1/2})t \in I+\mathbb{R}$ by the argument preceding Lemma 3.1 and $i+r$ is regular in $I+\mathbb{R}$, so we must have $1-(|i+r|^{1/2}+|j+s|^{1/2})t = 0$. Now $Z(i+r) \cap Z(j+s) \neq \emptyset$ implies that $1 = 0$, a contradiction. □

Corollary 3.3 *For each z-ideal I of $C(X)$, every regular sequence in $I+\mathbb{R}$ has length at most 1 and consequently depth$(I+\mathbb{R}) \leq 1$.*

As we already mentioned, X is an almost P-space if and only if depth$(C(X) = 0$, but this is not the case for subrings of the form $I+\mathbb{R}$, even if I is a z-ideal. In the following examples we show that whenever X is an almost P-space, then depth$(I+\mathbb{R})$ may not be zero, and whenever X is not an almost P-space, then depth$(I+\mathbb{R})$ may be zero. To see this, we first prove the following proposition.

Proposition 3.4 *Let $f \in C(X)$ and $Z(f) \neq \emptyset$. Then the following statements hold.*

1. *If depth$(M_f + \mathbb{R}) = 0$, then $Z(f)$ is open.*
2. *If Z(f) is open and cozf is an almost P-space, then depth$(M_f + \mathbb{R}) = 0$.*

Proof (1) Using Corollary 3.3, it is enough to show that whenever $Z(f)$ is not open, then $\text{depth}(M_f + \mathbb{R}) = 1$. First we observe that the nonunit element f of $M_f + \mathbb{R}$ is regular. To see this, we show that f is not a zero divisor in $M_f + \mathbb{R}$. If so, then $f(g + r) = 0$ for some $g \in M_f$ and $r \in \mathbb{R}$ such that $g + r \neq 0$. If $r = 0$, then $fg = 0$ and $Z(f) \subseteq Z(g)$ imply that $g = 0$, whence $g + r = 0$, a contradiction. Hence, $r \neq 0$. Now $f(g + r) = 0$ implies that $Z(f) \cup Z(g + r) = X$, $Z(f) \cap Z(g + r) = \emptyset$, and this means that $Z(f)$ is open, which contradicts our hypothesis. Next, it is clear that $f(M_f + \mathbb{R}) \neq M_f + \mathbb{R}$, so $\text{depth}(M_f + \mathbb{R}) = 1$ by Corollary 3.3.

(2) If $h \in M_f$, then h is a zero divisor of $M_f + \mathbb{R}$. To see this, it is enough to define $g \in C(X)$ such that $g(Z(f)) = 0$ and $g(X \setminus Z(f)) = 1$. Clearly $g \in M_f$, $0 \neq g - 1 \in M_f + \mathbb{R}$, and $h(g - 1) = 0$. Now suppose that $h + r \in M_f + \mathbb{R}$ is a nonunit, where $h \in M_f$ and $0 \neq r \in \mathbb{R}$. Then $\emptyset \neq Z(h + r) = (X \setminus Z(f)) \cap Z(h + r) = Z((h + r)|_{\text{coz} f})$. However, $\text{coz} f$ is an almost P-space, and then $\text{int}_{\text{coz} f} Z((h + r)|_{\text{coz} f}) \neq \emptyset$, whence $\text{int}_X Z(h + r) \neq \emptyset$, because $\text{coz} f$ is open. Take $x \in \text{int}_X Z(h + r)$ and define $g \in C(X)$ such that $g(x) = 1$ and $g(X \setminus \text{int}_X Z(h + r)) = 0$. Since $Z(f) \subseteq X \setminus \text{int}_X Z(h + r)$ $(Z(h) \cap Z(h + r) = \emptyset)$, we have $Z(f) \subseteq Z(g)$ and hence $0 \neq g \in M_f$. Moreover, $g(h + r) = 0$, i.e. $h + r$ is a zero divisor, so $\text{depth}(M_f + \mathbb{R}) = 0$. □

Example 3.5 *(a) Suppose that X is an almost P-space that is not a P-space. Then there is $f \in C(X)$ such that $Z(f)$ is not open. Now using part (1) of Proposition 3.4, depth$(M_f + \mathbb{R}) = 1$, whereas depth$(C(X)) = 0$.*

(b) Suppose X and Y are two disjoint spaces such that X is an almost P-space but Y is not and take $T = X \cup Y$ as a free union of X and Y. Since Y is a closed-open subset of T, we may consider it as a zero-set, say $Z(f)$, where $f \in C(T)$. Since $Z(f) = Y$ is open and $\text{coz} f = X$ is an almost P-space, using part (2) of Proposition 3.4, the subring $M_f + \mathbb{R}$ of $C(T)$ has depth zero, whereas depth$(C(T)) = 1$, because T is not an almost P-space.

What we have shown so far is that the lengths of regular sequences in two large classes of subrings of $C(X)$ are at most 1. This fact also holds for some other subrings of $C(X)$ outside of these two classes. For instance, whenever S is a closed-open subset of a space X, then $C(X)$ is isomorphic to the direct sum of $C(S)$ and $C(X \setminus S)$; see 1B(6) in [5]. Clearly, $C(S)$ is a subring of $C(X)$ that is neither an intermediate subring nor a subring of the form $I + \mathbb{R}$ for some z-ideal I of $C(X)$. Now, using Corollary 2.7 in [2] and our Corollary 2.4, the subring $C(S)$ does not contain a regular sequence of length ≥ 2.

Similar to $C(X)$ and some subrings of $C(X)$, it seems that the length of every regular sequence in each subring of $C(X)$ is ≤ 1, or equivalently, $\text{depth}(S) \leq 1$ for every subring S of $C(X)$. We were unable to prove or disprove this assertion, so we cite it here as a conjecture.

Conjecture. If S is a subring of $C(X)$, then $\text{depth}(S) \leq 1$.

4. Regular sequences in the trivial ring extension of $C(X)$

Not only for a large class of subrings of $C(X)$ but for some other rings that contain $C(X)$ as a subring, the length of regular sequences does not exceed 1. In this section, we show this pretension in trivial ring extension of $C(X)$ by $C(X)$ itself, which is the ring $C^2(X) = C(X) \times C(X)$ with addition and multiplication defined as follows:

$$(f,g) + (h,k) = (f+h, g+k),$$

$$(f,g)(h,k) = (fh, fk+gh),$$

for all $f,g,h,k \in C(X)$. First we need the following lemma.

Lemma 4.1 *Let $f,g,h,k \in C(X)$. Then the following statements hold.*

1. *(f,g) is a unit element of $C^2(X)$ if and only if f is a unit in $C(X)$, i.e. $Z(f) = \emptyset$.*
2. *(f,g) is a regular element of $C^2(X)$ if and only if $int_X Z(f) = \emptyset$ (i.e. f is a regular element of $C(X)$).*
3. *Whenever (f,g) is a regular element of $C^2(X)$, then (h,k) is a unit in $C^2(X)/(f,g)C^2(X)$ if and only if $Z(f) \cap Z(h) = \emptyset$.*

Proof (1) If (f,g) is a unit in $C^2(X)$, then clearly f is a unit in $C(X)$. Conversely, if f is a unit in $C(X)$, then $(f,g)(\frac{1}{f}, \frac{-g}{f^2}) = (1,0)$, so (f,g) is a unit in $C^2(X)$.

(2) If (f,g) is regular in $C^2(X)$, then for each $0 \neq h \in C(X)$, we have $(f,g)(0,h) = (0, fh) \neq (0,0)$. This means that f is regular in $C(X)$, so $\text{int}_X Z(f) = \emptyset$ by Lemma 1.1. Conversely, if $\text{int}_X Z(f) = \emptyset$, then f is regular in $C(X)$ by the same lemma. Now if $(h,k) \neq (0,0)$, then clearly $(f,g)(h,k) = (fh, fk+gh) \neq (0,0)$. In fact, if $h \neq 0$, then $fh \neq 0$ and if $h = 0$ but $k \neq 0$, then $fk \neq 0$ because f is regular in $C(X)$. In any case, we have $(f,g)(h,k) \neq (0,0)$, so (f,g) is regular in $C^2(X)$.

(3) Let (h,k) be a unit in $C^2(X)/(f,g)C^2(X)$. Then there exist $t,s,u,v \in C(X)$ such that $(h,k)(t,s) = (1,0) + (f,g)(u,v)$. Hence, $ht - fu = 1$ implies that $Z(h) \cap Z(f) = \emptyset$. Conversely, suppose that $Z(h) \cap Z(f) = \emptyset$, i.e. $f^2 + h^2 = u$ is a unit by Lemma 1.1. Thus, $\frac{f^2}{u} + \frac{h^2}{u} = 1$ and hence $(h,k)(\frac{h}{u}, 0) = (1, \frac{kh}{u} + \frac{fg}{u}) + (f,g)(-\frac{f}{u}, 0)$. Now, multiplying both sides of the equality by $(1, -\frac{fg}{u} - \frac{hk}{u})$, we get

$$(h,k)(\frac{h}{u}, -\frac{fgh}{u^2} - \frac{kh^2}{u^2}) = (1,0) + (f,g)(-\frac{f}{u}, \frac{f^2 g}{u^2} + \frac{fhk}{u^2}).$$

Therefore, (h,k) is a unit in $C^2(X)/(f,g)C^2(X)$. □

Proposition 4.2 *Let $f,g \in C(X)$ and (f,g) be regular in $C^2(X)$. Then every nonunit element of $C^2(X)/(f,g)C^2(X)$ is a zero divisor.*

Proof Let $(h,k) + (f,g)C^2(X)$ be a nonunit in $C^2(X)/(f,g)C^2(X)$. Then $Z(f) \cap Z(h) \neq \emptyset$ by Lemma 3.6(3), and part (1) of the same lemma implies that h is a nonunit in $C(X)$. Since (f,g) is regular in $C^2(X)$, f is regular in $C(X)$ by Lemma 4.1(2), but $\text{depth}(C(X)) \leq 1$ by Corollary 2.4, so h is a zero divisor of $C(X)/(f)$. Thus, there exists $t \in C(X)$ such that $th = fu$ for some $u \in C(X)$, where $t \notin (f)$. Now $(h,k)(0,t) = (f,g)(0,u)$, and if we show that $(0,t) \notin (f,g)C^2(X)$, then $(h,k) + (f,g)C^2(X)$ will be a zero divisor in $C^2(X)/(f,g)C^2(X)$. To this end, suppose, on the contrary, that $(0,t) = (f,g)(m,n)$ for some $m,n \in C(X)$. Therefore, we have $fm = 0$, which implies $m = 0$ because f is regular in $C(X)$. On the other hand, $fn + gm = t$ implies $fn = t$, i.e. $t \in (f)$, a contradiction. □

Corollary 4.3 *depth*$(C^2(X)) \leq 1$.

Remark 4.4 *The results of this article can be extended to the case of complex-valued continuous functions. It seems that many arguments are valid in this case and the techniques applied in the proof of the results also work in the case of complex-valued continuous functions. We leave this to the interested readers.*

Acknowledgment

We would like to thank the referee for the time he/she devoted to reading our article and providing some comments and corrections towards the improvement of the paper.

References

[1] Auslander M. Selected Works of Maurice Auslander. Part 1. Edited and with a Foreword by Idun Reiten, Sverre O. Smal$\varnothing$, and $\varnothing$yvind Solberg. Providence, RI, USA: AMS, 1999.

[2] Azarpanah F, Esmaeilvandi D, Salehi AR. Depth of ideals of $C(X)$. Journal of Algebra 2019; 528: 474-496. doi: 10.1016/j.jalgebra.2019.03.018

[3] Azarpanah F, Namdari M, Olfati AR. On subrings of the form $I + \mathbb{R}$ of $C(X)$. Journal of Commutative Algebra 2019; 11 (4): 479-508. doi: 10.1216/JCA.2019.11.4.479

[4] Dominguez JM, Gomez J, Mulero MA. Intermediate algebras between $C^*(X)$ and $C(X)$ as rings of quotients of $C^*(X)$. Topology and Its Applications 1997; 77: 115-130.

[5] Gillman L, Jerison M. Rings of Continuous Functions. Berlin, Germany: Springer-Verlag, 1976.

[6] Leuschke GJ, Wiegand R. Cohen-Macaulay Representations Type. Providence, RI, USA: AMS, 2012.

[7] Levy R. Almost P-spaces. Canadian Journal of Mathematics 1977; 29 (2): 284-288.

[8] Rudd D. On structure spaces of ideals in rings of continuous functions. Transactions of the American Mathematical Society 1974; 190 (4): 393-403.

[9] Rudd D. An example of a Φ-algebra whose uniform closure is ring of continuous functions. Fundamenta Mathematicae 1972; 77: 1-4.

[10] Serre JP. Sur la dimension cohomologique des anneaux et des modules noetheriens. In: Proceedings of the International Symposium on Algebra, Number Theory, Tokyo, 1955, pp. 176-189 (in French).