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The strong 3-rainbow index of edge-amalgamation of some graphs

Zata Yumni AWANIS[1,*], Anm SALMAN[1], Suhadi Wido SAPUTRO[1]
Martin BAČA[2], Andrea SEMANIČOVÁ-FEŇOVČÍKOVÁ[2]
[1]Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia
[2]Department of Applied Mathematics and Informatics, Technical University, Košice, Slovakia

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Abstract: Let G be a nontrivial, connected, and edge-colored graph of order $n \geq 3$, where adjacent edges may be colored the same. Let k be an integer with $2 \leq k \leq n$. A tree T in G is a rainbow tree if no two edges of T are colored the same. For $S \subseteq V(G)$, the Steiner distance $d(S)$ of S is the minimum size of a tree in G containing S. An edge-coloring of G is called a strong k-rainbow coloring if for every set S of k vertices of G there exists a rainbow tree of size $d(S)$ in G containing S. The minimum number of colors needed in a strong k-rainbow coloring of G is called the strong k-rainbow index $srx_k(G)$ of G. In this paper, we study the strong 3-rainbow index of edge-amalgamation of graphs. We provide a sharp upper bound for the srx_3 of edge-amalgamation of graphs. We also determine the srx_3 of edge-amalgamation of some graphs.

Key words: Edge-amalgamation, rainbow coloring, rainbow tree, strong k-rainbow index

1. Introduction

All graphs considered in this paper are simple, finite, and connected. We follow the terminology and notation of Diestel [5]. For simplifying, we define $[a, b]$ as a set of all integers x with $a \leq x \leq b$. Let G be an edge-colored graph of order $n \geq 3$, where adjacent edges may be colored the same. A tree T in G is a *rainbow tree* if no two edges of T receive the same color. For $S \subseteq V(G)$, a *rainbow S-tree* is a rainbow tree that contains the vertices of S. Let k be an integer with $k \in [2, n]$. An edge-coloring of G is called a *k-rainbow coloring* if for every set S of k vertices of G there exists a rainbow S-tree in G. The *k-rainbow index* $rx_k(G)$ of G, introduced by Chartrand et al. [3], is the minimum number of colors needed in a k-rainbow coloring of G. Thus, if $k = 2$, then $rx_2(G)$ is the *rainbow connection number* $rc(G)$ of G, which was first introduced by Chartrand et al. in 2008 [2]. Some known results about the rainbow connection number of graphs can be found in [2, 6–9, 11–13]. For every nontrivial connected graph G of order n, it is easy to see that $rx_2(G) \leq rx_3(G) \leq ... \leq rx_n(G)$.

The concept of the k-rainbow index has an interesting application in transferring classified information in communication networks security. One of the things that can be done to make a secure transfer line between k agencies (which may have other agencies as intermediaries) in communication networks is to assign a large enough number of passwords to the line so that no password is repeated. An immediate question arises: What is the minimum number of passwords needed that allows one secure line between every k agencies so that the

*Correspondence: zata.yumni@s.itb.ac.id
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passwords along the line are distinct? This situation can be modeled by a graph and the minimum number of these passwords is represented by the k-rainbow index of a graph.

The *Steiner distance* $d(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S. Such a tree is called a *Steiner S-tree*. The maximum Steiner distance of S among all sets S of k vertices of G is called the *k-Steiner diameter* $sdiam_k(G)$ of G. Chartrand et al. [3] stated that for every connected graph G of order $n \geq 3$ and each integer k with $k \in [3, n]$, $k-1 \leq sdiam_k(G) \leq rx_k(G) \leq n-1$. In [3], they showed that trees are composed of a class of graphs whose k-rainbow index attains the upper bound for $rx_k(G)$. They also determined the k-rainbow index of cycles and the 3-rainbow index of complete graphs. Chen et al. [4] provided the 3-rainbow index of regular complete bipartite and multipartite graphs and wheels. In [1], we determined the 3-rainbow index of amalgamation of some graphs with diameter 2. Liu and Hu in 2014 [10] studied the 3-rainbow index with respect to three important graph product operations, namely the Cartesian product, strong product, and lexicographic product, and also other graph operations. Graph operations are an interesting subject, which can be used to understand structures of graphs.

We generalized the concept of the k-rainbow index of G called the *strong k-rainbow index* of G*. A *strong k-rainbow coloring* of G is an edge-coloring of G having the property that for every set S of k vertices of G there exists a rainbow tree of size $d(S)$ containing S. Such a rainbow tree is called a *rainbow Steiner S-tree*. The minimum number of colors needed in a strong k-rainbow coloring of G is the strong k-rainbow index of G, denoted by $srx_k(G)$. Thus, we have $rx_k(G) \leq srx_k(G)$ for every connected graph G. If $k = 2$, then $srx_2(G)$ is the *strong rainbow connection number* $src(G)$ of G [2]. Chartrand et al. [2] gave lower and upper bounds for the strong rainbow connection number; that is, $diam(G) \leq rc(G) \leq src(G) \leq |E(G)|$.

Note that every coloring that assigns distinct colors to all edges of a connected graph is a strong k-rainbow coloring. Thus, the strong k-rainbow index is defined for every connected graph G. Furthermore, if G is a nontrivial connected graph of size $|E(G)|$ whose k-Steiner diameter is $sdiam_k(G)$, then it is easy to check that

$$sdiam_k(G) \leq rx_k(G) \leq srx_k(G) \leq |E(G)|. \tag{1.1}$$

We have determined the strong 3-rainbow index of some certain graphs. We also provided a sharp upper bound for the strong 3-rainbow index of amalgamation of graphs and determined the exact values of the strong 3-rainbow index of amalgamation of some graphs*. The following results are needed.

Theorem 1.1 * *Let T_n be a tree of order $n \geq 3$. For each integer $k \in [3, n]$, $srx_k(T_n) = |E(T_n)| = n-1$.*

Theorem 1.2 * *For $n \geq 3$, let L_n be a ladder graph of order $2n$. Then $srx_3(L_n) = sdiam_3(L_n) = n$.*

Theorem 1.3 * *For $n \geq 3$, let $K_{n,n}$ be a regular complete bipartite graph of order $2n$. Then $srx_3(K_{n,n}) = n$.*

Theorem 1.4 * *Let C_n be a cycle of order $n \geq 3$. Then:*

$$srx_3(C_n) = \begin{cases} 2, & \text{for } n = 3; \\ n-2, & \text{for } n \in [4,6] \text{ or } n = 8; \\ n, & \text{for } n = 7 \text{ or } n \geq 9. \end{cases}$$

For illustration, strong 3-rainbow colorings of C_3, C_4, C_5, C_6, and C_8 are given in Figure 1.

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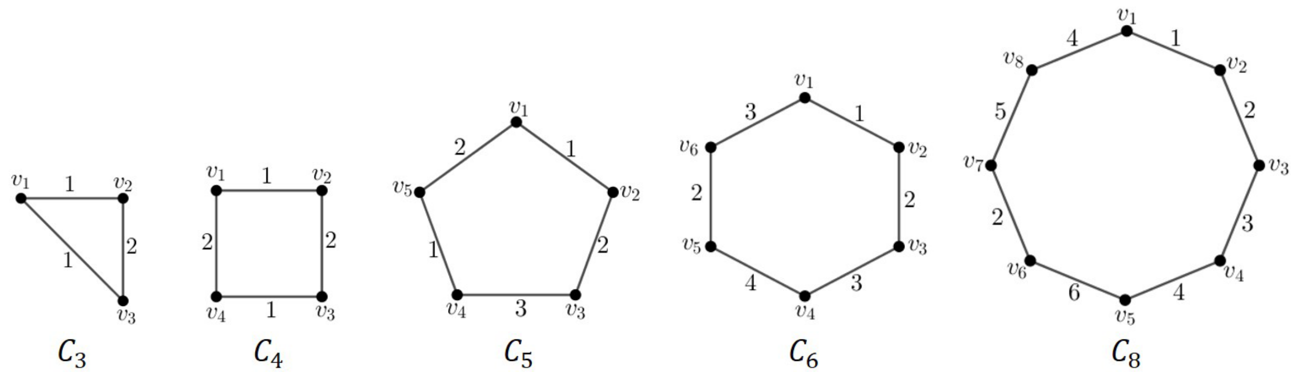

Figure 1. Strong 3-rainbow colorings of C_3, C_4, C_5, C_6, and C_8.

For an integer $t \geq 2$, let $\{G_1, G_2, ..., G_t\}$ be a collection of finite, simple, and connected graphs and each G_i has a fixed edge e_{o_i} called a *terminal edge*. Assume that each terminal edge has an orientation. The *edge-amalgamation* of $G_1, G_2, ..., G_t$, denoted by $Edge - Amal\{G_i; e_{o_i}\}$, is a graph obtained by taking all the G_i's and identifying their terminal edges with the same orientation. If for each $i \in [1, t]$, $G_i \cong G$ and $e_{o_i} = e$, then $Edge - Amal\{G_i; e_{o_i}\}$ is denoted by $Edge - Amal(G, e, t)$.

In this paper, we study graphs of type $Edge - Amal(G, e, t)$. It is needed when we want to make a larger and complex communication networks and some agencies must pass through one or two centers in order to transfer information or communicate with each other safely. We focus on $k = 3$. We determine a sharp upper bound for the strong 3-rainbow index of $Edge - Amal(G, e, t)$. We also determine the exact values of the strong 3-rainbow index of $Edge - Amal(G, e, t)$ for some connected graphs G.

2. Main results

Let G be a simple connected graph of order $n \geq 3$ and let e be a terminal edge of G, which has an orientation. Given c as a strong 3-rainbow coloring of G and $X \subseteq E(G)$, let $c(X)$ denote the set of colors assigned to all edges of X. For $t \geq 2$, consider graphs $Edge - Amal(G, e, t)$. Let $V\,(Edge - Amal(G, e, t)) = \{u, v\} \cup \{v_i^p | i \in [1, t], p \in [1, n-2]\}$ and uv be the identified edge of $Edge - Amal(G, e, t)$. For further discussion, given a tree T of size m as a subgraph of $Edge - Amal(G, e, t)$, let $T = \{e_1, e_2, ..., e_m\}$ denote the tree with edge set $\{e_1, e_2, ..., e_m\}$.

2.1. Sharp upper bound for $srx_3(Edge - Amal(G, e, t)$

In the following theorem, we provide an upper bound for the strong 3-rainbow index of $Edge - Amal(G, e, t)$.

Theorem 2.1 *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let G be a simple connected graph of order n and e be a terminal edge of G. Then:*

$$srx_3(Edge - Amal(G, e, t)) \leq min\,\{t\,(|E(G)| - 1) + 1,\; t\,(srx_3(G))\}\,.$$

Proof Following (1.1), we know that $|E(Edge - Amal(G, e, t))| = t\,(|E(G)| - 1) + 1$ is the natural upper bound for $srx_3(Edge - Amal(G, e, t))$. Now, let c' be a strong 3-rainbow coloring of G. We show that

$srx_3(Edge\!-\!Amal(G,e,t)) \le t\,(srx_3(G))$ by defining a strong 3-rainbow coloring $c: E(Edge\!-\!Amal(G,e,t)) \to [1, t\,(srx_3(G))]$ as follows:

$$c(e') = \begin{cases} c'(e'), & e' \in E(G_1); \\ c'(e') + (q-1)\,srx_3(G), & e' \in E(G_q)\backslash\{e\} \text{ for each } q \in [2,t]. \end{cases}$$

Observe that the coloring c above maintains the position of colors in G_i and assigns distinct colors in $E(G_i)$ and $E(G_j)$ for distinct i and j in $[1,t]$. Therefore, it is not difficult to find a rainbow Steiner S-tree for every set S of three vertices of $Edge-Amal(G,e,t)$. □

The upper bound in Theorem 2.1 is sharp. It can be proven by providing some connected graphs G such that $srx_3(Edge-Amal(G,e,t))$ attains the upper bound. Theorems 2.2 and 2.4 show that $srx_3(Edge-Amal(G,e,t)) = t\,(|E(G)|-1)+1$ where G is a tree or a cycle of order odd $n \ge 9$. Meanwhile, Theorem 2.8 shows that $srx_3(Edge-Amal(G,e,t)) = t\,(srx_3(G))$ where G is a fan.

Theorem 2.2 *Let t and n be two integers with $t \ge 2$ and $n \ge 3$. Let T_n be a tree of order n and e be an arbitrary edge of T_n. Then $srx_3(Edge-Amal(T_n,e,t)) = t\,(n-2)+1$.*

Proof Note that the edge-amalgamation of trees is also a tree with $|E(Edge-Amal(T_n,e,t))| = t\,(|E(T_n)|-1)+1$. It follows by Theorem 1.1 that $srx_3(Edge-Amal(T_n,e,t)) = |E(Edge-Amal(T_n,e,t))| = t\,(|E(T_n)|-1)+1 = t\,(n-2)+1$. □

Let C_n be a cycle of order $n \ge 3$. Consider graphs $Edge-Amal(C_n,e,t)$ where e is an arbitrary edge of C_n. Let $V(Edge-Amal(C_n,e,t)) = \{u,v\} \cup \{v_i^p | i \in [1,t], p \in [1,n-2]\}$ such that $E(Edge-Amal(C_n,e,t)) = \{uv\} \cup \{uv_i^1, vv_i^{n-2} | i \in [1,t]\} \cup \{v_i^p v_i^{p+1} | i \in [1,t], p \in [1,n-3]\}$. We start with the following observation, which will be used to prove the lower bound in Theorem 2.4.

Observation 2.3 *Let t and n be two integers at least 2 and n is odd. For $i \in [1,t]$, let A_i be a set of edges of path $uv_i^1 v_i^2 ... v_i^{\lfloor \frac{n}{2} \rfloor -1} v_i^{\lfloor \frac{n}{2} \rfloor}$ and B_i be a set of edges of path $vv_i^{n-2} v_i^{n-3} ... v_i^{\lfloor \frac{n}{2} \rfloor +1} v_i^{\lfloor \frac{n}{2} \rfloor}$. If c is a strong 3-rainbow coloring of $Edge-Amal(C_n,e,t)$, then:*

1. *$c(A_i) \cap c(A_j) = \emptyset$ and $c(B_i) \cap c(B_j) = \emptyset$ for distinct i and j in $[1,t]$;*
2. *for $n \ge 9$, $c(A_i) \cap c(B_j) = \emptyset$ for distinct i and j in $[1,t]$.*

Proof Let i and j be two distinct integers in $[1,t]$.

1. Since path $v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor -1} ... v_i^1 u v_j^1 ... v_j^{\lfloor \frac{n}{2} \rfloor -1} v_j^{\lfloor \frac{n}{2} \rfloor}$ is the only possible rainbow Steiner $\{u, v_i^{\lfloor \frac{n}{2} \rfloor}, v_j^{\lfloor \frac{n}{2} \rfloor}\}$-tree, we have $c(A_i) \cap c(A_j) = \emptyset$. Similarly, by considering $\{v, v_i^{\lfloor \frac{n}{2} \rfloor}, v_j^{\lfloor \frac{n}{2} \rfloor}\}$, we have $c(B_i) \cap c(B_j) = \emptyset$.
2. By considering $\{v_i^{\lfloor \frac{n}{4} \rfloor}, v_i^{\lfloor \frac{n}{2} \rfloor}, v_j^{\lfloor \frac{n}{2} \rfloor +1}\}$ and $\{v_i^{\lfloor \frac{n}{2} \rfloor}, v_j^{\lfloor \frac{n}{2} \rfloor -2}, v_j^{\lfloor \frac{n}{2} \rfloor +1}\}$, we obtain that no edges of the paths $v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor -1} ... v_i^{\lfloor \frac{n}{4} \rfloor} ... v_i^1 uvv_j^{n-2} ... v_j^{\lfloor \frac{n}{2} \rfloor +2} v_j^{\lfloor \frac{n}{2} \rfloor +1}$ and $v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor -1} ... v_i^1 uv_j^1 ... v_j^{\lfloor \frac{n}{2} \rfloor -2} v_j^{\lfloor \frac{n}{2} \rfloor -1} v_j^{\lfloor \frac{n}{2} \rfloor} v_j^{\lfloor \frac{n}{2} \rfloor +1}$ are colored the same. Thus, we have $c(A_i) \cap c(B_j) = \emptyset$.

□

Theorem 2.4 *Let t be an integer at least 2 and n be an odd integer at least 9. Let C_n be a cycle of order n and e be an arbitrary edge of C_n. Then $srx_3(Edge - Amal(C_n, e, t)) = t\,(n-1)+1$.*

Proof Since $t\,(srx_3(C_n)) = t\,n$ by Theorem 1.4 and $t\,(|E(C_n)|-1)+1 = t\,(n-1)+1$, it follows by Theorem 2.1 that $srx_3(Edge - Amal(C_n, e, t)) \leq t\,(n-1)+1$. Thus, we only need to prove the lower bound. Following Theorem 1.4, $c(uv) \notin c(A_i) \cup c(B_i)$ and we need at least $n-1$ distinct colors assigned to all edges in $A_i \cup B_i$ for each $i \in [1,t]$. Hence, by using Observation 2.3, $srx_3(Edge - Amal(C_n, e, t)) \geq t\,(n-1)+1$. □

For $n \geq 3$, recall that a *fan* F_n of order $n+1$ is a graph constructed by joining a vertex v to every vertex of a path $P_n : v_1 v_2 ... v_n$. The edges of P_n are called the *rims* of F_n and the edges connecting v to the vertices of P_n are called the *spokes* of F_n. Before we proceed to Theorem 2.8, we first determine the strong 3-rainbow index of F_n. We start with the following lemma.

Lemma 2.5 *For $n \geq 3$, let c be a strong 3-rainbow coloring of F_n. Then at most two spokes of F_n may be colored the same. Moreover, if $c(vv_i) = c(vv_j)$ for distinct i and j in $[1,n]$, then v_i and v_j are adjacent.*

Proof Suppose that there are three spokes of F_n, vv_i, vv_j, and vv_k, such that $c(vv_i) = c(vv_j) = c(vv_k)$. Note that two of the three vertices v_i, v_j, and v_k are not adjacent. Without loss of generality, assume that v_i and v_j are not adjacent. Observe that $T = \{vv_i, vv_j\}$ is the only possible rainbow Steiner $\{v, v_i, v_j\}$-tree, but $c(vv_i) = c(vv_j)$, a contradiction. Hence, at most two spokes of F_n may be colored the same. Furthermore, if $c(vv_i) = c(vv_j)$ for distinct i and j in $[1,n]$, then v_i and v_j are adjacent. □

The following theorem is an immediate consequence of Lemma 2.5.

Theorem 2.6 *For $n \geq 3$, let F_n be a fan of order $n+1$. Then:*

$$srx_3(F_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{for } n = 3 \text{ or } n \geq 5; \\ 3, & \text{for } n = 4. \end{cases}$$

Proof Let $V(F_n) = \{v\} \cup \{v_i | i \in [1,n]\}$ such that $E(F_n) = \{vv_i | i \in [1,n]\} \cup \{v_i v_{i+1} | i \in [1, n-1]\}$.

For $n \in [3,4]$, since $sdiam_3(F_3) = 2$ and $sdiam_3(F_4) = 3$, we have $srx_3(F_3) \geq 2$ and $srx_3(F_4) \geq 3$ by (1.1). Next, we show that $srx_3(F_3) \leq 2$ by defining a strong 3-rainbow coloring $c : E(F_3) \to [1,2]$, which can be obtained by assigning the color 1 to the edges vv_1, vv_2, and v_2v_3, and the color 2 to the edges vv_3 and v_1v_2. We show that $srx_3(F_4) \leq 3$ by defining a strong 3-rainbow coloring $c : E(F_4) \to [1,3]$, which can be obtained by assigning the color 1 to the edges vv_1, vv_2, and v_2v_3, the color 2 to the edges vv_3, vv_4, and v_1v_2, and the color 3 to the edge v_3v_4. By these two colorings, it is easy to find a rainbow Steiner S-tree for every set S of three vertices of F_n for $n \in [3,4]$.

For $n \geq 5$, it follows by Lemma 2.5 that $srx_3(F_n) \geq \lceil \frac{n}{2} \rceil$. Now we show that $srx_3(F_n) \leq \lceil \frac{n}{2} \rceil$ by defining a strong 3-rainbow coloring $c : E(F_n) \to [1, \lceil \frac{n}{2} \rceil]$ as follows:

$$c(vv_i) = \lceil \frac{i}{2} \rceil \text{ for } i \in [1,n];$$

$$\text{for odd } n, \quad c(v_i v_{i+1}) = \begin{cases} \frac{i+1}{2} + 1, & \text{for odd } i \in [1, n-1]; \\ \frac{i}{2}, & \text{for even } i \in [1, n-1]; \end{cases}$$

$$\text{for even } n, \quad c(v_iv_{i+1}) = \begin{cases} \frac{i+1}{2}+1, & \text{for odd } i \in [1, n-3]; \\ \frac{n}{2}-1, & \text{for } i = n-1; \\ \frac{i}{2}, & \text{for even } i \in [1, n-4]; \\ \frac{n}{2}-2, & \text{for } i = n-2. \end{cases}$$

Now we show that c is a strong 3-rainbow coloring of F_n. Let S be a set of three vertices of F_n. Let i, j, and k be three distinct integers in $[1, n]$. We consider two cases.

Case 1 $S = \{v_i, v_j, v_k\}$

If $v_iv_j, v_jv_k \in E(F_n)$, then $T = \{v_iv_j, v_jv_k\}$ is a rainbow Steiner S-tree. If i is odd, $j = i+1$ with $v_iv_j \in E(F_n)$, and $k = i-2$ or $k = j+2$, then there exists distinct $l \in [1, n]$ such that $v_kv_l, v_lv_i \in E(F_n)$ or $v_jv_l, v_lv_k \in E(F_n)$. Thus, the rainbow Steiner S-tree is a path of order 4, which contains vertices v_i, v_j, v_k, and v_l. If i is odd, $j = i+1$ with $v_iv_j \in E(F_n)$, and $k \leq i-3$ or $k \geq j+3$, then $T = \{vv_i, v_iv_{i+1}, vv_k\}$ is a rainbow Steiner S-tree. For other values of i, j, and k, $T = \{vv_i, vv_j, vv_k\}$ is a rainbow Steiner S-tree.

Case 2 $S = \{v, v_i, v_j\}$

If i is odd and $j = i+1$ with $v_iv_j \in E(F_n)$, then $T = \{vv_i, v_iv_{i+1}\}$ is a rainbow Steiner S-tree. For other values of i and j, $T = \{vv_i, vv_j\}$ is a rainbow Steiner S-tree. □

Now we consider graphs $Edge-Amal(F_n, e, t)$ where $e = vv_s$ is an arbitrary spoke of F_n. By symmetry, we only consider for $s \in \left[1, \lceil \frac{n}{2} \rceil\right]$. Let $V(Edge-Amal(F_n, e, t)) = \{u, v\} \cup \{v_i^p | i \in [1, t], p \in [1, n-1]\}$ such that $E(Edge-Amal(F_n, e, t)) = \{uv\} \cup \{vv_i^p | i \in [1, t], p \in [1, n-1]\} \cup \{uv_i^s | i \in [1, t]\} \cup \{v_i^pv_i^{p+1} | i \in [1, t], p \in [s, n-2]\} \cup E^1$ where

$$E^1 = \begin{cases} \emptyset, & \text{if } s = 1; \\ \{uv_i^1 | i \in [1, t]\}, & \text{if } s = 2; \\ \{uv_i^{s-1} | i \in [1, t]\} \cup \{v_i^pv_i^{p+1} | i \in [1, t], p \in [1, s-2]\}, & \text{otherwise.} \end{cases}$$

The following observation is also an immediate consequence of Lemma 2.5.

Observation 2.7 *Let t, n, and s be three integers with $t \geq 2$, $n \geq 3$, and $s \in \left[1, \lceil \frac{n}{2} \rceil\right]$. For $i \in [1, t]$, let A_i be a set of spokes vv_i^p for $p \in [1, s-1]$ and B_i be a set of spokes vv_i^p for $p \in [s, n-1]$. If c is a strong 3-rainbow coloring of $Edge-Amal(F_n, e, t)$, then $c(A_i) \cap c(A_j) = \emptyset$ and $c(B_i) \cap c(B_j) = \emptyset$ for distinct i and j in $[1, t]$, and $c(A_i) \cap c(B_j) = \emptyset$ for all i and j in $[1, t]$.*

Theorem 2.8 *Let t, n, and s be three integers with $t \geq 2$, $n \geq 3$, and $s \in \left[1, \lceil \frac{n}{2} \rceil\right]$. Let F_n be a fan of order $n+1$ and $e = vv_s$ be an arbitrary spoke of F_n. For odd n and even s, or even $n \geq 6$, $srx_3(Edge-Amal(F_n, e, t)) = t\left(\lceil \frac{n}{2} \rceil\right)$.*

Proof Since $t\left(srx_3(F_n)\right) = t\left(\lceil \frac{n}{2} \rceil\right)$ by Theorem 2.6 and $t\left(|E(F_n)| - 1\right) + 1 = t\left(2n-2\right) + 1$, it follows by Theorem 2.1 that $srx_3(Edge-Amal(F_n, e, t)) \leq t\left(\lceil \frac{n}{2} \rceil\right)$. Thus, we only need to prove the lower bound. Let c be a strong 3-rainbow coloring of $Edge-Amal(F_n, e, t)$. For $i \in [1, t]$, let A_i be a set of spokes vv_i^p for $p \in [1, s-1]$ and B_i be a set of spokes vv_i^p for $p \in [s, n-1]$. Hence, $|A_i| = s-1$ and $|B_i| = n-s$.

For odd n and even s, we have that for each $i \in [1, t]$, both $|A_i|$ and $|B_i|$ are odd. Hence, by using Lemma 2.5, $|c(A_i)| \geq \lceil \frac{s-1}{2} \rceil = \frac{s}{2}$ and $|c(B_i)| \geq \lceil \frac{n-s}{2} \rceil = \frac{n-s+1}{2}$. It follows by Observation 2.7 that $srx_3(Edge-Amal(F_n, e, t)) \geq t\left(\lceil \frac{n}{2} \rceil\right)$.

For even $n \geq 6$, if s is odd, then for each $i \in [1,t]$, $|A_i|$ is even and $|B_i|$ is odd. It follows by Lemma 2.5 that $|c(A_i)| \geq \frac{s-1}{2}$ and $|c(B_i)| \geq \lceil \frac{n-s}{2} \rceil = \frac{n-s+1}{2}$. Hence, by using Observation 2.7, we have $srx_3(Edge - Amal(F_n,e,t)) \geq t\left(\frac{n}{2}\right)$. Similarly, we have $srx_3(Edge - Amal(F_n,e,t)) \geq t\left(\frac{n}{2}\right)$ if s is even. □

2.2. The strong 3-rainbow index of $Edge - Amal(G,e,t)$ for some connected graphs G

In this subsection, we determine the strong 3-rainbow index of $Edge - Amal(G,e,t)$ for some connected graphs G. In particular, we consider G as a cycle, a fan, a ladder, and a regular complete bipartite graph. First, we consider graphs $Edge - Amal(C_n,e,t)$ where e is an arbitrary edge of C_n. In Theorem 2.4, we determine $srx_3(Edge - Amal(C_n,e,t))$ for odd $n \geq 9$. In the next theorem, we determine $srx_3(Edge - Amal(C_n,e,t))$ for other values of n. First, we verify the following observation.

Observation 2.9 *Let t be an integer at least 2 and n be an even integer at least 4. For $i \in [1,t]$, let A_i be a set of edges of path $uv_i^1v_i^2...v_i^{\frac{n}{2}-2}v_i^{\frac{n}{2}-1}$ and B_i be a set of edges of path $vv_i^{n-2}v_i^{n-3}...v_i^{\frac{n}{2}+1}v_i^{\frac{n}{2}}$. If c is a strong 3-rainbow coloring of $Edge - Amal(C_n,e,t)$, then:*

1. *$c(A_i) \cap c(A_j) = \emptyset$ and $c(B_i) \cap c(B_j) = \emptyset$ for distinct i and j in $[1,t]$;*
2. *for $n \geq 10$, $c(A_i) \cap c(B_j) = \emptyset$ for distinct i and j in $[1,t]$.*

Proof Let i and j be two distinct integers in $[1,t]$.

1. Since path $v_i^{\frac{n}{2}-1}v_i^{\frac{n}{2}-2}...v_i^1uv_j^1...v_j^{\frac{n}{2}-2}v_j^{\frac{n}{2}-1}$ is the only possible rainbow Steiner $\{u, v_i^{\frac{n}{2}-1}, v_j^{\frac{n}{2}-1}\}$-tree, we have $c(A_i) \cap c(A_j) = \emptyset$. Similarly, by considering $\{v, v_i^{\frac{n}{2}}, v_j^{\frac{n}{2}}\}$, we have $c(B_i) \cap c(B_j) = \emptyset$.
2. By considering $\{u, v_i^{\frac{n}{2}-1}, v_j^{\frac{n}{2}+1}\}$, we obtain that no edge of path $v_i^{\frac{n}{2}-1}v_i^{\frac{n}{2}-2}...v_i^1uvv_j^{n-2}...v_j^{\frac{n}{2}+1}$ is colored the same. Also, by considering $\{v_i^{\frac{n}{2}-1}, v_j^{\frac{n}{2}-2}, v_j^{\frac{n}{2}+1}\}$, no edge of path $v_i^{\frac{n}{2}-1}v_i^{\frac{n}{2}-2}...v_i^1uv_j^1..v_j^{\frac{n}{2}-2}v_j^{\frac{n}{2}-1}v_j^{\frac{n}{2}}v_j^{\frac{n}{2}+1}$ is colored the same. Thus, we have $c(A_i) \cap c(B_j) = \emptyset$.

□

Theorem 2.10 *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let C_n be a cycle of order n and e be an arbitrary edge of C_n. Then:*

$$srx_3(Edge - Amal(C_n,e,t)) = \begin{cases} t\,(srx_3(C_n) - 1), & \text{for } n = 3, \text{ or } n = 5 \text{ and } t \geq 3; \\ t\,(srx_3(C_n) - 1) + 1, & \text{for } n = 4, \text{ or } n = 5 \text{ and } t = 2; \\ t\,(srx_3(C_n) - 2) + 2, & \text{for even } n \geq 6, \text{ or } n = 7 \text{ and } t = 2; \\ 5t + 1, & \text{for } n = 7 \text{ and } t \geq 3. \end{cases}$$

Proof For each $i \in [1,t]$, let C_n^i denote the ith cycle C_n in $Edge - Amal(C_n,e,t)$. For simplyfing the proof, we define path $v_pv_qv_qv_r = v_pv_qv_r$.

Case 1 $n = 3$

Note that $srx_3(C_3) = 2$ by Theorem 1.4. Let c be a strong 3-rainbow coloring of $Edge - Amal(C_3,e,t)$. Then $srx_3(Edge - Amal(C_3,e,t)) \geq t$ by Observation 2.3. Now we show that $srx_3(Edge - Amal(C_3,e,t)) \leq t$ by defining a strong 3-rainbow coloring $c : E(Edge - Amal(C_3,e,t)) \to [1,t]$. This coloring can be obtained by assigning the colors i to the edges uv_i^1 for all $i \in [1,t]$, the colors $i+1$ to the edges vv_i^1 for all $i \in [1,t-1]$, and the color 1 to the edges uv and vv_t^1. Now we show that c is a strong 3-rainbow coloring of $Edge-Amal(C_3,e,t)$. Let S be a set of three vertices of $Edge-Amal(C_3,e,t)$. We can find a rainbow Steiner S-tree as shown in Table 1.

Table 1. A rainbow Steiner S-tree of $Edge-Amal(C_3,e,t)$.

A set of three vertices S	Condition	A rainbow Steiner S-tree
$\{u,v,v_i^1\}$	$i \in [1,t-1]$	$\{uv,vv_i^1\}$
$\{u,v,v_t^1\}$		$\{uv,uv_t^1\}$
$\{u,v_i^1,v_j^1\}$	$i \neq j$	$\{uv_i^1,uv_j^1\}$
$\{v,v_i^1,v_j^1\}$	$i \neq j$	$\{vv_i^1,vv_j^1\}$
$\{v_i^1,v_j^1,v_k^1\}$	$i<j<k$	$\{uv_i^1,uv_j^1,uv_k^1\}$

Case 2 $n=4$

By using Theorem 1.4, $srx_3(C_4)=2$. Let c be a strong 3-rainbow coloring of $Edge-Amal(C_4,e,t)$. Since $c(uv) \neq c(uv_i^1)$ for all $i \in [1,t]$, it follows by Observation 2.9 that $srx_3(Edge-Amal(C_4,e,t)) \geq t+1$.

Next, we show that $srx_3(Edge-Amal(C_4,e,t)) \leq t+1$. We define an edge-coloring $c: E(Edge-Amal(C_4,e,t)) \rightarrow [1,t+1]$, which can be obtained by assigning the color 1 to edges uv and $v_i^1v_i^2$ for all $i \in [1,t]$ and the colors $i+1$ to edges uv_i^1 and vv_i^2 for all $i \in [1,t]$. Now we show that c is a strong 3-rainbow coloring of $Edge-Amal(C_4,e,t)$. Let S be a set of three vertices of $Edge-Amal(C_4,e,t)$. Observe that the coloring above assigns two colors to C_4^i and has the same pattern as the strong 3-rainbow coloring of C_4 as shown in Figure 1. It follows by Theorem 1.4 that we can find a rainbow Steiner S-tree if the vertices of S are contained on the same cycle C_4^i for some $i \in [1,t]$. Hence, we may assume that vertices of S are not contained on the same cycle C_4^i. Let i, j, and k be three distinct integers in $[1,t]$. By symmetry, we consider six subcases as shown in Table 2.

Table 2. A rainbow Steiner S-tree of $Edge-Amal(C_4,e,t)$.

A set of three vertices S	A rainbow Steiner S-tree
$\{u,v_i^1,v_j^1\}$	$\{uv_i^1,uv_j^1\}$
$\{v,v_i^1,v_j^1\}$	$\{uv,uv_i^1,uv_j^1\}$
$\{u,v_i^1,v_j^2\}$	$\{uv_i^1,uv,vv_j^2\}$
$\{v_i^1,v_i^2,v_j^1\}$	$\{uv_i^1,v_i^1v_i^2,uv_j^1\}$
$\{v_i^1,v_j^1,v_k^1\}$	$\{uv_i^1,uv_j^1,uv_k^1\}$
$\{v_i^1,v_j^1,v_k^2\}$	$\{uv_i^1,uv_j^1,uv,vv_k^2\}$

Case 3 $n=5$

Note that $srx_3(C_5)=3$ by Theorem 1.4. For $t=2$, since $sdiam_3(Edge-Amal(C_5,e,2))=5$, we have $srx_3(Edge-Amal(C_5,e,2)) \geq 5$ by (1.1). Next, we show that $srx_3(Edge-Amal(C_5,e,2)) \leq 5$ by defining a strong 3-rainbow coloring of $Edge-Amal(C_5,e,2)$ as shown in Figure 2.

For $t \geq 3$, let c be a strong 3-rainbow coloring of $Edge-Amal(C_5,e,t)$. It follows by Observation 2.3 that $srx_3(Edge-Amal(C_5,e,t)) \geq 2t$. Next, we show that $srx_3(Edge-Amal(C_5,e,t)) \leq 2t$ by defining a

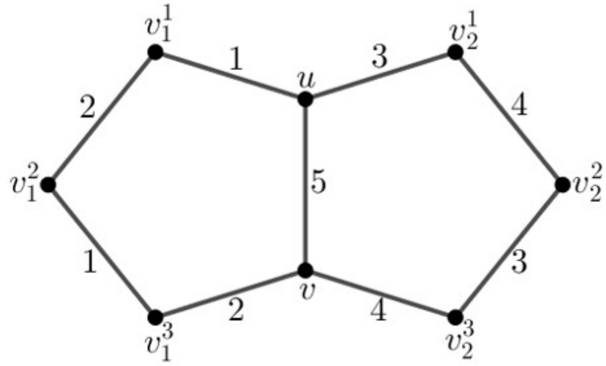

Figure 2. A strong 3-rainbow coloring of $Edge-Amal(C_5,e,2)$.

strong 3-rainbow coloring $c: E(Edge-Amal(C_5,e,t)) \to [1,2t]$ as follows:

$$\begin{aligned} c(uv) &= 1; \\ c(uv_i^1) &= c(vv_i^3) = 2+2(i-1) \text{ for } i \in [1,t]; \\ c(v_i^1v_i^2) &= 1+2(i-1) \text{ for } i \in [1,t]; \\ c(v_i^2v_i^3) &= \begin{cases} 1+2i, & \text{for } i \in [1,t-1]; \\ 1, & \text{for } i = t. \end{cases} \end{aligned}$$

Now we show that c is a strong 3-rainbow coloring of $Edge-Amal(C_5,e,t)$. Let S be a set of three vertices of $Edge-Amal(C_5,e,t)$. We consider three subcases.

– The vertices of S belong to the same cycle C_5^i for some $i \in [1,t]$
  Since the coloring above assigns three colors to C_5^i and has the same pattern as the strong 3-rainbow coloring of C_5 as shown in Figure 1, it follows by Theorem 1.4 that we can find a rainbow Steiner S-tree.

– Two vertices of S belong to the same cycle C_5^i for some $i \in [1,t]$
  Let $j \in [1,t]$ with $j \neq i$. First, consider $S=\{u,v_i^p,v_j^q\}$. If $p,q \in [1,2]$, then $P=v_i^pv_i^1uv_j^1v_j^q$ is a rainbow Steiner S-tree. If $p=3$ and $q \in [1,2]$, then $P=v_i^3vuv_j^1v_j^q$ is a rainbow Steiner S-tree. If $p=q=3$, then $T=\{uv,vv_i^3,vv_j^3\}$ is a rainbow Steiner S-tree. A similar argument applies for $S=\{v,v_i^p,v_j^q\}$. Next, consider $S=\{v_i^p,v_i^q,v_j^r\}$. We can find a rainbow Steiner S-tree as shown in Table 3.

– Each vertex of S belongs to distinct cycles C_5^i, C_5^j, and C_5^k for some $i,j,k \in [1,t]$
  Let $S=\{v_i^p,v_j^q,v_k^r\}$. We can find a rainbow Steiner S-tree as shown in Table 4.

Case 4 even $n \geq 6$

Subcase 4.1 $n=6$

Note that $srx_3(C_6)=4$ by Theorem 1.4. First, we prove the lower bound. Assume to the contrary that $srx_3(Edge-Amal(C_6,e,t)) \leq 2t+1$. Then there exists a strong 3-rainbow coloring $c: E(Edge-Amal(C_6,e,t)) \to [1,2t+1]$. Let i and j be two distinct integers in $[1,t]$. By using Observation 2.9, we need at least $2t$ distinct colors assigned to the edges uv_i^1 and $v_i^1v_i^2$ for all $i \in [1,t]$. This implies we have at most one color

Table 3. A rainbow Steiner $\{v_i^p, v_i^q, v_j^r\}$-tree of $Edge-Amal(C_5,e,t)$.

p	q	r	Condition	A rainbow Steiner $\{v_i^p, v_i^q, v_j^r\}$-tree
1	2	1, 2		$v_i^2 v_i^1 u v_j^1 v_j^r$
1	2	3		$\{v_i^1 v_i^2, v_i^2 v_i^3, vv_i^3, vv_j^3\}$
1	3	1		$\{uv_i^1, v_i^1 v_i^2, v_i^2 v_i^3, uv_j^1\}$
1	3	2	$i \in [1,t-1]$ and $j = i+1$ $i = t$ and $j = 1$	$\{v_i^1 v_i^2, v_i^2 v_i^3, vv_i^3, vv_j^3, v_j^2 v_j^3\}$
			others i and j	$\{uv_i^1, v_i^1 v_i^2, v_i^2 v_i^3, uv_j^1, v_j^1 v_j^2\}$
1	3	3		$\{v_i^1 v_i^2, v_i^2 v_i^3, vv_i^3, vv_j^3\}$
2	3	1, 2, 3	The proof is similar to the case $p=1$, $q=2$, $r \in [1,3]$	

Table 4. A rainbow Steiner $\{v_i^p, v_j^q, v_k^r\}$-tree of $Edge-Amal(C_5,e,t)$.

p	q	r	Condition	A rainbow Steiner $\{v_i^p, v_j^q, v_k^r\}$-tree
1, 2	1, 2	1, 2		$uv_i^1 v_i^p \cup uv_j^1 v_j^q \cup uv_k^1 v_k^r$
1	1	3		$\{uv, uv_i^1, uv_j^1, vv_k^3\}$
2	2	3		$\{vv_i^3, v_i^2 v_i^3, vv_j^3, v_j^2 v_j^3, vv_k^3\}$
3	3	1		$\{uv, vv_i^3, vv_j^3, uv_k^1\}$
3	3	2		$\{vv_i^3, vv_j^3, vv_k^3, v_k^2 v_k^3\}$
3	3	3		$\{vv_i^3, vv_j^3, vv_k^3\}$
1	2	3	$j = 1$	$\{uv, uv_i^1, vv_j^3, v_j^2 v_j^3, vv_k^3\}$
			$j \neq 1$	$\{uv, uv_i^1, uv_j^1, v_j^1 v_j^2, v_k^3\}$

left, say color a. Note that the only possible rainbow Steiner $\{v_i^1, v_i^4, v_j^2\}$-tree is $T = \{uv, uv_i^1, vv_i^4, uv_j^1, v_j^1 v_j^2\}$ where $\{c(uv), c(vv_i^4)\} \subseteq \{c(v_i^1 v_i^2), a\}$. Since $c(uv) \neq c(v_i^1 v_i^2)$, this forces $c(uv) = a$ and $c(vv_i^4) = c(v_i^1 v_i^2)$. Next, by considering $\{v_i^2, v_i^3, v_j^2\}$ and $\{v_i^3, v_i^4, v_j^p\}$ for $p \in \{1,4\}$, we have $c(v_i^2 v_i^3) = a$ and $c(v_i^3 v_i^4) = c(uv_i^1)$. Hence, $srx_3(C_6^i) \leq 3$, contradicting Theorem 1.4.

Next, we show that $srx_3(Edge-Amal(C_6,e,t)) \leq 2t+2$ by defining a strong 3-rainbow coloring $c : E(Edge-Amal(C_6,e,t)) \to [1, 2t+2]$. This coloring can be obtained by assigning the color 1 to the edge uv, the color 2 to the edges $v_i^2 v_i^3$ for all $i \in [1,t]$, and the colors $3, 4, ..., 2t+2$ to the remaining $4t$ edges where $c(uv_i^1) = c(v_i^3 v_i^4)$ and $c(v_i^1 v_i^2) = c(vv_i^4)$ for all $i \in [1,t]$. Now we show that c is a strong 3-rainbow coloring of $Edge-Amal(C_6,e,t)$. Let S be a set of three vertices of $Edge-Amal(C_6,e,t)$. Since the coloring above assigns four distinct colors to C_6^i and has the same pattern as the strong 3-rainbow coloring of C_6 as shown in Figure 1, if the vertices of S belong to the same cycle C_6^i for some $i \in [1,t]$, then by using Theorem 1.4, there exists a rainbow Steiner S-tree by coloring c. Therefore, we consider the following subcases.

- Two vertices of S belong to the same cycle C_6^i for some $i \in [1,t]$

  Let $j \in [1,t]$ with $j \neq i$. First, consider $S = \{u, v_i^p, v_j^q\}$. If $p, q \in [1,2]$, then $P = v_i^p v_i^1 u v_j^1 v_j^q$ is a rainbow Steiner S-tree. If $p \in [1,2]$ and $q \in [3,4]$, then $P = v_i^p v_i^1 uvv_j^4 v_j^q$ is a rainbow Steiner S-tree. If $p, q \in [3,4]$, then $T = v_i^p v_i^4 vv_j^4 v_j^q \cup \{uv\}$ is a rainbow Steiner S-tree. A similar argument applies for

$S = \{v, v_i^p, v_j^q\}$. Next, consider $S = \{v_i^p, v_i^q, v_j^r\}$. We can find a rainbow Steiner S-tree as shown in Table 5.

Table 5. A rainbow Steiner $\{v_i^p, v_i^q, v_j^r\}$-tree of $Edge - Amal(C_6, e, t)$.

p	q	r	Condition	A rainbow Steiner $\{v_i^p, v_i^q, v_j^r\}$-tree
$1,2,3$	$1,2,3$	$1,2$	$p < q$	$v_i^q v_i^{q-1} ... v_i^p ... uv_j^1 v_j^r$
1	4	$1,2$		$v_i^1 uvv_i^4 \cup uv_j^1 v_j^r$
$2,3$	4	$1,2$		$v_i^p v_i^{p+1} v_i^4 vuv_j^1 v_j^r$
1	$2,3$	$3,4$		$v_i^q v_i^{q-1} v_i^1 uvv_j^4 v_j^r$
1	4	$3,4$		$v_i^1 uvv_i^4 \cup vv_j^4 v_j^r$
$2,3,4$	$2,3,4$	$3,4$	$p < q$	$v_i^p v_i^{p+1} ... v_i^q ... vv_j^4 v_j^r$

– Each vertex of S belongs to distinct cycles C_6^i, C_6^j, and C_6^k for some $i, j, k \in [1, t]$
Let $S = \{v_i^p, v_j^q, v_k^r\}$. By symmetry, we consider two cases. If $p, q, r \in [1, 2]$, then $T = uv_i^1 v_i^p \cup uv_j^1 v_j^q \cup uv_k^1 v_k^r$ is a rainbow Steiner S-tree. If $p, q \in [1, 2]$ and $r \in [3, 4]$, then $T = uv_i^1 v_i^p \cup uv_j^1 v_j^q \cup uvv_k^4 v_k^r$ is a rainbow Steiner S-tree.

Subcase 4.2 $n = 8$

Note that $srx_3(C_8) = 6$ by Theorem 1.4. For the lower bound, assume to the contrary that $srx_3(Edge - Amal(C_8, e, t)) \leq 4t+1$. Then there exists a strong 3-rainbow coloring $c : E(Edge - Amal(C_8, e, t)) \to [1, 4t+1]$. By using Observation 2.9, without loss of generality, let $c(uv_i^1) = 3i-2$, $c(v_i^1 v_i^2) = 3i-1$, and $c(v_i^2 v_i^3) = 3i$ for all $i \in [1, t]$. Next, by considering $\{u, v, v_i^3\}$ for all $i \in [1, t]$, we have $c(uv) \notin [1, 3t]$. Hence, write $c(uv) = 3t + 1$. This implies we have t colors left. Let $A = [3t + 2, 4t + 1]$ be the set of these t colors. For an arbitrary $i \in [1, t]$, consider $\{u, v_i^5, v_j^3\}$ for all $j \in [1, t]$ with $j \neq i$. Since path $v_i^5 v_i^6 vuv_j^1 v_j^2 v_j^3$ is the only possible rainbow Steiner tree connecting these three vertices, $c(v_i^5 v_i^6) \notin \{c(uv_i^1), c(v_i^2 v_i^3)\}$, and $c(vv_i^6) \notin \{c(uv_i^1), c(v_i^1 v_i^2)\}$, we have $c(v_i^5 v_i^6) \in \{c(v_i^1 v_i^2)\} \cup A$ and $c(vv_i^6) \in \{c(v_i^2 v_i^3)\} \cup A$, with condition $c(v_i^5 v_i^6) = c(v_i^1 v_i^2)$ if and only if $c(vv_i^6) \neq c(v_i^2 v_i^3)$. It follows by Observation 2.9 that we have used all available colors. For the next steps, let i and j be two distinct integers in $[1, t]$. First, consider $\{v_i^3, v_i^4, v_j^p\}$ for $p \in [3, 4]$. We obtain that $c(v_i^3 v_i^4) \notin [1, 3t] \cup A$, which means $c(v_i^3 v_i^4) = c(uv)$ for all $i \in [1, t]$. This implies for each $i \in [1, t]$, $c(vv_i^6) \neq c(v_i^2 v_i^3)$, since if $c(vv_i^6) = c(v_i^2 v_i^3)$ for some $i \in [1, t]$, then there is no rainbow Steiner $\{v, v_i^2, v_i^4\}$-tree. Hence, we have $c(v_i^5 v_i^6) = c(v_i^1 v_i^2)$ and $c(vv_i^6) = 3t + 1 + i$ for all $i \in [1, t]$. Next, consider $\{v_i^2, v_i^5, v_j^3\}$. Note that the rainbow Steiner tree connecting these three vertices should be the path $v_i^5 v_i^4 v_i^3 v_i^2 v_i^1 uv_j^1 v_j^2 v_j^3$, which implies $c(v_i^4 v_i^5) \notin [1, 3t]$. Also, by considering $\{v_i^2, v_i^5, v_j^4\}$, we have $c(v_i^4 v_i^5) \notin A$. Hence, $c(v_i^4 v_i^5) = c(uv)$, but there is no rainbow Steiner $\{v_i^3, v_i^4, v_i^5\}$-tree, a contradiction. Thus, $srx_3(Edge - Amal(C_8, e, t)) \geq 4t + 2$.

Next, we show that $srx_3(Edge - Amal(C_8, e, t)) \leq 4t + 2$. We define an edge-coloring $c : E(Edge - Amal(C_8, e, t)) \to [1, 4t + 2]$, which can be obtained by assigning the color 1 to the edge uv, the color 2 to the edges $v_i^3 v_i^4$ for all $i \in [1, t]$, and the colors $3, 4, ..., 4t + 2$ to the remaining $6t$ edges where $c(uv_i^1) = c(v_i^4 v_i^5)$ and $c(v_i^2 v_i^3) = c(vv_i^6)$ for all $i \in [1, t]$. By using an argument similar to that used in the proof for $n = 6$, we can find a rainbow Steiner S-tree for every set S of three vertices of $Edge - Amal(C_8, e, t)$.

Subcase 4.3 $n \geq 10$

By using Theorem 1.4, assume to the contrary that $srx_3(Edge-Amal(C_n,e,t)) \leq t(n-2)+1$. Then there exists a strong 3-rainbow coloring $c: E(Edge-Amal(C_n,e,t)) \to [1, t(n-2)+1]$. For each $i \in [1,t]$, let A_i be a set of edges of path $uv_i^1v_i^2...v_i^{\frac{n}{2}-1}$ and B_i be a set of edges of path $vv_i^{n-2}v_i^{n-3}...v_i^{\frac{n}{2}}$. It follows by Theorem 1.4 and Observation 2.9 that $\sum_{i=1}^{t}|c(A_i)|+|c(B_i)| \geq t(n-2)$. This implies we have at most one color left. Note that by using Theorem 1.4 and by considering $\{v_1^{\frac{n}{2}-1}, v_1^{\frac{n}{2}}, v_i^p\}$ for all $i \in [2,t]$ and $p \in [\frac{n}{2}-1, \frac{n}{2}]$, we have $c(uv) \neq c(v_1^{\frac{n}{2}-1}v_1^{\frac{n}{2}})$ and $\{c(uv), c(v_1^{\frac{n}{2}-1}v_1^{\frac{n}{2}})\} \nsubseteq c(A_i) \cup c(B_i)$ for all $i \in [1,t]$. It means we need two new distinct colors assigned to the edges uv and $v_1^{\frac{n}{2}-1}v_1^{\frac{n}{2}}$, which is impossible. Thus, $srx_3(Edge-Amal(C_n,e,t)) \geq t(n-2)+2$.

Next, we prove the upper bound. We define a strong 3-rainbow coloring $c: E(Edge-Amal(C_n,e,t)) \to [1, t(n-2)+2]$, which can be obtained by assigning the color 1 to the edge uv, the color 2 to the edges $v_i^{\frac{n}{2}-1}v_i^{\frac{n}{2}}$ for all $i \in [1,t]$, and the colors $3, 4, ..., t(n-2)+2$ to the remaining $t(n-2)$ edges of $Edge-Amal(C_n,e,t)$. Now we show that c is a strong 3-rainbow coloring of $Edge-Amal(C_n,e,t)$. Let S be a set of three vertices of $Edge-Amal(C_n,e,t)$. If the vertices of S belong to the same cycle C_n^i for some $i \in [1,t]$, then there exists a rainbow Steiner S-tree since the coloring above assigns distinct colors to C_n^i. Hence, we assume that the vertices of S are not contained on the same cycle C_n^i. By this coloring, we know that each edge of $Edge-Amal(C_n,e,t)$ is colored with distinct colors, except edges $v_i^{\frac{n}{2}-1}v_i^{\frac{n}{2}}$, i.e. $c(v_i^{\frac{n}{2}-1}v_i^{\frac{n}{2}}) = c(v_j^{\frac{n}{2}-1}v_j^{\frac{n}{2}})$ for distinct i and j in $[1,t]$. Hence, it is not difficult to find a rainbow Steiner S-tree in $Edge-Amal(C_n,e,t)$.

Case 5 $n=7$

By using Theorem 1.4, we have $srx_3(C_7)=7$. For $t \geq 3$, let c be a strong 3-rainbow coloring of $Edge-Amal(C_7,e,t)$. By using Theorem 1.4 and Observation 2.3, and by considering $\{v_i^1, v_i^3, v_j^4\}$ for distinct i and j in $[1,t]$, we need at least $5t+1$ distinct colors assigned to all edges of $Edge-Amal(C_7,e,t)$ except edges $v_i^3v_i^4$ for all $i \in [1,t]$. Hence, $srx_3(Edge-Amal(C_7,e,t)) \geq 5t+1$. For $t=2$, assume to the contrary that $srx_3(Edge-Amal(C_7,e,2)) \leq 11$. Similarly, we need at least 11 distinct colors assigned to all edges of $Edge-Amal(C_7,e,2)$ except edges $v_1^3v_1^4$ and $v_2^3v_2^4$. It is easy to check that we need one new distinct color assigned to these two edges, which is impossible. Thus, $srx_3(Edge-Amal(C_7,e,2)) \geq 12$.

Next, we prove the upper bound. We show that $srx_3(Edge-Amal(C_7,e,2)) \leq 12$ by defining a strong 3-rainbow coloring $c: E(Edge-Amal(C_7,e,2)) \to [1,12]$. This coloring can be obtained by assigning the color 1 to the edge uv, the color 2 to the edges $v_1^3v_1^4$ and $v_2^2v_2^3$, and the colors $3, 4, ..., 12$ to the remaining 10 edges. For $t \geq 3$, we show that $srx_3(Edge-Amal(C_7,e,t)) \leq 5t+1$ by defining a strong 3-rainbow coloring $c: E(Edge-Amal(C_7,e,t)) \to [1, 5t+1]$, which can be obtained by assigning the color 1 to the edge uv and the colors $2, 3, ..., 5t+1$ to the remaining $6t$ edges where $c(v_i^3v_i^4) = c(v_{i+1}^2v_{i+1}^3)$ for $i \in [1, t-1]$ and $c(v_t^3v_t^4) = c(v_1^2v_1^3)$. Now we show that c is a strong 3-rainbow coloring of $Edge-Amal(C_7,e,t)$. Let S be a set of three vertices of $Edge-Amal(C_7,e,t)$. If the vertices of S belong to the same cycle C_7^i for some $i \in [1,t]$, then there exists a rainbow Steiner S-tree since the coloring above assigns distinct colors to C_7^i. Hence, we assume that the vertices of S are not contained on the same cycle C_7^i. By the coloring above, we know that each edge of $Edge-Amal(C_7,e,t)$ has distinct colors, except edges $v_i^2v_i^3$ and $v_i^3v_i^4$, i.e. $c(v_i^3v_i^4) = c(v_{i+1}^2v_{i+1}^3)$ for $i \in [1, t-1]$ and $c(v_t^3v_t^4) = c(v_1^2v_1^3)$. Hence, it is not difficult to find a rainbow Steiner S-tree in $Edge-Amal(C_7,e,t)$. □

Next, we consider graphs $Edge-Amal(F_n, e, t)$ where $e = vv_s$ is an arbitrary spoke of F_n. In Theorem 2.8, we provide the exact values of $srx_3(Edge-Amal(F_n, e, t))$ for certain values of n and s. The next theorem provides $srx_3(Edge-Amal(F_n, e, t))$ for other values of n and s.

Theorem 2.11 *Let t, n, and s be three integers with $t \geq 2$, $n \geq 3$, and $s \in \left[1, \lceil \frac{n}{2} \rceil\right]$. Let F_n be a fan of order $n+1$ and $e = vv_s$ be an arbitrary spoke of F_n. Then:*

$$srx_3(Edge-Amal(F_n, e, t)) = \begin{cases} t\left(\frac{n-1}{2}\right) + 1, & \text{for odd } n \text{ and odd } s; \\ 2t, & \text{for } n = 4. \end{cases}$$

Proof We consider two cases.

Case 1 n and s are odd

Assume to the contrary that $srx_3(Edge-Amal(F_n, e, t)) \leq t\left(\frac{n-1}{2}\right)$. Then there exists a strong 3-rainbow coloring $c : E(Edge-Amal(F_n, e, t)) \to \left[1, t\left(\frac{n-1}{2}\right)\right]$. For $i \in [1, t]$, let A_i be a set of spokes vv_i^p for $p \in [1, s-1]$ and B_i be a set of spokes vv_i^p for $p \in [s, n-1]$. Thus, both $|A_i|$ and $|B_i|$ are even with $|A_i| = s-1$ and $|B_i| = n-s$. It follows by Observation 2.7 that we need at least $t\left(\frac{n-1}{2}\right)$ distinct colors assigned to all spokes of A_i and B_i for all $i \in [1, t]$, which means we have used all available colors. Next, consider spoke uv. Since $|A_i|$ and $|B_i|$ are even, by using Lemma 2.5, $c(uv) \neq c(vv_i^p)$ for all $i \in [1, t]$ and $p \in [1, n-1]$. It means we need one new distinct color assigned to the spoke uv, which is impossible. Thus, $srx_3(Edge-Amal(F_n, e, t)) \geq t\left(\frac{n-1}{2}\right) + 1$.

Next, we show that $srx_3(Edge-Amal(F_n, e, t)) \leq t\left(\frac{n-1}{2}\right) + 1$. Let $i \in [1, t]$. We define an edge-coloring $c : E(Edge-Amal(F_n, e, t)) \to \left[1, t\left(\frac{n-1}{2}\right) + 1\right]$ as follows:

$$\begin{aligned} c(uv) &= 1; \\ c(vv_i^p) &= \left\lceil \frac{p}{2} \right\rceil + 1 + (i-1)\left(\frac{n-1}{2}\right) \text{ for } p \in [1, n-1]; \\ c(uv_i^p) &= c(vv_i^p) \text{ for } p \in [s-1, s]; \\ c(v_i^{s-2}v_i^{s-1}) &= c(v_i^s v_i^{s+1}) = 1; \\ c(v_i^p v_i^{p+1}) &= \begin{cases} \frac{p+1}{2} + 2 + (i-1)(\frac{n-1}{2}), & \text{for odd } p \in [1, s-3]; \\ \frac{p+1}{2} + (i-1)(\frac{n-1}{2}), & \text{for odd } p \in [s+1, n-2]; \\ \lceil \frac{p+1}{2} \rceil + (i-1)(\frac{n-1}{2}), & \text{for even } p \in [1, s-3]; \\ \lceil \frac{p+1}{2} \rceil + 1 + (i-1)(\frac{n-1}{2}), & \text{for even } p \in [s+1, n-2]. \end{cases} \end{aligned}$$

By the coloring above, it is not difficult to find a rainbow Steiner S-tree for every set S of three vertices of $Edge-Amal(F_n, e, t)$.

Case 2 $n = 4$

By using an argument similar to that used in the proof of the lower bound for even $n \geq 6$, we have $srx_3(Edge-Amal(F_4, e, t)) \geq 2t$. Now we show that $srx_3(Edge-Amal(F_4, e, t)) \leq 2t$ by defining a strong 3-rainbow coloring $c : E(Edge-Amal(F_4, e, t)) \to [1, 2t]$ as follows:

$$\begin{aligned} c(uv) &= 1; \\ c(vv_i^1) &= c(v_i^2 v_i^3) = 1 + 2(i-1) \text{ for } i \in [1, t]; \\ c(vv_i^2) &= c(vv_i^3) = c(uv_i^1) = 2 + 2(i-1) \text{ for } i \in [1, t]; \end{aligned}$$

$$\text{for } s=1, \quad c(v_i^1 v_i^2) = \begin{cases} 1+2i, & \text{for } i \in [1, t-1]; \\ 1, & \text{for } i = t; \end{cases}$$

$$\text{for } s=2, \quad c(uv_i^2) = \begin{cases} 1+2i, & \text{for } i \in [1, t-1]; \\ 1, & \text{for } i = t. \end{cases}$$

By the coloring above, it is easy to find a rainbow Steiner S-tree for every set S of three vertices of $Edge-Amal(F_4, e, t)$. □

A ladder graph L_n is a *Cartesian product* of a P_n and a P_2, where P_n is a path of order n. Let $V(L_n) = \{v_i | i \in [1, 2n]\}$ such that $E(L_n) = \{v_i v_{i+1} | i \in [1, n-1] \cup [n+1, 2n-1]\} \cup \{v_i v_{i+n} | i \in [1, n]\}$. In the following theorem, we determine the strong 3-rainbow index of $Edge - Amal(L_n, e, t)$ where e is an arbitrary edge of L_n.

Theorem 2.12 *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let L_n be a ladder of order $2n$ and e be an arbitrary edge of L_n. Then $srx_3(Edge - Amal(L_n, e, t)) = t(n-1) + 1$.*

Proof Without loss of generality, we consider two cases.

Case 1 $e = v_s v_{s+1}$ for $s \in \left[1, \lfloor \frac{n}{2} \rfloor\right]$

Let $V(Edge-Amal(L_n, e, t)) = \{u, v\} \cup \{v_i^p | i \in [1, t], p \in [1, 2n-2]\}$ such that $E(Edge-Amal(L_n, e, t)) = \{uv\} \cup \{vv_i^s | i \in [1, t]\} \cup \{v_i^p v_i^{p+1} | i \in [1, t], p \in [s, n-3] \cup [n-1, 2n-3]\} \cup \{uv_i^{n+s-2}, vv_i^{n+s-1} | i \in [1, t]\} \cup \{v_i^p v_i^{p+n} | i \in [1, t], p \in [s, n-2]\} \cup E^2$ where

$$E^2 = \begin{cases} \emptyset, & \text{if } s = 1; \\ \{uv_i^1, v_i^1 v_i^{n-1} | i \in [1, t]\}, & \text{if } s = 2; \\ \{uv_i^{s-1} | i \in [1, t]\} \cup \{v_i^p v_i^{p+1} | i \in [1, t], p \in [1, s-2]\} \cup \{v_i^p v_i^{p+n-2} | i \in [1, t], p \in [1, s-1]\}, & \text{otherwise.} \end{cases}$$

First, we prove the lower bound. Assume to the contrary that $srx_3(Edge - Amal(L_n, e, t)) \leq t(n-1)$. Then there exists a strong 3-rainbow coloring $c : E(Edge - Amal(L_n, e, t)) \to [1, t(n-1)]$. For $i \in [1, t]$, let C_i be a set of colors assigned to the path $v_i^1 v_i^2 ... v_i^{s-1} u \cup vv_i^s v_i^{s+1} ... v_i^{n-2}$. Clearly, $|C_i| = n-2$. For distinct i and j in $[1, t]$, by considering $\{v_i^1, v_i^{n-2}, v_j^1\}$ and $\{v_i^1, v_i^{n-2}, v_j^{n-2}\}$, we have $C_i \cap C_j = \emptyset$. Thus, $\sum_{i=1}^{t} |C_i| \geq t(n-2)$. Next, consider edges uv_i^{n+s-2} for all $i \in [1, t]$. By considering $\{u, v_i^{n+s-2}, v_j^1\}$ and $\{u, v_i^{n+s-2}, v_j^{n-2}\}$ for all $j \in [1, t]$, we obtain $c(uv_i^{n+s-2}) \notin C_j$. Since $c(uv_i^{n+s-2}) \neq c(uv_j^{n+s-2})$ for distinct i and j in $[1, t]$, we need t new distinct colors assigned to the edges uv_i^{n+s-2} for all $i \in [1, t]$. This implies we have used all available colors. Next, consider edge uv. We can check that $c(uv) \notin C_i$ and $c(uv) \neq c(uv_i^{n+s-2})$ for all $i \in [1, t]$. This forces us to need one new distinct color assigned to the edge uv, which is impossible. Thus, $srx_3(Edge - Amal(L_n, e, t)) \geq t(n-1) + 1$.

Next, we show that $srx_3(Edge - Amal(L_n, e, t) \leq t(n-1) + 1$. Let $i \in [1, t]$. We define an edge-coloring

$c : E(Edge - Amal(L_n, e, t)) \to [1, t\,(n-1)+1]$ as follows:

$$\begin{aligned} c(v_i^p v_i^{p+1}) &= c(v_i^{p+n-2} v_i^{p+n-1}) = p + (i-1)(n-1) \text{ for } p \in [1, s-2]; \\ c(v_i^p v_i^{p+1}) &= c(v_i^{p+n} v_i^{p+n+1}) = p + 1 + (i-1)(n-1) \text{ for } p \in [s, n-3]; \\ c(uv_i^{s-1}) &= c(v_i^{n+s-3} v_i^{n+s-2}) = s - 1 + (i-1)(n-1); \\ c(vv_i^s) &= c(v_i^{n+s-1} v_i^{n+s}) = s + (i-1)(n-1); \\ c(uv) &= c(v_i^{n+s-2} v_i^{n+s-1}) = t\,(n-1) + 1; \\ c(v_i^p v_i^{p+n-2}) &= i\,(n-1) \text{ for } p \in [1, s-1]; \\ c(v_i^p v_i^{p+n}) &= i\,(n-1) \text{ for } p \in [s, n-2]; \\ c(uv_i^{n+s-2}) &= c(vv_i^{n+s-1}) = i\,(n-1). \end{aligned}$$

By the coloring above, it is not difficult to find a rainbow Steiner S-tree for every set S of three vertices of $Edge - Amal(L_n, e, t)$.

Case 2 $e = v_s v_{s+n}$ for $s \in \left[1, \lceil \frac{n}{2} \rceil\right]$

Let $V(Edge-Amal(L_n, e, t)) = \{u, v\} \cup \{v_i^p | i \in [1, t], p \in [1, 2n-2]\}$ such that $E(Edge-Amal(L_n, e, t)) = \{uv\} \cup \{v_i^p v_i^{p+n-1} | i \in [1, t], p \in [1, n-1]\} \cup \{uv_i^s, vv_i^{n+s-1} | i \in [1, t]\} \cup \{v_i^p v_i^{p+1} | i \in [1, t], p \in [s, n-2] \cup [n+s-1, 2n-3]\} \cup E^3$ where

$$E^3 = \begin{cases} \emptyset, & \text{if } s = 1; \\ \{uv_i^1, vv_i^n | i \in [1, t]\}, & \text{if } s = 2; \\ \{uv_i^{s-1}, vv_i^{n+s-2} | i \in [1, t]\} \cup \{v_i^p v_i^{p+1} | i \in [1, t], p \in [1, s-2] \cup [n, n+s-3]\}, & \text{otherwise.} \end{cases}$$

First, we prove the lower bound. Let c be a strong 3-rainbow coloring of $Edge - Amal(L_n, e, t)$. For $i \in [1, t]$, let D_i be a set of colors assigned to the path $v_i^1 v_i^2 ... v_i^{s-1} u v_i^s ... v_i^{n-1}$. Clearly, $|D_i| = n - 1$. By considering $\{v_i^1, v_i^{n-1}, v_j^1\}$ and $\{v_i^1, v_i^{n-1}, v_j^{n-1}\}$ for distinct i and j in $[1, t]$, we have $D_i \cap D_j = \emptyset$. Thus, $\sum_{i=1}^{t} |D_i| \geq t\,(n-1)$. Next, consider edge uv. We can check that $c(uv) \notin D_i$ for all $i \in [1, t]$, which means we need one new distinct colors assigned to the edge uv. Thus, $srx_3(Edge - Amal(L_n, e, t)) \geq t\,(n-1) + 1$.

Next, we show that $srx_3(Edge - Amal(L_n, e, t) \leq t\,(n-1) + 1$. Let $i \in [1, t]$. We define an edge-coloring $c : E(Edge - Amal(L_n, e, t)) \to [1, t\,(n-1)+1]$ as follows:

$$\begin{aligned} c(v_i^p v_i^{p+1}) &= c(v_i^{p+n-1} v_i^{p+n}) = \begin{cases} p + (i-1)(n-1), & \text{for } p \in [1, s-2]; \\ p + 1 + (i-1)(n-1), & \text{for } p \in [s, n-2]; \end{cases} \\ c(uv_i^p) &= c(vv_i^{p+n-1}) = p + (i-1)(n-1) \text{ for } p \in [s-1, s]; \\ c(uv) &= c(v_i^p v_i^{p+n-1}) = t\,(n-1) + 1 \text{ for } p \in [1, n-1]. \end{aligned}$$

By the coloring above, it is not difficult to find a rainbow Steiner S-tree for every set S of three vertices of $Edge - Amal(L_n, e, t)$. □

Following (1.1), $sdiam_3(Edge-Amal(G, e, t))$ is the natural lower bound for $srx_3(Edge-Amal(G, e, t))$. Consider the edge-amalgamation of ladders shown in Theorem 2.12. For $e = v_1 v_{1+n}$ (or $e = v_n v_{2n}$), we can check that $sdiam_3(Edge - Amal(L_n, e, 2)) = 2n - 1$ and $sdiam_3(Edge - Amal(L_n, e, t)) = 3n - 2$ for $t \geq 3$. Hence, following Theorem 2.12, we have $srx_3(Edge - Amal(L_n, e, t)) = sdiam_3(Edge - Amal(L_n, e, t))$ for $e = v_1 v_{1+n}$ (or $e = v_n v_{2n}$) and $t \in [2, 3]$.

Next, we consider graphs $Edge - Amal(K_{n,n}, e, t)$ where e is an arbitrary edge of $K_{n,n}$. We determine the strong 3-rainbow index of $Edge - Amal(K_{n,n}, e, t)$, which is given in the following theorem.

Theorem 2.13 *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let $K_{n,n}$ be a regular complete bipartite graph of order $2n$ and e be an arbitrary edge of $K_{n,n}$. Then $srx_3(Edge - Amal(K_{n,n}, e, t)) = t\,(n-1)+1$.*

Proof Let $V(Edge - Amal(K_{n,n}, e, t)) = \{u, v\} \cup \{u_i^p | i \in [1, t], p \in [1, n-1]\} \cup \{v_i^p | i \in [1, t], p \in [1, n-1]\}$ such that $E(Edge - Amal(K_{n,n}, e, t)) = \{uv\} \cup \{uv_i^p | i \in [1, t], p \in [1, n-1]\} \cup \{vu_i^p | i \in [1, t], p \in [1, n-1]\} \cup \{u_i^p v_i^q | i \in [1, t], p, q \in [1, n-1]\}$.

First, we prove the lower bound. Let c be a strong 3-rainbow coloring of $Edge - Amal(K_{n,n}, e, t)$. For all $i, j \in [1, t]$ and $p, q \in [1, n-1]$, by considering $\{u, v, v_i^p\}$ and $\{u, v_i^p, v_j^q\}$, we have $c(uv) \neq c(uv_i^p)$ and $c(uv_i^p) \neq c(uv_j^q)$. Since $d(u) = t\,(n-1)+1$, we have $srx_3(Edge - Amal(K_{n,n}, e, t)) \geq t\,(n-1)+1$.

Next, we show that $srx_3(Edge - Amal(K_{n,n}, e, t)) \leq t\,(n-1)+1$. Let $i \in [1, t]$ and $p, q \in [1, n-1]$. We define an edge-coloring $c : E(Edge - Amal(K_{n,n}, e, t)) \to [1, t\,(n-1)+1]$ as follows:

$$\begin{aligned} c(uv) &= c(u_i^p v_i^p) = 1; \\ c(uv_i^p) &= p + 1 + (i-1)(n-1); \\ c(vu_i^p) &= n - p + 1 + (i-1)(n-1); \\ c(u_i^p v_i^q) &= \begin{cases} q - p + 1 + (i-1)(n-1), & \text{if } p < q; \\ n + q - p + 1 + (i-1)(n-1), & \text{if } p > q. \end{cases} \end{aligned}$$

By the coloring above, it is not difficult to find a rainbow Steiner S-tree for every set S of three vertices of $Edge - Amal(K_{n,n}, e, t)$. □

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