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Nonlinear variants of the generalized Filbert and Lilbert matrices

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Abstract: In this paper, we present variants of the generalized Filbert and Lilbert matrices by products of the general Fibonacci and Lucas numbers whose indices are in certain nonlinear forms of the indices with certain integer parameters. We derive explicit formulæ for inverse matrix, LU-decomposition and inverse matrices L^{-1} and U^{-1} for all matrices. Generally, we present q-versions of these matrices and their related results.

Key words: Filbert and Lilbert matrices, inverse matrix determinant, generalized Pochhammer symbol, Fibonacci and Lucas numbers

1. Introduction

For $n \geq 2$, define the second order linear recurrences $\{U_n\}$ and $\{V_n\}$ by

$$U_n = pU_{n-1} + U_{n-2} \quad \text{and} \quad V_n = pV_{n-1} + V_{n-2}$$

with initials $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, resp. Especially when $p = 1$, $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number), resp.

The Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n \left(1 + q^n\right),$$

where $\alpha, \beta = \left(p \mp \sqrt{\Delta}\right)/2$ with $q = \beta/\alpha = -\alpha^2$ and $\Delta = p^2 + 4$, so that $\alpha = \mathbf{i}q^{-1/2}$, where $\mathbf{i} = \sqrt{-1}$.

The Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where $(x;q)_n$ is the q-Pochhammer symbol, $(x;q)_n = (1-x)(1-xq)\ldots\left(1-xq^{n-1}\right)$ (for more details, we refer to [2]).

Note that

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

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where $\binom{n}{k}$ is the usual binomial coefficient.

From the current literature, one could see many interesting and useful combinatorial matrices that are constructed via the binomial coefficients or their certain generalizations, namely, the Gaussian q-binomial coefficients or via well-known number sequences such as natural numbers, integers, or the Fibonacci and Lucas numbers, etc. [1, 3–7, 12, 14, 17–19].

For the reader's convenience, we recall some well-known combinatorial matrices:

- Chu and Di Claudio [4] studied the matrix $\left[\frac{(a)_{j+\lambda_i}}{(c)_{j+\lambda_i}}\right]_{0\le i,j\le n}$, where a, c and $\{\lambda_i\}_{i=0}^{n}$ are complex numbers and $(x)_n$ is the shifted factorial by $(x)_0 = 1$ and $(x)_n = x(x+1)\dots(x+n-1)$ for $n = 1, 2, \dots$. They also presented some variants of the above matrix.

- Zhou and Zhaolin [18] studied the g-circulant matrices whose elements consist of the Fibonacci and Lucas numbers, separately for nonnegative integer g.

- Hilbert matrix is defined with entries
$$\frac{1}{i+j-1}.$$

- As a Fibonacci analogue of the Hilbert matrix, Richardson [17] defined the Filbert matrix with entries
$$\frac{1}{F_{i+j-1}},$$
where F_n is the nth Fibonacci number.

- Kılıç and Prodinger [7] defined the generalized Filbert matrix $\mathcal{F}$ with entries $\frac{1}{F_{i+j+r}}$, where $r \ge -1$ is an integer parameter.

- After this, Prodinger [16] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters by taking its entries as $\frac{x^i y^j}{F_{\lambda(i+j)+r}}$, where $r \ge -1$ and $\lambda > 0$ are integers.

- Kılıç and Prodinger [8] defined a further generalization of the generalized Filbert Matrix $\mathcal{F}$ by defining the matrix Q with entries
$$q_{ij} = \frac{1}{F_{i+j+r}F_{i+j+r+1}\dots F_{i+j+r+t-1}},$$
where $r \ge -1$ and $t \ge 0$ are integers.

- In a recent paper [11], Kılıç and Prodinger introduced the matrix G as a parametric generalization of the matrix Q by
$$g_{ij} = \frac{1}{F_{\lambda(i+j)+r}F_{\lambda(i+j+1)+r}\dots F_{\lambda(i+j+t-1)+r}},$$
where $r \ge -1$, $t \ge 0$ and $\lambda > 0$ are integer parameters.

- Much recently, Kılıç and Prodinger [10] defined and studied the following four new generalizations and variants of the Filbert matrix $\mathcal{F}$ with entries

$$\frac{1}{F_{\lambda i+\mu j+r}}, \quad \frac{F_{\lambda i+\mu j+r}}{F_{\lambda i+\mu j+s}}, \quad \frac{1}{L_{\lambda i+\mu j+r}} \quad \text{and} \quad \frac{L_{\lambda i+\mu j+r}}{L_{\lambda i+\mu j+s}},$$

where s, r, λ, and μ are integer parameters such that $s \neq r$, $r, s \geq -1$ and $\lambda, \mu > 0$.

- As a Lucas analogue of the matrix G, Kılıç and Prodinger [12] defined the matrix S with entries

$$S_{ij} = \frac{1}{L_{\lambda(i+j)+r} L_{\lambda(i+j+1)+r} \ldots L_{\lambda(i+j+t-1)+r}},$$

where $r \geq -1$, $t \geq 0$ and $\lambda > 0$ are integer parameters.

Here before presenting a new item, we would like to take the attention of the readers to a point that the indices of the Fibonacci and Lucas numbers in the Filbert and Filbert-like matrices as well as all their all analogues are in various linear forms. For example, in the usual Filbert matrix $\left[\frac{1}{F_{i+j-1}}\right]$, the index of the matrix is $i+j-1$ which is a linear form of i and j.

The first known nonlinear form of the indices was studied in the following work, to the best of our knowledge of the literature.

- For the first time, Kılıç and Talha [13] defined nonlinear generalizations of the Filbert and Lilbert matrices with entries

$$M = \frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c}} \quad \text{and} \quad \check{M} = \frac{1}{V_{\lambda(i+r)^k+\mu(j+s)^m+c}},$$

respectively, where λ, μ, k, m are positive integers and r, s, c are integers such that $\lambda(i+r)^k + \mu(j+s)^m + c > 0$.

In all the studies mentioned above, the authors have studied various properties of the matrices such as LU and Cholesky decompositions, determinants, inverses. In many of them, the authors firstly converted the entries of the matrices into q-form and then proved all their claims in q-form by the celebrated q-Zeilberger algorithm (for details about the algorithm see [15] and for its usage see [7–9, 11, 12, 16]) or backward induction (see about its usage [10, 13]).

In this work, mainly inspired by the first example of the nonlinear generalizations of the Filbert and Lilbert matrices [13] and the ideas of including finite products of the Fibonacci and Lucas numbers [11, 12], we will continue to seek nonlinear variants of the Filbert and Lilbert matrices rather than their generalizations. For this purpose, we then present and study new nonlinear variants of the Filbert and Lilbert matrices, resp., by means of the products of Fibonacci and Lucas numbers in which their indices are in the nonlinear forms $\lambda(i+r)^k + \mu(j+s)^m + c_1$ and $\lambda(i+r)^k - \mu(j+s)^m + c_2$ for positive integers λ, μ, k, m and integers r, s, c_1 and c_2.

Much clearly we shall study two matrices W and Z having the entries

$$W_{ij} = \frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c_1} V_{\lambda(i+r)^k-\mu(j+s)^m+c_2}}$$

and

$$Z_{ij} = \frac{1}{V_{\lambda(i+r)^k+\mu(j+s)^m+c_1} V_{\lambda(i+r)^k-\mu(j+s)^m+c_2}}.$$

In the manner of nonlinear variants of the Filbert or Lilbert matrices, our results will satisfy valuable contribution to the current literature as the second example. These matrices will be the nonlinear variants of the generalized Filbert matrix M and Lilbert matrix $\check{M}$. Consequently they will be new analogues of the Filbert and Lilbert matrices given in [10].

In Sections 2 and 3, we define new variants of the Filbert and Lilbert matrices, respectively. For all the matrices will be studied we derive explicit formulæ for inverse matrix, LU-decomposition, and inverse matrices L^{-1} and U^{-1}.

In Section 4, we only prove the results of Section 2. The results of Section 3 could be similarly proven. We omit details here. In Section 5, we give q-forms of the results of Sections 2 and 3 for an indeterminate q without proof. These results are more general versions of the results given in Sections 2 and 3. For special values of q, one may obtain many special cases. For example, the results of Sections 2 and 3 are obtained when $q = \beta/\alpha$. In general, for each section, the size of the matrix does not really matter except the results about inverse matrix, so that we may think about an infinite matrix W and restrict it whenever necessary to the first N rows resp. columns and use the notation W_N.

Throughout the paper, we only assume that λ, μ, k, and m are positive integers, r, s, and c_1 are any integers such that $\lambda\left(i+r\right)^k + \mu\left(j+s\right)^m + c_1 > 0$ for all positive integers i and j because, while studying only the matrix W, we could encounter zero factors in its entries' denominators that could come from the Fibonacci factor, $U_{\lambda(i+r)^k+\mu(j+s)^m+c_1}$.

2. A nonlinear variant of the generalized Filbert matrix

In this section, we will present a nonlinear variant of the Filbert matrix by considering its first nonlinear generalization given in [13] and including an extra Lucas factor to its denominator, where the indices are in geometric progressions. Define the matrix W with entries

$$W_{ij} = \frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c_1} V_{\lambda(i+r)^k-\mu(j+s)^m+c_2}},$$

where U_n and V_n are nth generalized Fibonacci and Lucas numbers, resp.

For the matrix W, we give its inverse and LU-decomposition as well as we derive explicit formulæ for the matrices L^{-1} and U^{-1}. We obtain the LU-decomposition $W = LU$:

Theorem 2.1 *For $i, j \geq 1$,*

$$L_{ij} = \frac{\left(\prod_{t=1}^{j} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{j} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)}$$

$$\times\frac{\left(\prod_{t=1}^{j-1}V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{j}V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{j-1}V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{j}V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)}$$

and

$$U_{ij}=(-1)^{\lambda(i+r)-\mu(j+s)+c_2}\ \frac{\prod_{t=1}^{i-1}U_{\lambda(i+r)^k-\lambda(t+r)^k}}{\prod_{t=1}^{i}U_{\mu(j+s)^m+\lambda(t+r)^k+c_1}}$$

$$\times\frac{\left(\prod_{t=1}^{i-1}U_{\mu(j+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{i-1}V_{\mu(t+s)^m+\mu(j+s)^m+c_1-c_2}\right)\left(\prod_{t=1}^{i-1}V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}{\left(\prod_{t=1}^{i-1}U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{i}V_{\mu(j+s)^m-\lambda(t+r)^k-c_2}\right)\left(\prod_{t=1}^{i-1}V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)}.$$

We also determine the inverses of the matrices L and U:

Theorem 2.2 *For* $i,j\geq 1$,

$$L_{ij}^{-1}=(-1)^{\lambda i(j+1)+\binom{j-i+n}{n}+1}\ \frac{V_{2\lambda(j+r)^k+c_1+c_2}}{V_{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}}$$

$$\times\frac{\left(\prod_{t=1}^{i-1}V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{i-1}V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}{\left(\prod_{t=1}^{i-1}V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{i-1}V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}$$

$$\times\frac{\left(\prod_{t=1}^{i-j-1}U_{\lambda(t+j+r)^k-\lambda(i+r)^k}\right)\left(\prod_{t=1}^{j-1}U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{i-1}U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{i-j-1}U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)\left(\prod_{t=1}^{j-1}U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{i-1}U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)}$$

and

$$U_{ij}^{-1}=(-1)^{\lambda(i+r)^k+\mu(j+s)^m+(\lambda+\mu)(i+j)+c_2}\ \frac{1}{\left(\prod_{t=1}^{i-1}V_{\mu(i+s)^m+\mu(t+s)^m+c_1-c_2}\right)}$$

$$\times\frac{\left(\prod_{t=1}^{j-1}U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{j}U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{j-1}U_{\lambda(t+r)^k-\lambda(j+r)^k}\right)\left(\prod_{t=1}^{j-i}U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{i-1}U_{\mu(t+s)^m-\mu(i+s)^m}\right)}$$

$$\times\frac{\left(\prod_{t=1}^{j-1}V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{j}V_{\mu(t+s)^m-\lambda(j+r)^k-c_2}\right)}{\left(\prod_{t=1}^{j-i}V_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)\left(\prod_{t=1}^{j-1}V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}.$$

Now we give the inverse of W. This time it depends on the dimension, so we compute W_n^{-1}.

Theorem 2.3 *For* $1 \leq i,j \leq n$,

$$\left(W_n^{-1}\right)_{ij} = \frac{(-1)^{\lambda r+\lambda j+\mu(i+s)+c_2+\mu\binom{n+1}{2}+n(\mu s-\lambda r+\lambda j-c_2)}\,V_{2\lambda(j+r)^k+c_1+c_2}}{V_{\lambda(j+r)^k-\mu(i+s)^m+c_2}U_{\lambda(j+r)^k+\mu(i+s)^m+c_1}}$$

$$\times \frac{\left(\prod_{t=1}^{n} U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{n} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{n-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)\left(\prod_{t=1}^{n-i} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{i-1} U_{\mu(t+s)^m-\mu(i+s)^m}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{n} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{n} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{n-i} V_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{n} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)}{\left(\prod_{t=1}^{i-1} V_{\mu(i+s)^m+\mu(t+s)^m+c_1-c_2}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(t+r)^k-\lambda(j+r)^k}\right)}.$$

3. A nonlinear variant of the generalized Lilbert matrix

As Lilbert (Lucas-Hilbert) analogue of the matrix W, we define the matrix Z as a nonlinear variant of the generalized Lilbert matrix $\check{M}$ with entries

$$Z_{ij} = \frac{1}{V_{\lambda(i+r)^k+\mu(j+s)^m+c_1}V_{\lambda(i+r)^k-\mu(j+s)^m+c_2}},$$

where V_n is the nth general Lucas number.

Similarly, we have the LU-decomposition of the matrix Z, the matrices L^{-1} and U^{-1} and the inverse matrix Z^{-1}

Theorem 3.1 *For* $i,j \geq 1$,

$$L_{ij} = \frac{\left(\prod_{t=1}^{j} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{j} V_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{j} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{j} V_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{j-1} U_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)}$$

and

$$U_{ij} = (-1)^{\lambda(i+r)-\mu(j+s)+c_2} \, \Delta^{2(i-1)} \frac{\left(\prod_{t=1}^{i-1} U_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}{\left(\prod_{t=1}^{i-1} V_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{i-1} U_{\mu(t+s)^m+\mu(j+s)^m+c_1-c_2}\right)\left(\prod_{t=1}^{i-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{i-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{i} V_{\mu(j+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{i} V_{\mu(j+s)^m-\lambda(t+r)^k-c_2}\right)\left(\prod_{t=1}^{i-1} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)},$$

where Δ *is defined as before.*

Theorem 3.2 *For* $i, j \geq 1$,

$$L_{ij}^{-1} = (-1)^{\lambda i(j+1)+\binom{j-i+n}{n}+1} \frac{U_{2\lambda(j+r)^k+c_1+c_2}}{U_{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}}$$

$$\times \frac{\left(\prod_{t=1}^{i-1} V_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{i-1} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{i-1} V_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{i-1} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{i-1} U_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{i-j-1} U_{\lambda(t+j+r)^k-\lambda(i+r)^k}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(t+r)^k-\lambda(i+r)^k}\right)}{\left(\prod_{t=1}^{i-1} U_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{i-j-1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(t+r)^k-\lambda(j+r)^k}\right)}$$

and

$$U_{ij}^{-1} = (-1)^{\lambda(i+r)^k+\mu(j+s)^m+(\lambda+\mu)(i+j)+c_2} \, \Delta^{2(1-j)} \frac{1}{\left(\prod_{t=1}^{i-1} U_{\mu(i+s)^m+\mu(t+s)^m+c_1-c_2}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{j-1} V_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{j} V_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{j-1} U_{\lambda(t+r)^k-\lambda(j+r)^k}\right)\left(\prod_{t=1}^{j-i} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{i-1} U_{\mu(t+s)^m-\mu(i+s)^m}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{j-1} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{j} V_{\mu(t+s)^m-\lambda(j+r)^k-c_2}\right)}{\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{j-i} U_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)},$$

where Δ *is defined as before.*

Theorem 3.3 *For $1 \le i, j \le n$,*

$$\left(Z_n^{-1}\right)_{ij} = (-1)^{\lambda r+\lambda j+\mu(i+s)+c_2+\mu\binom{n+1}{2}+n(\mu s-\lambda r+\lambda j-c_2)}$$
$$\times \frac{\Delta^{2(1-n)} U_{2\lambda(j+r)^k+c_1+c_2}}{V_{\lambda(j+r)^k+\mu(i+s)^m+c_1} V_{\lambda(j+r)^k-\mu(i+s)^m+c_2} \left(\prod_{t=1}^{n-i} U_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)}$$
$$\times \frac{\left(\prod_{t=1}^{n} V_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{n} V_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{n-i} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{n-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)\left(\prod_{t=1}^{i-1} U_{\mu(t+s)^m-\mu(i+s)^m}\right)}$$
$$\times \frac{\left(\prod_{t=1}^{n} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{n} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{n} U_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(t+r)^k-\lambda(j+r)^k}\right)\left(\prod_{t=1}^{i-1} U_{\mu(i+s)^m+\mu(t+s)^m+c_1-c_2}\right)},$$

where Δ is defined as before.

4. Proofs

As mentioned in Introduction, we shall only give the proofs of the results of Section 2. We omit proofs of the results of Section 3. We will need the following three lemmas for later use. The proofs of them could be derived from the Binet formulas of $\{U_n\}$ and $\{V_n\}$ but we will omit here.

Lemma 4.1 *For $i, j \ge 1$,*

$$\begin{aligned}
&(-1)^{\lambda(N+r)-\mu(j+s)+c_2} U_{\lambda(N+r)^k+\mu(N+s)^m+c_1} U_{\lambda(i+r)^k+\mu(j+s)^m+c_1} V_{\lambda(N+r)^k-\mu(N+s)^m+c_2}\\
&\times V_{\lambda(i+r)^k-\mu(j+s)^m+c_2} + U_{\lambda(i+r)^k-\lambda(N+r)^k} U_{\mu(j+s)^m-\mu(N+s)^m} V_{\lambda(i+r)^k+\lambda(N+r)^k+c_1+c_2}\\
&\times V_{\mu(N+s)^m+\mu(j+s)^m+c_1-c_2}\\
&= U_{\lambda(i+r)^k+\mu(N+s)^m+c_1} U_{\lambda(N+r)^k+\mu(j+s)^m+c_1} V_{\mu(j+s)^m-\lambda(N+r)^k-c_2} V_{\lambda(i+r)^k-\mu(N+s)^m+c_2}.
\end{aligned}$$

Lemma 4.2 *For $i, j \ge 1$,*

$$\frac{(-1)^{(\lambda+1)(N-1+j)+\lambda\binom{N-j-1}{2}}}{U_{\lambda(i+r)^k-\lambda(j+r)^k} V_{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}}$$
$$- \frac{(-1)^{(\lambda+1)(N+j)+\lambda\binom{N-j+1}{2}} V_{\lambda(N+r)^k-\mu(N+s)^m+c_2}}{V_{\lambda(N+r)^k+\lambda(j+r)^k+c_1+c_2} U_{\lambda(i+r)^k+\mu(N+s)^m+c_1} V_{\lambda(i+r)^k-\mu(N+s)^m+c_2}}$$
$$\times \frac{\left(\prod_{t=1}^{N} U_{\lambda(N+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(N+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{N-1-j} U_{\lambda(t+j+r)^k-\lambda(N+r)^k}\right)}{\left(\prod_{t=1}^{N-1} U_{\lambda(N+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{N-1} U_{\lambda(N+r)^k+\mu(t+s)^m+c_1}\right)}$$
$$= (-1)^{\lambda\binom{N-j}{2}+(\lambda+1)(N+j)} \frac{U_{\lambda(i+r)^k-\lambda(N+r)^k} V_{\lambda(j+r)^k-\mu(N+s)^m+c_2}}{U_{\lambda(N+r)^k-\lambda(j+r)^k} U_{\lambda(i+r)^k-\lambda(j+r)^k} V_{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}}$$
$$\times \frac{U_{\lambda(j+r)^k+\mu(N+s)^m+c_1} V_{\lambda(i+r)^k+\lambda(N+r)^k+c_1+c_2}}{U_{\lambda(i+r)^k+\mu(N+s)^m+c_1} V_{\lambda(i+r)^k-\mu(N+s)^m+c_2} V_{\lambda(j+r)^k+\lambda(N+r)^k+c_1+c_2}}.$$

Lemma 4.3 *For* $i, j \geq 1$,

$$\frac{1}{U_{\lambda(j+r)^k+\mu(i+s)^m+c_1} V_{\lambda(j+r)^k-\mu(i+s)^m+c_2}} + (-1)^{\mu N+(\mu s-\lambda r-c_2)+\lambda j}$$
$$\times \frac{U_{\lambda(N+r)^k+\mu(N+s)^m+c_1} V_{\lambda(N+r)^k-\mu(N+s)^m+c_2}}{U_{\mu(N+s)^m-\mu(i+s)^m} U_{\lambda(N+r)^k-\lambda(j+r)^k} V_{\lambda(j+r)^k+\lambda(N+r)^k+c_1+c_2} V_{\mu(i+s)^m+\mu(N+s)^m+c_1-c_2}}$$
$$= (-1)^{\mu N+(\mu s-\lambda r-c_2)+\lambda j+\binom{j-N+n}{n}} \frac{V_{\lambda(N+r)^k-\mu(i+s)^m+c_2} V_{\lambda(j+r)^k-\mu(N+s)^m+c_2}}{V_{\mu(i+s)^m+\mu(N+s)^m+c_1-c_2} V_{\lambda(j+r)^k-\mu(i+s)^m+c_2}}$$
$$\times \frac{U_{\mu(i+s)^m+\lambda(N+r)^k+c_1} U_{\lambda(j+r)^k+\mu(N+s)^m+c_1}}{U_{\lambda(j+r)^k+\mu(i+s)^m+c_1} U_{\mu(N+s)^m-\mu(i+s)^m} U_{\lambda(N+r)^k-\lambda(j+r)^k} V_{\lambda(j+r)^k+\lambda(N+r)^k+c_1+c_2}}.$$

Theorem 4.4 *For* $i, j \geq 1$,

$$\sum_{d=K}^{\min\{i,j\}} (-1)^{\lambda(d+r)-\mu(j+s)+c_2} U_{\lambda(d+r)^k+\mu(d+s)^m+c_1} V_{\lambda(d+r)^k-\mu(d+s)^m+c_2}$$
$$\times \frac{\left(\prod_{t=1}^{d-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{d-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{d} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{d} U_{\mu(j+s)^m+\lambda(t+r)^k+c_1}\right)}$$
$$\times \frac{\left(\prod_{t=1}^{d-1} V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{d-1} V_{\mu(t+s)^m+\mu(j+s)^m+c_1-c_2}\right)}{\left(\prod_{t=1}^{d} V_{\mu(j+s)^m-\lambda(t+r)^k-c_2}\right)\left(\prod_{t=1}^{d} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)}$$
$$= \frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c_1} V_{\lambda(i+r)^k-\mu(j+s)^m+c_2}}$$
$$\times \frac{\left(\prod_{t=1}^{K-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-1} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K-1} U_{\lambda(t+r)^k+\mu(j+s)^m+c_1}\right)}$$
$$\times \frac{\left(\prod_{t=1}^{K-1} V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{K-1} V_{\mu(t+s)^m+\mu(j+s)^m+c_1-c_2}\right)}{\left(\prod_{t=1}^{K-1} V_{\mu(j+s)^m-\lambda(t+r)^k-c_2}\right)\left(\prod_{t=1}^{K-1} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)}.$$

Proof We shall use backward induction. Denote the sum just above by $\mathtt{SUM}_K^{(1)}$ and its summand by S_d for brevity. First, we assume that $i \geq j$. Thus, when $K = j$, the claim is obvious. For the case $j > i$, we give the proof. The backward induction step amounts to show that

$$\mathtt{SUM}_{K-1}^{(1)} = \mathtt{SUM}_K^{(1)} + S_{K-1}.$$

By the definitions of $\texttt{SUM}_K^{(1)}$ and S_{K-1}, consider the RHS of the above equality

$$
\begin{aligned}
&(-1)^{\lambda(K-1+r)-\mu(j+s)+c_2}\\
&\times U_{\lambda(K-1+r)^k+\mu(K-1+s)^m+c_1}\frac{\left(\prod\limits_{t=1}^{K-2}U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod\limits_{t=1}^{K-2}U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod\limits_{t=1}^{K-1}U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod\limits_{t=1}^{K-1}U_{\mu(j+s)^m+\lambda(t+r)^k+c_1}\right)}\\
&\times V_{\lambda(K-1+r)^k-\mu(K-1+s)^m+c_2}\frac{\left(\prod\limits_{t=1}^{K-2}V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod\limits_{t=1}^{K-2}V_{\mu(t+s)^m+\mu(j+s)^m+c_1-c_2}\right)}{\left(\prod\limits_{t=1}^{K-1}V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod\limits_{t=1}^{K-1}V_{\mu(j+s)^m-\lambda(t+r)^k-c_2}\right)}\\
&+\frac{\prod\limits_{t=1}^{K-1}U_{\lambda(i+r)^k-\lambda(t+r)^k}}{U_{\lambda(i+r)^k+\mu(j+s)^m+c_1}V_{\lambda(i+r)^k-\mu(j+s)^m+c_2}\left(\prod\limits_{t=1}^{K-1}U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)}\\
&\times\frac{\left(\prod\limits_{t=1}^{K-1}U_{\mu(j+s)^m-\mu(t+s)^m}\right)\left(\prod\limits_{t=1}^{K-1}V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod\limits_{t=1}^{K-1}V_{\mu(t+s)^m+\mu(j+s)^m+c_1-c_2}\right)}{\left(\prod\limits_{t=1}^{K-1}U_{\lambda(t+r)^k+\mu(j+s)^m+c_1}\right)\left(\prod\limits_{t=1}^{K-1}V_{\mu(j+s)^m-\lambda(t+r)^k-c_2}\right)\left(\prod\limits_{t=1}^{K-1}V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)},
\end{aligned}
$$

which, after some simplifications, equals

$$
\begin{aligned}
&\frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c_1}V_{\lambda(i+r)^k-\mu(j+s)^m+c_2}}\\
&\times\frac{\left(\prod\limits_{t=1}^{K-2}V_{\mu(t+s)^m+\mu(j+s)^m+c_1-c_2}\right)\left(\prod\limits_{t=1}^{K-2}V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}{\left(\prod\limits_{t=1}^{K-1}V_{\mu(j+s)^m-\lambda(t+r)^k-c_2}\right)\left(\prod\limits_{t=1}^{K-1}V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)}\\
&\times\frac{\left(\prod\limits_{t=1}^{K-2}U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod\limits_{t=1}^{K-2}U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod\limits_{t=1}^{K-1}U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod\limits_{t=1}^{K-1}U_{\mu(j+s)^m+\lambda(t+r)^k+c_1}\right)}\\
&\Big((-1)^{\lambda(K-1+r)-\mu(j+s)+c_2}U_{\lambda(K-1+r)^k+\mu(K-1+s)^m+c_1}\\
&\times U_{\lambda(i+r)^k+\mu(j+s)^m+c_1}V_{\lambda(i+r)^k-\mu(j+s)^m+c_2}V_{\lambda(K-1+r)^k-\mu(K-1+s)^m+c_2}\\
&+U_{\lambda(i+r)^k-\lambda(K-1+r)^k}U_{\mu(j+s)^m-\mu(K-1+s)^m}\\
&\times V_{\lambda(i+r)^k+\lambda(K-1+r)^k+c_1+c_2}V_{\mu(K-1+s)^m+\mu(j+s)^m+c_1-c_2}\Big).
\end{aligned}
$$

If we arrange the expression in the parenthesis at the end of the last three lines just above by taking $N=K-1$

in Lemma 4.1, then the last statement equals

$$\frac{\prod_{t=1}^{K-2} U_{\lambda(i+r)^k-\lambda(t+r)^k}}{U_{\lambda(i+r)^k+\mu(j+s)^m+c_1} V_{\lambda(i+r)^k-\mu(j+s)^m+c_2} \left(\prod_{t=1}^{K-2} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)}$$
$$\times \frac{\left(\prod_{t=1}^{K-2} U_{\mu(j+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{K-2} V_{\mu(t+s)^m+\mu(j+s)^m+c_1-c_2}\right)\left(\prod_{t=1}^{K-2} V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}{\left(\prod_{t=1}^{K-2} U_{\lambda(t+r)^k+\mu(j+s)^m+c_1}\right)\left(\prod_{t=1}^{K-2} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{K-2} V_{\mu(j+s)^m-\lambda(t+r)^k-c_2}\right)},$$

which is equal to $\mathtt{SUM}_{K-1}^{(1)}$, as claimed. □

Theorem 4.5 *For* $i, j \geqslant 1$,

$$\sum_{d=j}^{K} (-1)^{\lambda d(j+1)+\binom{j-d+n}{n}+1} V_{\lambda(d+r)^k-\mu(d+s)^m+c_2} U_{\lambda(d+r)^k+\mu(d+s)^m+c_1}$$
$$\times \frac{\left(\prod_{t=1}^{d-1} V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{d-j-1} V_{\lambda(d+r)^k+\lambda(t+j+r)^k+c_1+c_2}\right)}{\left(\prod_{t=1}^{d-1} V_{\lambda(d+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{d-j-1} V_{\lambda(j+r)^k+\lambda(t+j+r)^k+c_1+c_2}\right)}$$
$$\times \frac{\left(\prod_{t=1}^{d-1} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(t+r)^k-\lambda(d+r)^k}\right)\left(\prod_{t=1}^{d-1} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{d} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{d-1} U_{\lambda(d+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{d-j-1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$
$$\times \frac{\left(\prod_{t=1}^{d-j-1} U_{\lambda(t+j+r)^k-\lambda(d+r)^k}\right)\left(\prod_{t=1}^{j-1} V_{\lambda(d+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{d-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{d} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)}$$
$$= \frac{(-1)^{j+1+\lambda\binom{j}{2}} \prod_{t=1}^{K} V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}}{U_{\lambda(i+r)^k-\lambda(j+r)^k} V_{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}\left(\prod_{t=1}^{K-j} V_{\lambda(j+t+r)^k+\lambda(j+r)^k+c_1+c_2}\right)}$$
$$\times \frac{\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{K} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(j+r)^k-\lambda(j+r+t)^k}\right)\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)}.$$

Proof Denote the sum just above by $\mathtt{SUM}_K^{(2)}$ and the summand term by S_d. By using induction, the case $K = j$ is obvious. Thus, the induction step amounts to show that

$$\mathtt{SUM}_{K+1}^{(2)} = \mathtt{SUM}_K^{(2)} + S_{K+1}.$$

Consider

$$\begin{aligned}
&\mathtt{SUM}_K^{(2)} + S_{K+1} = (-1)^{\lambda\binom{K-j}{2}+(\lambda+1)(K+j)} \\
&\times \frac{V_{\lambda(j+r)^k+\lambda(j+r)^k+c_1+c_2}\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{U_{\lambda(i+r)^k-\lambda(j+r)^k}V_{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}\left(\prod_{t=1}^{K-j} U_{\lambda(j+t+r)^k-\lambda(j+r)^k}\right)} \\
&\times \frac{\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{K} V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}{\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)} \\
&+ (-1)^{\lambda\binom{K+1-j}{2}+(\lambda+1)(K+1+j)} \\
&\times \frac{V_{\lambda(j+r)^k+\lambda(j+r)^k+c_1+c_2}\left(\prod_{t=1}^{K+1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{U_{\lambda(i+r)^k-\lambda(j+r)^k}V_{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}\left(\prod_{t=1}^{K+1-j} U_{\lambda(j+t+r)^k-\lambda(j+r)^k}\right)} \\
&\times \frac{\left(\prod_{t=1}^{K+1} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}{\left(\prod_{t=1}^{K+1} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)},
\end{aligned}$$

which, after some simplifications, equals

$$\begin{aligned}
&\frac{V_{2\lambda(j+r)^k+c_1+c_2}\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K} V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}{\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)} \\
&\times \frac{\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{K} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)} \\
&\times \left[\frac{(-1)^{\lambda\binom{K-j}{2}+(\lambda+1)(K+j)}}{U_{\lambda(i+r)^k-\lambda(j+r)^k}V_{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}}\right. \\
&-\frac{(-1)^{(\lambda+1)(K+j+1)+\lambda\binom{K-j+2}{2}}V_{\lambda(K+1+r)^k-\mu(K+1+s)^m+c_2}\left(\prod_{t=1}^{K+1} U_{\lambda(K+r+1)^k+\mu(t+s)^m+c_1}\right)}{V_{\lambda(K+1+r)^k+\lambda(j+r)^k+c_1+c_2}U_{\lambda(i+r)^k+\mu(K+1+s)^m+c_1}} \\
&\left.\times \frac{\left(\prod_{t=1}^{j-1} U_{\lambda(K+r+1)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(t+j+r)^k-\lambda(K+r+1)^k}\right)}{V_{\lambda(i+r)^k-\mu(K+1+s)^m+c_2}\left(\prod_{t=1}^{K} U_{\lambda(K+1+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K} U_{\lambda(K+r+1)^k+\mu(t+s)^m+c_1}\right)}\right].
\end{aligned}$$

If we take $N = K+1$ in Lemma 4.2, then we write

$$\begin{aligned}
\mathtt{SUM}_K^{(2)} + S_{K+1} &= (-1)^{\lambda\binom{K+1-j}{2}+(\lambda+1)(K+1+j)} \\
&\times \frac{V_{2\lambda(j+r)^k+c_1+c_2}\left(\prod_{t=1}^{K+1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{U_{\lambda(i+r)^k-\lambda(j+r)^k} V_{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}\left(\prod_{t=1}^{K+1-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)} \\
&\times \frac{\left(\prod_{t=1}^{K+1} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(i+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{K+1} U_{\lambda(i+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(i+r)^k-\mu(t+s)^m+c_2}\right)}, \\
&= \mathtt{SUM}_{K+1}^{(2)},
\end{aligned}$$

as wanted. Thus, the proof is complete. □

Theorem 4.6 *For $i, j \geqslant 1$,*

$$\begin{aligned}
&\sum_{d=\max\{i,j\}}^{K} (-1)^{d(\mu s-\lambda r+\lambda j-c_2)+\mu\binom{d+1}{2}+\binom{j-d+n}{n}} \\
&\times U_{\lambda(d+r)^k+\mu(d+s)^m+c_1} \frac{\left(\prod_{t=1}^{d-1} U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{d-1} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{d-i} U_{-\mu(i+s)^m+\mu(t+i+s)^m}\right)\left(\prod_{t=1}^{d-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)} \\
&\times V_{\lambda(d+r)^k-\mu(d+s)^m+c_2} \frac{\left(\prod_{t=1}^{d-1} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{d-1} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{d-i} V_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)\left(\prod_{t=1}^{d} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)} \\
&= (-1)^{\mu\binom{K+1}{2}+K(\mu s-\lambda r-c_2)+K\lambda j} \\
&\times \frac{\left(\prod_{t=1}^{K} U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{U_{\lambda(j+r)^k+\mu(i+s)^m+c_1}\left(\prod_{t=1}^{K-i} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)} \\
&\times \frac{\left(\prod_{t=1}^{K} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{V_{\lambda(j+r)^k-\mu(i+s)^m+c_2}\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{K-i} V_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)}.
\end{aligned}$$

Proof Denote the above sum by $\mathtt{SUM}_K^{(3)}$. If $j \geq i$, the case $K=j$ easily follows. If $i > j$, then

$$\mathtt{SUM}_i^{(3)} = (-1)^{\mu\binom{i+1}{2}+i(\mu s-\lambda r-c_2)+\lambda ij} \times \frac{\left(\prod_{t=1}^{i} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{i-1} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{i} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)} \times \frac{\left(\prod_{t=1}^{i} U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{i-1} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{i-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$= (-1)^{\mu\binom{i+1}{2}+i(\mu s-\lambda r-c_2)+\lambda ij} \frac{\left(\prod_{t=1}^{i} U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)}{U_{\lambda(j+r)^k+\mu(i+s)^m+c_1} V_{\lambda(j+r)^k-\mu(i+s)^m+c_2}} \times \frac{\left(\prod_{t=1}^{i} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)\left(\prod_{t=1}^{i} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{i} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{i-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)\left(\prod_{t=1}^{i} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)}.$$

Thus, the first step of induction is complete. For the next step of induction, we write

$$\begin{aligned}
\mathtt{SUM}_K^{(3)} + S_{K+1} &= (-1)^{\mu\binom{K+1}{2}+K(\mu s-\lambda r-c_2)+K\lambda j} \frac{1}{U_{\lambda(j+r)^k+\mu(i+s)^m+c_1} V_{\lambda(j+r)^k-\mu(i+s)^m+c_2}} \\
&\times \frac{\left(\prod_{t=1}^{K} U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{K-i} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)} \\
&\times \frac{\left(\prod_{t=1}^{K} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{K-i} V_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)} \\
&+ (-1)^{\mu\binom{K+2}{2}+(K+1)(\mu s-\lambda r-c_2)+\lambda j(K+1)} \\
&\times U_{\lambda(K+1+r)^k+\mu(K+1+s)^m+c_1} V_{\lambda(K+1+r)^k-\mu(K+1+s)^m+c_2} \\
&\times \frac{\left(\prod_{t=1}^{K} U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{K-i+1} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K-j+1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)} \\
&\times \frac{\left(\prod_{t=1}^{K} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{K} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{K-i+1} V_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)},
\end{aligned}$$

which, after some simplifications, equals

$$(-1)^{\mu\binom{K+1}{2}+K(\mu s-\lambda r-c_2)+K\lambda j}\frac{\left(\prod_{t=1}^{K}U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{K}U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{K-i}U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K-j}U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$\times\frac{\left(\prod_{t=1}^{K}V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{K}V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{K}V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{K-i}V_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)}$$

$$\times\left[\frac{1}{U_{\lambda(j+r)^k+\mu(i+s)^m+c_1}V_{\lambda(j+r)^k-\mu(i+s)^m+c_2}}+\frac{(-1)^{\mu(K+1)+(\mu s-\lambda r-c_2)+\lambda j}}{U_{\mu(K+1+s)^m-\mu(i+s)^m}U_{\lambda(K+1+r)^k-\lambda(j+r)^k}}\right.$$

$$\left.\times\frac{U_{\lambda(K+1+r)^k+\mu(K+1+s)^m+c_1}V_{\lambda(K+1+r)^k-\mu(K+1+s)^m+c_2}}{V_{\lambda(j+r)^k+\lambda(K+1+r)^k+c_1+c_2}V_{\mu(i+s)^m+\mu(K+1+s)^m+c_1-c_2}}\right].$$

If we take $N=K+1$ in Lemma 4.3, then we write

$$\mathtt{SUM}_K^{(3)}+S_{K+1}=(-1)^{\mu\binom{K+1}{2}+K(\mu s-\lambda r-c_2)+K\lambda j+\mu(K+1)+(\mu s-\lambda r-c_2)+\lambda j}$$

$$\times\frac{\left(\prod_{t=1}^{K}U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{K}U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{K-i}U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K-j}U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$\times\frac{\left(\prod_{t=1}^{K}V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{K}V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{K}V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{K-i}V_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)}$$

$$\times\frac{U_{\mu(i+s)^m+\lambda(K+1+r)^k+c_1}U_{\lambda(j+r)^k+\mu(K+1+s)^m+c_1}}{U_{\lambda(j+r)^k+\mu(i+s)^m+c_1}U_{\mu(K+1+s)^m-\mu(i+s)^m}U_{\lambda(K+1+r)^k-\lambda(j+r)^k}}$$

$$\times\frac{V_{\lambda(K+1+r)^k-\mu(i+s)^m+c_2}V_{\lambda(j+r)^k-\mu(K+1+s)^m+c_2}}{V_{\lambda(j+r)^k-\mu(i+s)^m+c_2}V_{\lambda(j+r)^k+\lambda(K+1+r)^k+c_1+c_2}V_{\mu(i+s)^m+\mu(K+1+s)^m+c_1-c_2}}$$

$$=\frac{(-1)^{\mu\binom{K+2}{2}+(K+1)(\mu s-\lambda r-c_2)+(K+1)\lambda j+\binom{j-K-1+n}{n}}}{U_{\lambda(j+r)^k+\mu(i+s)^m+c_1}V_{\lambda(j+r)^k-\mu(i+s)^m+c_2}}$$

$$\times\frac{\left(\prod_{t=1}^{K+1}U_{\mu(i+s)^m+\lambda(t+r)^k+c_1}\right)\left(\prod_{t=1}^{K+1}U_{\lambda(j+r)^k+\mu(t+s)^m+c_1}\right)}{\left(\prod_{t=1}^{K-i+1}U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K-j+1}U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{K+1} V_{\lambda(t+r)^k-\mu(i+s)^m+c_2}\right)\left(\prod_{t=1}^{K+1} V_{\lambda(j+r)^k-\mu(t+s)^m+c_2}\right)}{\left(\prod_{t=1}^{K+1} V_{\lambda(j+r)^k+\lambda(t+r)^k+c_1+c_2}\right)\left(\prod_{t=1}^{K-i+1} V_{\mu(i+s)^m+\mu(t+i+s)^m+c_1-c_2}\right)}$$

$$= \mathtt{SUM}_{K+1}^{(3)},$$

which completes the proof. □

Proofs of the results of Section 2.

For the matrices L and L^{-1}, it is obvious $L_{ii}L_{ii}^{-1} = 1$. For $i > j$, by Theorem 4.5, we write

$$\sum_{j \le d \le i} L_{id} L_{dj}^{-1} = \mathtt{SUM}_i^{(2)} = 0.$$

Thus, we conclude $\sum_{j \le d \le i} L_{id} L_{dj}^{-1} = \delta_{ij}$, where δ_{ij} is Kronecker delta, as desired.

Here we omit the proof of $UU^{-1} = I$, it could be similarly done by constructing a proper lemma.

For LU-decomposition of the matrix W, we have to prove that

$$\sum_{1 \le d \le \min\{i,j\}} L_{id} U_{dj} = W_{ij}.$$

By Theorem 4.4, we write

$$\sum_{1 \le d \le \min\{i,j\}} L_{id} U_{dj} = \mathtt{SUM}_1^{(1)} = \frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c_1} V_{\lambda(i+r)^k-\mu(j+s)^m+c_2}},$$

which completes the proof.

For the inverse matrix W_n^{-1}, we use the fact $W_n^{-1} = U_n^{-1} L_n^{-1}$. Consider

$$\sum_{\max\{i,j\} \le d \le n} U_{id}^{-1} L_{dj}^{-1}$$

$$= \frac{(-1)^{\lambda r+\lambda j+\mu(i+s)+c_2+1} V_{2\lambda(j+r)^k+c_1+c_2}}{\left(\prod_{t=1}^{i-1} U_{\mu(t+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(t+r)^k-\lambda(j+r)^k}\right)\left(\prod_{t=1}^{i-1} V_{\mu(i+s)^m+\mu(t+s)^m+c_1-c_2}\right)} \times \mathtt{SUM}_n^{(3)}$$

$$= \left(W_n^{-1}\right)_{ij}.$$

Thus, the proofs of the results of Section 2 are now complete.

5. Q-forms of the nonlinear variants of the generalized Filbert and Lilbert matrices

In this section, we present q-forms of the results of Sections 2 and 3. Thus, the results for the matrices W and Z given previously shall come out as corollaries of the results of this section for the special choice of $q = \beta/\alpha$. We omit the proofs not to bore the readers, they could be similarly done by finding q-analogues of Theorems 7–9. Here we would like to note that mechanic summation method or q-Zeilberger algorithm will not work due to the unhypergeometric summand terms.

Denote q-analogues of the matrices W and Z by $\mathcal{W}$ and $\mathcal{Z}$, resp. In that case, we have

$$\mathcal{W}_{ij} = (-1)^{-\lambda(r+i)^k + \frac{1-c_1-c_2}{2}} \frac{q^{\lambda(r+i)^k + \frac{c_1+c_2-1}{2}} (1-q)}{\left(1 - q^{\lambda(i+r)^k + \mu(j+s)^m + c_1}\right)\left(1 + q^{\lambda(i+r)^k - \mu(j+s)^m + c_2}\right)}$$

and

$$\mathcal{Z}_{ij} = (-1)^{-\lambda(r+i)^k - \frac{c_1+c_2}{2}} \frac{q^{\lambda(r+i)^k + \frac{c_1+c_2}{2}}}{\left(1 + q^{\lambda(i+r)^k + \mu(j+s)^m + c_1}\right)\left(1 + q^{\lambda(i+r)^k - \mu(j+s)^m + c_2}\right)},$$

resp.

Now we shall recall a useful definition from [13] on a generalization of the q-Pochhammer symbol with two additional parameters in which one of them is in geometric progression as follows

$$(a;q)_n^{(r,k)} := \left(1 - aq^{(r+1)^k}\right)\left(1 - aq^{(2+r)^k}\right)\dots\left(1 - aq^{(n+r)^k}\right) = \prod_{t=1}^{n}\left(1 - aq^{(t+r)^k}\right),$$

where a is a real number, r is an integer, and n, k are positive integers with $(a;q)_0^{(r,k)} = 1$. Some special cases are

$$\begin{aligned}
(1;q)_n^{(0,2)} &= (1-q)\left(1-q^4\right)\dots\left(1-q^{n^2}\right),\\
\left(a;q^2\right)_n^{(1,2)} &= \left(1-aq^8\right)\left(1-aq^{18}\right)\dots\left(1-aq^{2\times(n+1)^2}\right),\\
(-q;q)_n^{(-1,3)} &= (1+q)\left(1+q^2\right)\left(1+q^9\right)\cdots\left(1+q^{(n-1)^3+1}\right),\\
\left(a;q^\lambda\right)_n^{(0,1)} &= \left(1-aq^\lambda\right)\left(1-aq^{2\lambda}\right)\dots\left(1-aq^{n\lambda}\right) = \left(aq^\lambda;q^\lambda\right)_n.
\end{aligned}$$

In general, we have the relation between the q-Pochhammer symbol and the general q-Pochhammer notation is

$$(x;q)_n = (x;q)_n^{(-1,1)}.$$

As the q-analogue of the results given in Section 2, firstly we give the LU-decomposition and inverse of the matrix $\mathcal{W}$:

Theorem 5.1 *For $i, j \geq 1$,*

$$L_{i,j} = (-q)^{\lambda\left((i+r)^k - (j+r)^k\right)} \frac{\left(q^{\lambda(i+r)^k}; q^{-\lambda}\right)_{j-1}^{(r,k)}}{\left(q^{\lambda(j+r)^k}; q^{-\lambda}\right)_{j-1}^{(r,k)}}$$
$$\times \frac{\left(q^{\lambda(j+r)^k + c_1}; q^\mu\right)_j^{(s,m)} \left(-q^{\lambda(j+r)^k + c_2}; q^{-\mu}\right)_j^{(s,m)} \left(-q^{\lambda(i+r)^k + c_1 + c_2}; q^\lambda\right)_{j-1}^{(r,k)}}{\left(q^{\lambda(i+r)^k + c_1}; q^\mu\right)_j^{(s,m)} \left(-q^{\lambda(i+r)^k + c_2}; q^{-\mu}\right)_j^{(s,m)} \left(-q^{\lambda(j+r)^k + c_1 + c_2}; q^\lambda\right)_{j-1}^{(r,k)}},$$

$$U_{ij}=(-\mathbf{1})^{-\mu(j+s)^m-\frac{c_1-c_2-1}{2}}\,q^{\mu(j+s)^m+\frac{c_1-c_2-1}{2}}\,(1-q)\,\frac{\left(q^{\lambda(i+r)^k};q^{-\lambda}\right)_{i-1}^{(r,k)}}{\left(q^{\lambda(i+r)^k+c_1};q^{\mu}\right)_{i-1}^{(s,m)}}$$

$$\times\frac{\left(q^{\mu(j+s)^m};q^{-\mu}\right)_{i-1}^{(s,m)}\left(-q^{\lambda(i+r)^k+c_1+c_2};q^{\lambda}\right)_{i-1}^{(r,k)}\left(-q^{\mu(j+s)^m+c_1-c_2};q^{\mu}\right)_{i-1}^{(s,m)}}{\left(q^{\mu(j+s)^m+c_1};q^{\lambda}\right)_{i}^{(r,k)}\left(-q^{\lambda(i+r)^k+c_2};q^{-\mu}\right)_{i-1}^{(s,m)}\left(-q^{\mu(j+s)^m-c_2};q^{-\lambda}\right)_{i}^{(r,k)}},$$

$$L_{ij}^{-1}=(-1)^{\lambda i+\lambda ij+\binom{j-i+n}{n}+1+\lambda\left((i-j)(j+r)^k+(1-i+j)(r+i)^k\right)}\,q^{\lambda\left((i-j)(j+r)^k+(1-i+j)(r+i)^k\right)}$$

$$\times\frac{\left(1+q^{2\lambda(j+r)^k+c_1+c_2}\right)\left(-q^{\lambda(j+r)^k+c_2};q^{-\mu}\right)_{i-1}^{(s;m)}\left(q^{\lambda(j+r)^k+c_1};q^{\mu}\right)_{i-1}^{(s;m)}}{\left(1+q^{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}\right)\left(-q^{\lambda(i+r)^k+c_2};q^{-\mu}\right)_{i-1}^{(s;m)}\left(q^{\lambda(i+r)^k+c_1};q^{\mu}\right)_{i-1}^{(s;m)}}$$

$$\times\frac{\left(-q^{\lambda(i+r)^k+c_1+c_2};q^{\lambda}\right)_{i-1}^{(r;k)}\left(q^{\lambda(i+r)^k};q^{-\lambda}\right)_{j-1}^{(r;k)}\left(q^{-\lambda(i+r)^k};q^{\lambda}\right)_{i-j-1}^{(j+r;k)}}{\left(-q\lambda\,(j+r)^k+c_1+c_2;q^{\lambda}\right)_{i-1}^{(r;k)}\left(q^{\lambda(j+r)^k};q^{-\lambda}\right)_{j-1}^{(r;k)}\left(q^{-\lambda(j+r)^k};q^{\lambda}\right)_{i-j-1}^{(j+r;k)}},$$

$$U_{ij}^{-1}=(-\mathbf{1})^{\lambda(i+r)^k+\mu(j+s)^m+(\lambda+\mu)(i+j)+c_2+\mu(i+s)^m+\frac{c_1-c_2-1}{2}}\,\frac{q^{-\mu(i+s)^m-\frac{c_1-c_2-1}{2}}}{(1-q)}$$

$$\times\frac{\left(q^{\mu(i+s)^m+c_1};q^{\lambda}\right)_{j-1}^{(r,k)}\left(q^{\lambda(j+r)^k+c_1};q^{\mu}\right)_{j}^{(s,m)}}{\left(q^{-\lambda(j+r)^k};q^{\lambda}\right)_{j-1}^{(r,k)}\left(q^{-\mu(i+s)^m};q^{\mu}\right)_{j-i}^{(i+s,m)}\left(q^{-\mu(i+s)^m};q^{\mu}\right)_{i-1}^{(s,m)}}$$

$$\times\frac{\left(-q^{-\mu(i+s)^m+c_2};q^{\lambda}\right)_{j-1}^{(r,k)}\left(-q^{-\lambda(j+r)^k-c_2};q^{\mu}\right)_{j}^{(s,m)}}{\left(-q^{\mu(i+s)^m+c_1-c_2};q^{\mu}\right)_{i-1}^{(s,m)}\left(-q^{\lambda(j+r)^k+c_1+c_2};q^{\lambda}\right)_{j-1}^{(r,k)}\left(-q^{\mu(i+s)^m+c_1-c_2};q^{\mu}\right)_{j-i}^{(i+s,m)}}$$

and

$$\left(\mathcal{W}_n^{-1}\right)_{ij}=(-\mathbf{1})^{\lambda(r+j)+\mu(i+s)+c_2+\mu(i+s)^m+\frac{c_1-c_2-1}{2}}\,q^{-\mu(i+s)^m-\frac{c_1-c_2-1}{2}}\,\frac{1}{1-q}$$

$$\times\left(1+q^{2\lambda(j+r)^k+c_1+c_2}\right)\left(1+q^{-\lambda(j+r)^k+\mu(i+s)^m-c_2}\right)\left(-q^{-\lambda(j+r)^k-c_2};q^{\mu}\right)_{i-1}^{(s,m)}$$

$$\times\frac{\left(-q^{-\mu(i+s)^m+c_2};q^{\lambda}\right)_{j-1}^{(r,k)}\left(-q^{-\mu(i+s)^m+c_2};q^{\lambda}\right)_{n-j}^{(j+r,k)}}{\left(-q^{\mu(i+s)^m+c_1-c_2};q^{\mu}\right)_{i-1}^{(s,m)}\left(-q^{\mu(i+s)^m+c_1-c_2};q^{\mu}\right)_{n-i}^{(i+s,m)}}$$

$$\times\frac{\left(q^{\lambda(j+r)^k+c_1};q^{\mu}\right)_{i-1}^{(s,m)}\left(-q^{-\lambda(j+r)^k-c_2};q^{\mu}\right)_{n-i}^{(i+s,m)}}{\left(q^{-\mu(i+s)^m};q^{\mu}\right)_{i-1}^{(s,m)}\left(-q^{\lambda(j+r)^k+c_1+c_2};q^{\lambda}\right)_{n}^{(r,k)}}$$

$$\times\frac{\left(q^{\lambda(j+r)^k+c_1};q^{\mu}\right)_{n-i}^{(i+s,m)}\left(q^{\mu(i+s)^m+c_1};q^{\lambda}\right)_{n}^{(r,k)}}{\left(q^{-\mu(i+s)^m};q^{\mu}\right)_{n-i}^{(i+s,m)}\left(q^{-\lambda(j+r)^k};q^{\lambda}\right)_{j-1}^{(r,k)}\left(q^{-\lambda(j+r)^k};q^{\lambda}\right)_{n-j}^{(j+r,k)}}.$$

As the q-analogue of the results given in Section 3, we present the results for the matrix $\mathcal{Z}$:

Theorem 5.2 *For* $i, j \geq 1$,

$$L_{ij} = (-q)^{\lambda\left((i+r)^k-(j+r)^k\right)} \frac{\left(q^{\lambda(i+r)^k}; q^{-\lambda}\right)_{j-1}^{(r,k)}}{\left(q^{\lambda(j+r)^k}; q^{-\lambda}\right)_{j-1}^{(r,k)}}$$

$$\times \frac{\left(q^{\lambda(i+r)^k+c_1+c_2}; q^{\lambda}\right)_{j-1}^{(r,k)} \left(-q^{\lambda(j+r)^k+c_2}; q^{-\mu}\right)_{j}^{(s,m)} \left(-q^{\lambda(j+r)^k+c_1}; q^{\mu}\right)_{j}^{(s,m)}}{\left(q^{\lambda(j+r)^k+c_1+c_2}; q^{\lambda}\right)_{j-1}^{(r,k)} \left(-q^{\lambda(i+r)^k+c_2}; q^{-\mu}\right)_{j}^{(s,m)} \left(-q^{\lambda(i+r)^k+c_1}; q^{\mu}\right)_{j}^{(s,m)}},$$

$$U_{ij} = (-1)^{\lambda(i+r)+c_2+(c_2-c_1)/2} q^{\mu(j+s)^m-(c_2-c_1)/2} \frac{\left(q^{\mu(j+s)^m}; q^{-\mu}\right)_{i-1}^{(s,m)}}{\left(-q^{\lambda(i+r)^k+c_2}; q^{-\mu}\right)_{i-1}^{(s,m)}}$$

$$\times \frac{\left(q^{\lambda(i+r)^k+c_1+c_2}; q^{\lambda}\right)_{i-1}^{(r,k)} \left(q^{\lambda(i+r)^k}; q^{-\lambda}\right)_{i-1}^{(r,k)} \left(q^{\mu(j+s)^m+c_1-c_2}; q^{\mu}\right)_{i-1}^{(s,m)}}{\left(-q^{\mu(j+s)^m+c_1}; q^{\lambda}\right)_{i}^{(r,k)} \left(-q^{\lambda(i+r)^k+c_1}; q^{\mu}\right)_{i-1}^{(s,m)} \left(-q^{\mu(j+s)^m-c_2}; q^{-\lambda}\right)_{i}^{(r,k)}},$$

$$L_{ij}^{-1} = (-1)^{\lambda i(j+1)+\binom{j-i+n}{n}+1+\lambda(i-1)\left((j+r)^k-(i+r)^k\right)} q^{-\lambda(i-1)\left((j+r)^k-(i+r)^k\right)}$$

$$\times \frac{(1-q^{2\lambda(j+r)^k+c_1+c_2}) \left(-q^{\lambda(j+r)^k+c_2}; q^{-\mu}\right)_{i-1}^{(s;m)} \left(-q^{\lambda(j+r)^k+c_1}; q^{\mu}\right)_{i-1}^{(s;m)}}{(1-q^{\lambda(i+r)^k+\lambda(j+r)^k+c_1+c_2}) \left(-q^{\lambda(i+r)^k+c_2}; q^{-\mu}\right)_{i-1}^{(s;m)} \left(-q^{\lambda(i+r)^k+c_1}; q^{\mu}\right)_{i-1}^{(s;m)}}$$

$$\times \frac{\left(q^{-\lambda(i+r)^k}; q^{\lambda}\right)_{j-1}^{(r;k)} \left(q^{-\lambda(i+r)^k}; q^{\lambda}\right)_{i-j-1}^{(j+r;k)} \left(q^{\lambda(i+r)^k+c_1+c_2}; q^{\lambda}\right)_{i-1}^{(r;k)}}{\left(q^{-\lambda(j+r)^k}; q^{\lambda}\right)_{j-1}^{(r;k)} \left(q^{-\lambda(j+r)^k}; q^{\lambda}\right)_{i-j-1}^{(j+r;k)} \left(q^{\lambda(j+r)^k+c_1+c_2}; q^{\lambda}\right)_{i-1}^{(r;k)}},$$

$$U_{ij}^{-1} = (-\mathbf{1})^{\lambda(i+r)^k+\mu(j+s)^m+(\lambda+\mu)(i+j)+c_2+\mu(i+s)^m+\frac{c_1-c_2}{2}} q^{-\mu(i+s)^m-\frac{c_1-c_2}{2}}$$

$$\times \frac{\left(-q^{\mu(i+s)^m+c_1}; q^{\lambda}\right)_{j-1}^{(r,k)} \left(-q^{\lambda(j+r)^k+c_1}; q^{\mu}\right)_{j}^{(s,m)}}{\left(q^{-\lambda(j+r)^k}; q^{\lambda}\right)_{j-1}^{(r,k)} \left(q^{-\mu(i+s)^m}; q^{\mu}\right)_{j-i}^{(i+s,m)} \left(q^{-\mu(i+s)^m}; q^{\mu}\right)_{i-1}^{(s,m)}}$$

$$\times \frac{\left(-q^{-\mu(i+s)^m+c_2}; q^{\lambda}\right)_{j-1}^{(r,k)} \left(-q^{-\lambda(j+r)^k-c_2}; q^{\mu}\right)_{j}^{(s,m)}}{\left(q^{\lambda(j+r)^k+c_1+c_2}; q^{\lambda}\right)_{j-1}^{(r,k)} \left(q^{\mu(i+s)^m+c_1-c_2}; q^{\mu}\right)_{i-1}^{(s,m)} \left(q^{\mu(i+s)^m+c_1-c_2}; q^{\mu}\right)_{j-i}^{(i+s,m)}}$$

and

$$\left(\mathcal{Z}_n^{-1}\right)_{ij} = (-1)^{\lambda(r+j)+\frac{c_1+c_2}{2}} q^{-\mu(i+s)^m-\frac{c_1-c_2}{2}}$$
$$\times \frac{1-q^{2\lambda(j+r)^k+c_1+c_2}}{\left(1+q^{\lambda(j+r)^k+\mu(i+s)^m+c_1}\right)\left(1+q^{\lambda(j+r)^k-\mu(i+s)^m+c_2}\right)}$$
$$\times \frac{\left(-q^{\mu(i+s)^m+c_1};q^{\lambda}\right)_n^{(r,k)}\left(-q^{\lambda(j+r)^k+c_1};q^{\mu}\right)_n^{(s,m)}}{\left(q^{-\mu(i+s)^m};q^{\mu}\right)_{n-i}^{(i+s,m)}\left(q^{-\lambda(j+r)^k};q^{\lambda}\right)_{n-j}^{(j+r,k)}\left(q^{-\mu(i+s)^m};q^{\mu}\right)_{i-1}^{(s,m)}\left(q^{-\lambda(j+r)^k};q^{\lambda}\right)_{j-1}^{(r,k)}}$$
$$\times \frac{\left(-q^{-\mu(i+s)^m+c_2};q^{\lambda}\right)_n^{(r,k)}\left(-q^{-\lambda(j+r)^k-c_2};q^{\mu}\right)_n^{(s,m)}}{\left(q^{\mu(i+s)^m+c_1-c_2};q^{\mu}\right)_{n-i}^{(i+s,m)}\left(q^{\lambda(j+r)^k+c_1+c_2};q^{\lambda}\right)_n^{(r,k)}\left(q^{\mu(i+s)^m+c_1-c_2};q^{\mu}\right)_{i-1}^{(s,m)}}.$$

Thus, the results of Sections 2 and 3 are now consequence of Theorems 10 and 11 when $q=\beta/\alpha$, resp. Especially when $q=\left(1-\sqrt{5}\right)/\left(1+\sqrt{5}\right)$, the results of Sections 2 and 3 turns the case $p=1$ which gives us the results related with the usual Fibonacci and Lucas numbers.

As a final remark, if we let $q\to 1$, then the entries of the matrix $\mathcal{W}$ takes the form

$$\lim_{q\to 1}\mathcal{W}_{ij} = \frac{(-1)^{-\lambda(r+i)^k+\frac{1-c_1-c_2}{2}}}{\left(\lambda(i+r)^k+\mu(j+s)^m+c_1\right)\left(\lambda(i+r)^k-\mu(j+s)^m+c_2\right)}.$$

Since the sign function is separable with regard to the variables i and j, clearly depend only on i, and using some algebraic manipulations and Theorem 10, we can obtain similar results for the matrix $\tilde{\mathcal{W}}$ whose entries are given as

$$\tilde{\mathcal{W}}_{ij} = \frac{1}{\left(\lambda(i+r)^k+\mu(j+s)^m+c_1\right)\left(\lambda(i+r)^k-\mu(j+s)^m+c_2\right)},$$

which is a nonlinear variant of the Hilbert matrix.

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