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A class of warped product submanifolds of Kenmotsu manifolds

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Abstract: In 2018 Naghi et al. studied warped product skew CR-submanifold of the form $M = M_1 \times_f M_\perp$ of a Kenmotsu manifold $\bar{M}$ (throughout the paper), where $M_1 = M_T \times M_\theta$ and $M_T, M_\perp, M_\theta$ represents invariant, antiinvariant, proper slant submanifold of $\bar{M}$. Next, in 2019 Hui et al. studied another class of warped product skew CR-submanifold of the form $M = M_2 \times_f M_T$ of $\bar{M}$, where $M_2 = M_\perp \times M_\theta$. The present paper deals with the study of a class of warped product submanifold of the form $M = M_3 \times_f M_\theta$ of $\bar{M}$, where $M_3 = M_T \times M_\perp$ and $M_T, M_\perp, M_\theta$ represents invariant, antiinvariant and proper pointwise slant submanifold of $\bar{M}$. A characterization is given on the existence of such warped product submanifolds which generalizes the characterization of warped product contact CR-submanifolds of the form $M_\perp \times_f M_T$, studied by Uddin et al. in 2017 and also generalizes the characterization of warped product semi-slant submanifolds of the form $M_T \times_f M_\theta$, studied by Uddin in the same year. Beside that some inequalities on the squared norm of the second fundamental form are obtained which are also generalizations of the inequalities obtained in the just above two mentioned papers respectively.

Key words: Kenmotsu manifold, pointwise slant submanifolds, warped product submanifolds

1. Introduction

The third class of Tanno's classification [30] is characterized by Kenmotsu [20]. This class is known as Kenmotsu manifold. We refer the reader to [13–15] for further study.

The concept of slant submanifolds in a Hermitian manifold was initiated in [7]. Then Lotta [23] defined and studied slant immersions of a Riemannian manifold into an almost contact metric manifold. As a natural generalization of slant submanifold, Etayo [11] defined pointwise slant submanifolds under the name of quasislant submanifolds. Pointwise slant submanifolds in almost contact metric manifolds were studied in [24, 28].

As a generalization of Riemannian product manifold, Bishop and O'Neill [5] defined warped product manifolds. The warped product submanifold was initiated in [8–10]. Then many authors studied warped product submanifolds of different ambient manifolds, see [16, 17, 19]. Warped product submanifolds of Kenmotsu manifolds are studied in ([1–3], [21, 22, 25, 26], [32–35]).

In [29] Sahin studied skew CR-warped product submanifolds of Kaehler manifolds. Then Tastan [31] studied warped product skew semiinvariant submanifolds of order 1 of a locally product Riemannian manifold.

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Recently in [27], warped product skew CR-submanifold of the form $M = M_1 \times_f M_\perp$ of $\bar{M}$ has been studied, where $M_1 = M_T \times M_\theta$ and M_T, $M_\perp$, M_θ stands for an invariant, an antiinvariant, and a proper slant submanifold of $\bar{M}$. Moreover, in [18], a warped product submanifold of $\bar{M}$ of the form $M = M_2 \times_f M_T$ where $M_2 = M_\perp \times M_\theta$ is studied. Following the same, here we have considered the warped product submanifold of $\bar{M}$ of the form $M = M_3 \times_f M_\theta$, where $M_3 = M_T \times M_\perp$ and M_T, $M_\perp$, M_θ represents invariant, antiinvariant, and proper pointwise slant submanifolds of $\bar{M}$, respectively. Section 2 deals with some preliminaries of almost contact metric manifolds and submanifolds. In Sections 3 and 4, we have studied respectively submanifolds and warped product submanifolds of $\bar{M}$. We have characterized warped product submanifolds of said form in Section 5. In the last section two generalized inequalities of the squared norm of the second fundamental form are obtained.

2. Preliminaries

An odd dimensional smooth manifold $\bar{M}^{2m+1}$ is said to be an almost contact metric manifold [4] if it admits a $(1,1)$ tensor field φ, a vector field ξ, an 1-form η, and a Riemannian metric g such that

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on $\bar{M}$.

An almost contact metric manifold $\bar{M}^{2m+1}(\varphi, \xi, \eta, g)$ is said to be Kenmotsu manifold [20] if:

$$\bar{\nabla}_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\bar{\nabla}_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.5}$$

where $\bar{\nabla}$ denotes the Riemannian connection of g.

Let M be an n-dimensional submanifold of $\bar{M}$. Let ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M respectively. Then the Gauss and Weingarten formulae are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.6}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{2.7}$$

where h and A_V are second fundamental form and the shape operator such that $g(h(X,Y), V) = g(A_V X, Y)$ for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where g is the Riemannian metric on $\bar{M}$ as well as on M.

The mean curvature H of M is given by $H = \frac{1}{n}\text{trace}\, h$. A submanifold M of $\bar{M}$ is said to be totally umbilical if $h(X,Y) = g(X,Y)H$ for any $X, Y \in \Gamma(TM)$. If $h(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$, then M is totally geodesic and if $H = 0$ then M is minimal in $\bar{M}$.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of the tangent bundle TM and $\{e_{n+1}, \cdots, e_{2m+1}\}$ be that of the normal bundle $T^{\perp}M$. Set

$$h_{ij}^r = g(h(e_i, e_j), e_r) \text{ and } \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)), \tag{2.8}$$

for $i, j \in \{1, \cdots, n\}$ and $r \in \{n+1, \cdots, 2m+1\}$. For $f \in C^{\infty}(M)$, the gradient $\boldsymbol{\nabla} f$ is defined by

$$g(\boldsymbol{\nabla} f, X) = Xf \tag{2.9}$$

for any $X \in \Gamma(TM)$. As a consequence, we get

$$\|\boldsymbol{\nabla} f\|^2 = \sum_{i=1}^{n} (e_i(f))^2. \tag{2.10}$$

For any $X \in TM$, we write

$$\varphi X = PX + QX. \tag{2.11}$$

Here $PX = tan(\varphi X)$ and $QX = nor(\varphi X)$. Similarly, for any $N \in T^{\perp}M$, we write

$$\varphi N = bN + cN \tag{2.12}$$

where $bN = tan(\varphi N)$ and $cN = nor(\varphi N)$.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be slant if for each nonzero vector $X \in T_pM$, the angle θ between φX and T_pM is constant, i.e. it does not depend on the choice of $p \in M$ and $X \in T_pM$.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be pointwise slant [11] if for any nonzero vector $X \in T_pM$ at $p \in M$, such that X is not proportional to ξ_p, the angle $\theta(X)$ between φX and $T_p^*M = T_pM - \{0\}$ is independent of the choice of nonzero $X \in T_p^*M$.

For pointwise slant submanifold, θ is a function on M, which is known as slant function on M. Invariant and antiinvariant submanifolds are particular cases of pointwise slant submanifolds with slant function $\theta = 0$ and $\frac{\pi}{2}$, respectively. Also a pointwise slant submanifold M will be slant if and only if θ is constant on M. Thus, a pointwise slant submanifold is proper if neither $\theta = 0, \frac{\pi}{2}$ nor constant. It may be noted that M is pointwise slant [24] if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.13}$$

Furthermore, $\lambda = \cos^2 \theta$ for slant function θ. If M is a pointwise slant submanifold of $\bar{M}$, then [33]:

$$bQX = \sin^2 \theta\{-X + \eta(X)\xi\}, \quad cQX = -QPX. \tag{2.14}$$

The warped product [5] between two Riemannian manifolds (N_1, g_1) and (N_2, g_2) is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2), \tag{2.15}$$

where π_1 and π_2 are canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively and $\pi_i^*(g_i)$ is the pullback of g_i via π_i for $i = 1,\ 2$ and $f \in C^\infty(M)$. A warped product manifold $N_1 \times_f N_2$ is said to be trivial if f is constant. For $M = N_1 \times_f N_2$, we have [5]

$$\nabla_U X = \nabla_X U = (X \ln f)U \tag{2.16}$$

for any $X \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$.

3. Submanifolds of $\bar{M}$

We consider a submanifold M of $\bar{M}$ such that

$$TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \{\xi\},$$

where $\mathcal{D}^T$, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ are mutually orthogonal distributions such that $\mathcal{D}^T$ is invariant, $\mathcal{D}^\perp$ is antiinvariant and $\mathcal{D}^\theta$ is pointwise slant with slant function θ. Then the normal bundle $T^\perp M$ can be written as

$$T^\perp M = \varphi\mathcal{D}^\perp \oplus Q\mathcal{D}^\theta \oplus \nu,$$

where ν is a φ-invariant normal subbundle of $T^\perp M$.

Now for the sake of further study we obtain the following useful results.

Lemma 3.1 *Let M be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$ and ξ is orthogonal to $\mathcal{D}^\theta$. Then the following relations hold:*

$$\sin^2\theta g(\nabla_X Y, U) = g(h(X, \varphi Y), QU) - g(h(X, Y), QPU), \tag{3.1}$$

$$\cos^2\theta g(\nabla_X Z, U) = g(h(X, Z), QPU) - g(h(X, PU), \varphi Z), \tag{3.2}$$

$$\sin^2\theta g(\nabla_Z X, U) = g(h(Z, \varphi X), QU) - g(h(X, Z), QPU), \tag{3.3}$$

$$\cos^2\theta g(\nabla_Z W, U) = g(h(Z, PU), \varphi W) - g(h(Z, W), QPU) \tag{3.4}$$

for all $X, Y \in \Gamma(\mathcal{D}^T)$, $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$.

Proof For any $X, Y \in \Gamma(\mathcal{D}^T)$, $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$, we have from (2.3) and (2.11) that

$$\begin{aligned} g(\nabla_X Y, U) &= g(\bar{\nabla}_X \varphi Y, \varphi U) - g((\bar{\nabla}_X \varphi)Y, \varphi U) \\ &= g(\bar{\nabla}_X \varphi Y, PU) + g(\bar{\nabla}_X \varphi Y, QU) \\ &= g(\bar{\nabla}_X P^2 U, Y) + g(\bar{\nabla}_X QPU, Y) + g(\bar{\nabla}_X \varphi Y, QU). \end{aligned}$$

Using (2.13) in the above equation, we obtain

$$\begin{aligned} g(\nabla_X Y, U) &= -\cos^2\theta g(\bar{\nabla}_X U, Y) + \sin 2\theta X(\theta) g(U, Y) \\ &+ g(\bar{\nabla}_X QPU, Y) + g(\bar{\nabla}_X \varphi Y, QU). \end{aligned} \tag{3.5}$$

Using (2.6) and (2.7) in (3.5), we get (3.1).
Moreover,

$$\begin{aligned} g(\nabla_X Z, U) &= g(\bar{\nabla}_X \varphi Z, \varphi U) - g((\bar{\nabla}_X \varphi) Z, \varphi U) \\ &= g(\bar{\nabla}_X \varphi Z, PU) + g(\bar{\nabla}_X \varphi Z, QU) \\ &= g(\bar{\nabla}_X \varphi Z, PU) + g(\bar{\nabla}_X bQU, Z) + g(\bar{\nabla}_X cQU, Z). \end{aligned}$$

By virtue of (2.14) the above relation yields

$$\begin{aligned} g(\nabla_X Z, U) &= g(\bar{\nabla}_X \varphi Z, PU) - \sin^2\theta g(\bar{\nabla}_X U, Z) \\ &\quad - \sin 2\theta X(\theta) g(U, Z) - g(\bar{\nabla}_X QPU, Z). \end{aligned} \tag{3.6}$$

Using (2.7) in (3.6), we get (3.2).
Again we have

$$\begin{aligned} g(\nabla_Z X, U) &= g(\bar{\nabla}_Z \varphi X, \varphi U) - g((\bar{\nabla}_Z \varphi) X, \varphi U) \\ &= g(\bar{\nabla}_Z \varphi X, PU) + g(\bar{\nabla}_Z \varphi X, QU) \\ &= g(\bar{\nabla}_Z P^2 U, X) + g(\bar{\nabla}_Z QPU, X) + g(\bar{\nabla}_Z \varphi X, QU). \end{aligned}$$

Using (2.13) in the above relation, we find

$$\begin{aligned} g(\nabla_Z X, U) &= -\cos^2\theta g(X, \bar{\nabla}_Z U) + \sin 2\theta Z(\theta) g(U, X) \\ &\quad + g(\bar{\nabla}_Z QPU, X) + g(\bar{\nabla}_Z \varphi X, QU). \end{aligned} \tag{3.7}$$

Thus, (3.3) follows from (3.7).
Moreover,

$$\begin{aligned} g(\nabla_Z W, U) &= g(\bar{\nabla}_Z \varphi W, PU) + g(\bar{\nabla}_Z \varphi W, QU) - g((\bar{\nabla}_Z \varphi) W, \varphi U) \\ &= g(\bar{\nabla}_Z \varphi W, PU) + g(\bar{\nabla}_Z bQU, W) + g(\bar{\nabla}_Z cQU, W). \end{aligned}$$

By virtue of (2.14), the above relation yields

$$\begin{aligned} g(\nabla_Z W, U) &= g(\bar{\nabla}_Z \varphi W, PU) - \sin^2\theta g(\bar{\nabla}_Z U, W) \\ &\quad - \sin 2\theta Z(\theta) g(U, W) - g(\bar{\nabla}_Z QPU, W). \end{aligned} \tag{3.8}$$

From above (3.4) follows. □

Lemma 3.2 *Let M be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$ and ξ is orthogonal to $\mathcal{D}^\theta$. Then the relations*

$$\begin{aligned} \cos^2\theta g(\nabla_U V, Z) &= g(h(U, PV), \varphi Z) - g(h(U, Z), QPV) \\ &\quad - \cos^2\theta \eta(Z) g(U, V) \end{aligned} \tag{3.9}$$

and

$$\sin^2\theta g(\nabla_U V, X) = g(h(X,U), QPV) - g(h(U,\varphi X), QV) - \sin^2\theta\eta(X)g(U,V) \tag{3.10}$$

hold for all $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$ *and* $U,\ V \in \Gamma(\mathcal{D}^\theta)$.

Proof For any $U, V \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have from (2.3) and (2.11) that

$$\begin{aligned} g(\nabla_U V, Z) &= g(\bar\nabla_U \varphi V, \varphi Z) - g((\bar\nabla_U \varphi)V, \varphi Z) - \eta(Z)g(U,V) \\ &= g(\bar\nabla_U PV, \varphi Z) + g(\bar\nabla_U QV, \varphi Z) - \eta(Z)g(U,V) \\ &= g(\bar\nabla_U PV, \varphi Z) - g(\bar\nabla_U \varphi QV, Z) + g((\bar\nabla_U \varphi)QV, Z) - g(U,V)\eta(Z), \\ &= g(\bar\nabla_U PV, \varphi Z) - g(\bar\nabla_U bQV, Z) - g(\bar\nabla_U cQV, Z) - \cos^2\theta\eta(Z)g(U,V). \end{aligned}$$

Using (2.14) in the above relation, we get

$$g(\nabla_U V, Z) = g(\bar\nabla_U PV, \varphi Z) + \sin^2\theta g(\bar\nabla_U V, Z) + \sin 2\theta U(\theta)g(V,Z) + g(\bar\nabla_U QPV, Z) - \cos^2\theta\eta(Z)g(U,V). \tag{3.11}$$

By use of (2.6) and (2.7) in (3.11), we get (3.9).
Also for $U, V \in \Gamma(\mathcal{D}^\theta)$ and $X \in \Gamma(\mathcal{D}^T)$, we have from (2.3) and (2.11) that

$$\begin{aligned} g(\nabla_U V, X) &= g(\bar\nabla_U PV, \varphi X) + g(\bar\nabla_U QV, \varphi X) - \eta(X)g(U,V) \\ &= -g(\bar\nabla_U P^2 V, X) - g(\bar\nabla_U QPV, X) + g(\bar\nabla_U QV, \varphi X) - \sin^2\theta\eta(X)g(U,V). \end{aligned}$$

Using (2.13) in the last relation, we obtain

$$g(\nabla_U V, X) = \cos^2\theta g(\bar\nabla_U V, X) - \sin 2\theta U(\theta)g(V,X) - g(\bar\nabla_U QPV, X) + g(\bar\nabla_U QV, \varphi X). \tag{3.12}$$

Thus, (3.10) follows from (3.12). □

4. Warped product submanifolds of $\bar M$

In this section we study warped product submanifolds $M = M_3 \times_f M_\theta$ of $\bar M$ such that $M_3 = M_T \times M_\perp$ and ξ is tangent to M_3, where M_T, $M_\perp$ and M_θ stands for invariant, antiinvariant, and proper pointwise-slant submanifolds of $\bar M$ respectively. We now construct an example of such warped product submanifold of $\bar M$ for showing the existence.

Example 4.1 *Consider the Euclidean 13-space* $\mathbb{R}^{13}$ *with its cartesian coordinates* $(x_1, y_1, x_2, y_2, ..., x_6, y_6, t)$ *and the almost contact structure* (φ, ξ, η, g) *given by*

$$\varphi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i},\quad \varphi(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j} \text{ and } \varphi(\frac{\partial}{\partial t}) = 0,\ 1 \le i,\ j \le 6.$$

Then it is clear that $\mathbb{R}^{13}$ *is an almost contact metric manifold with respect to the Euclidean metric tensor of* $\mathbb{R}^{13}$.
Consider a submanifold M *of* $\mathbb{R}^{13}$ *defined by the immersion* χ *as follows:*
$\chi(u, v, \theta, \varphi, r, s, t)$
$= (u\cos\theta,\ u\sin\theta,\ v\cos\theta,\ v\sin\theta,\ u\cos\varphi,\ u\sin\varphi,\ v\cos\varphi, v\sin\varphi,\ 2\theta+3\varphi,\ 3\theta+2\varphi,\ r,\ s,\ t).$
The local orthonormal frame of TM *is spanned by the following:*

$$\begin{aligned}
Z_1 &= \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial y_1} + \cos\varphi\frac{\partial}{\partial x_3} + \sin\varphi\frac{\partial}{\partial y_3},\\
Z_2 &= \cos\theta\frac{\partial}{\partial x_2} + \sin\theta\frac{\partial}{\partial y_2} + \cos\varphi\frac{\partial}{\partial x_4} + \sin\varphi\frac{\partial}{\partial y_4},\\
Z_3 &= -u\sin\theta\frac{\partial}{\partial x_1} + u\cos\theta\frac{\partial}{\partial y_1} - v\sin\theta\frac{\partial}{\partial x_2} + v\cos\theta\frac{\partial}{\partial y_2} + 2\frac{\partial}{\partial x_5} + 3\frac{\partial}{\partial y_5},\\
Z_4 &= -u\sin\varphi\frac{\partial}{\partial x_3} + u\cos\varphi\frac{\partial}{\partial y_3} - v\sin\varphi\frac{\partial}{\partial x_4} + v\cos\varphi\frac{\partial}{\partial y_4} + 3\frac{\partial}{\partial x_5} + 2\frac{\partial}{\partial y_5},
\end{aligned}$$

$$Z_5 = \frac{\partial}{\partial x_6}, \quad Z_6 = \frac{\partial}{\partial y_6}, \ \text{and} \quad Z_7 = \frac{\partial}{\partial t}.$$

Moreover, we have

$$\begin{aligned}
\varphi Z_1 &= \cos\theta\frac{\partial}{\partial y_1} - \sin\theta\frac{\partial}{\partial x_1} + \cos\varphi\frac{\partial}{\partial y_3} - \sin\varphi\frac{\partial}{\partial x_3},\\
\varphi Z_2 &= \cos\theta\frac{\partial}{\partial y_2} - \sin\theta\frac{\partial}{\partial x_2} + \cos\varphi\frac{\partial}{\partial y_4} - \sin\varphi\frac{\partial}{\partial x_4},\\
\varphi Z_3 &= -u\sin\theta\frac{\partial}{\partial y_1} - u\cos\theta\frac{\partial}{\partial x_1} - v\sin\theta\frac{\partial}{\partial y_2} - v\cos\theta\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial y_5} - 3\frac{\partial}{\partial x_5},\\
\varphi Z_4 &= -u\sin\varphi\frac{\partial}{\partial y_3} - u\cos\varphi\frac{\partial}{\partial x_3} - v\sin\varphi\frac{\partial}{\partial y_4} - v\cos\varphi\frac{\partial}{\partial x_4} + 3\frac{\partial}{\partial y_5} - 2\frac{\partial}{\partial x_5},
\end{aligned}$$

$$\varphi Z_5 = \frac{\partial}{\partial y_6} \quad \text{and} \quad \varphi Z_6 = -\frac{\partial}{\partial x_6}.$$

We define $D^{\perp} = \{Z_1, Z_2\}$, $D^{\theta} = span\{Z_3, Z_4\}$ *and* $D^T = span\{Z_5, Z_6\}$. *Clearly* D^T *is invariant,* $\mathcal{D}^{\perp}$ *is antiinvariant and* D^{θ} *is pointwise-slant with slant function* $\cos^{-1}\frac{5}{u^2+v^2+13}$. *Moreover, it is clear that for* $\xi = \frac{\partial}{\partial t}$, $D^T \oplus \{\xi\}$, $D^{\perp}$ *and* D^{θ} *are integrable. If we denote the integral manifolds of* D^T, $D^{\perp}$, D^{θ} *by* M_T, $M_{\perp}$, M_{θ} *respectively and we write* $M_3 = M_T \times M_{\theta}$, *then the metric tensor* g_M *of* M *is given by*

$$\begin{aligned}
g_M &= 2(du^2 + dv^2) + dr^2 + ds^2 + dt^2 + (u^2 + v^2 + 13)(d\theta^2 + d\varphi^2)\\
&= g_{M_3} + (u^2 + v^2 + 13)(d\theta^2 + d\varphi^2).
\end{aligned}$$

Hence, $M = M_3 \times_f M_{\theta}$ *is a warped product submanifold of* $\bar{M}$ *with the warping function* $f = \sqrt{u^2 + v^2 + 13}$.

Next we prove the following lemmas for our further study.

Lemma 4.2 *Let $M = M_3 \times_f M_\theta$ be a warped product submanifold of $\bar{M}$ such that $\xi \in M_3$, where $M_3 = M_T \times M_\perp$, then the following relations hold:*

$$\begin{aligned} \xi lnf &= 1, &(4.1)\\ g(h(X,Y),QU) &= 0, &(4.2)\\ g(h(X,U),\varphi Z) &= g(h(X,Z),QU) = 0, &(4.3)\\ g(h(Z,PU),\varphi W) &= g(h(Z,W),QPU) &(4.4)\end{aligned}$$

for $X,\ Y \in \Gamma(M_T), Z,\ W \in \Gamma(M_\perp)$ and $U,\ V \in \Gamma(M_\theta)$.

Proof Relation (4.1) is already proved in [27].
For $X,\ Y \in \Gamma(M_T)$ and $U \in \Gamma(M_\theta)$, we find

$$g(h(\varphi X,Y),QU) = g(\nabla_X U,\varphi Y) + g(\nabla_X PU,Y). \tag{4.5}$$

By virtue of (2.16), (4.5) yields (4.2).
Also we find

$$g(h(X,U),\varphi Z) = -g(\nabla_U \varphi X,Z) + g((\nabla_U \varphi)X,Z). \tag{4.6}$$

Using (2.5) and (2.16) in (4.6), we get

$$g(h(X,U),\varphi Z) = 0. \tag{4.7}$$

Again we calculate

$$g(h(X,U),\varphi Z) = -g(\nabla_X PU,Z) - g(\nabla_X QU,Z). \tag{4.8}$$

By virtue of (2.7) and (2.16), (4.8) yields

$$g(h(X,U),\varphi Z) = g(h(X,Z),QU). \tag{4.9}$$

Thus, (4.3) follows from (4.7) and (4.9). Next we find

$$g(h(Z,U),\varphi W) = -g(\nabla_Z PU,W) - g(\nabla_Z QU,W). \tag{4.10}$$

Using (2.7) and (2.16) in (4.10), we get (4.4). □

Lemma 4.3 *Let $M = M_3 \times_f M_\theta$ be a warped product submanifold of $\bar{M}$, such that $\xi \in M_3$, where $M_3 = M_T \times M_\perp$, then*

$$\begin{aligned} g(h(X,U),QV) &= \{(Xlnf) - \eta(X)\}g(PU,V) - (\varphi Xlnf)g(U,V), &(4.11)\\ g(h(\varphi X,U),QV) &- g(h(X,U),QPV) = \sin^2\theta[(Xlnf) - \eta(X)]g(U,V), &(4.12)\\ g(h(U,PV),\varphi Z) &- g(h(U,Z),QPV) = -\cos^2\theta[(Zlnf) - \eta(Z)]g(U,V) &(4.13)\end{aligned}$$

for $X \in \Gamma(M_T)$, $Z \in \Gamma(M_\perp)$ and $U,\ V \in \Gamma(M_\theta)$.

Proof For $X \in \Gamma(M_T)$ and $U,\ V \in \Gamma(M_\theta)$, we have

$$g(h(X,U),QV) = g(\bar{\nabla}_U \varphi X,V) - g((\bar{\nabla}_U \varphi)X,V) - g(\bar{\nabla}_U X,PV). \tag{4.14}$$

Using (2.5) and (2.16) in (4.14), we get (4.11).
Replacing X by φX in (4.11), we obtain

$$g(h(\varphi X, U), QV) = (\varphi X \ln f) g(PU, V) - \{(X \ln f) - \eta(X)\} g(U, V). \tag{4.15}$$

Moreover, replacing U by PU in (4.11), we find

$$g(h(X, U), QPV) = \cos^2\theta \{(X lnf) - \eta(X)\} g(U, V) + (\varphi X lnf) g(PU, V), \tag{4.16}$$

Thus, (4.12) follows from (4.15) and (4.16).
Now for $Z \in \Gamma(M_\perp)$ and $U,\ V \in \Gamma(M_\theta)$, we calculate

$$g(h(U, PV), \varphi Z) = -g(\bar{\nabla}_U P^2 V, Z) - g(\bar{\nabla}_U QPV, Z) + \eta(Z) g(\varphi U, PV). \tag{4.17}$$

By virtue of (2.7) and (2.16), (4.17) yields

$$\begin{aligned} g(h(U, PV), \varphi Z) &= -\cos^2\theta (Z lnf) g(U, V) + g(h(U, Z), QPV) \\ &\quad + \cos^2\theta \eta(Z) g(U, V). \end{aligned} \tag{4.18}$$

Thus, (4.13) follows from (4.18). □

5. Characterization

In this section we characterized a warped product submanifold $M = M_3 \times_f M_\theta$ of $\bar{M}$ such that $\xi \in \Gamma(M_3)$, where $M_3 = M_T \times M_\perp$.

Theorem 5.1 *Let M be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$ with ξ is orthogonal to $\mathcal{D}^\theta$. Then M is locally a warped product submanifold of the form $M = M_3 \times_f M_\theta$ where $M_3 = M_T \times M_\perp$, and $M_T, M_\perp$ and M_θ are invariant, antiinvariant, and proper pointwise slant submanifolds of $\bar{M}$ respectively if and only if*

$$A_{QV}\varphi X - A_{QPV} X = \sin^2\theta [X\mu - \eta(X)] V, \tag{5.1}$$

$$A_{\varphi Z} PV - A_{QPV} Z = -\cos^2\theta [Z\mu - \eta(Z)] V, \tag{5.2}$$

$$(\xi\mu) = 1 \tag{5.3}$$

for every $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$, $V \in \Gamma(\mathcal{D}^\theta)$ and $(V\mu) = 0$ for some function μ on M.

Proof Let $M = M_3 \times_f M_\theta$ be a proper warped product submanifold of $\bar{M}$ such that $\xi \in \Gamma(M_3)$, where $M_3 = M_T \times M_\perp$. Denote the tangent space of M_T, $M_\perp$ and M_θ by $\mathcal{D}^T$, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ respectively. Then from (4.2), we have

$$g(A_{QPV} X - A_{QV}\varphi X, Y) = 0,$$

which implies that

$$A_{QPV} X - A_{QV}\varphi X \perp \mathcal{D}^T. \tag{5.4}$$

Similarly, from (4.3) we get,

$$g(A_{QPV} X - A_{QV}\varphi X, Z) = 0.$$

That means,

$$A_{QPV}X - A_{QV}\varphi X \perp \mathcal{D}^{\perp}. \tag{5.5}$$

Then from (5.4) and (5.5), we obtain

$$A_{QPV}X - A_{QV}\varphi X \in \mathcal{D}^{\theta}. \tag{5.6}$$

Thus, (5.1) follows from (4.12) and (5.6) .
Again from (4.3), we get

$$g(A_{\varphi Z}PV - A_{QPV}Z, X) = 0.$$

Therefore, we get

$$A_{\varphi Z}PV - A_{QPV}Z \perp \mathcal{D}^{T}. \tag{5.7}$$

Again from (4.4), we get

$$g(A_{\varphi Z}PV - A_{QPV}Z, W) = 0.$$

Hence,

$$A_{\varphi Z}PV - A_{QPV}Z \perp \mathcal{D}^{\perp}. \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$A_{\varphi Z}PV - A_{QPV}Z \in \mathcal{D}^{\theta}. \tag{5.9}$$

Thus, (5.2) follows from (4.13) and (5.9).
Also (5.3) follows directly from (4.1).
Conversely, let M be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^T \oplus \mathcal{D}^{\perp} \oplus \mathcal{D}^{\theta}$ with ξ is orthogonal to $\mathcal{D}^{\theta}$ and (5.1)-(5.3) holds. Then from (3.1), (3.2) and (5.1), we get

$$g(\nabla_X Y, U) = 0 \qquad \textit{and} \qquad g(\nabla_X Z, U) = 0 \tag{5.10}$$

for $X, Y \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^{\perp})$ and $U \in \Gamma(\mathcal{D}^{\theta})$. Again, from (3.3), (3.4) and (5.2), we get

$$g(\nabla_Z X, U) = 0 \qquad \textit{and} \qquad g(\nabla_Z W, U) = 0 \tag{5.11}$$

for $X \in \Gamma(\mathcal{D}^T)$, $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $U \in \Gamma(\mathcal{D}^{\theta})$.

Thus, from (5.10), (5.11) and the fact that $\nabla_X \xi = 0$ we conclude that $g(\nabla_E F, U) = 0$ for every $E, F \in \Gamma(\mathcal{D}^T \oplus \mathcal{D}^{\perp} \oplus \{\xi\})$. Hence, the leaves of $\mathcal{D}^T \oplus \mathcal{D}^{\perp} \oplus \{\xi\}$ are totally geodesic in M.
Now by virtue of (3.10), (5.1) yields

$$g([U, V], Z) = 0. \tag{5.12}$$

Also by virtue of (3.9), (5.2) yields

$$g([U, V], X) = 0. \tag{5.13}$$

Hence, from (5.12), (5.13) and the fact that $h(A, \xi) = 0$ for all $A \in \Gamma(TM)$, we conclude that

$$g([U, V], E) = 0 \tag{5.14}$$

for all $U, V \in \Gamma(\mathcal{D}^\theta)$ and $E \in \Gamma(\mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \{\xi\})$. Consequently the distribution $\mathcal{D}^\theta$ is integrable.

Let h^θ be the second fundamental form of M_θ in $\bar{M}$. Then for any $U, V \in \Gamma(\mathcal{D}^\theta)$ and $X \in \Gamma(\mathcal{D}^T)$, we have from (3.10) that

$$g(h^\theta(U,V),X) = \csc^2\theta g(A_{QPV}X - A_{QV}\varphi X, U) - \eta(X)g(U,V). \tag{5.15}$$

By virtue of (5.1), (5.15) yields

$$g(h^\theta(U,V),X) = -(X\mu)g(U,V). \tag{5.16}$$

Similarly for any $U, V \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$ we have from (3.9) that

$$g(h^\theta(U,V),Z) \;=\; \sec^2\theta g(A_{\varphi Z}PV - A_{QPV}Z, U) - \eta(Z)g(U,V). \tag{5.17}$$

Now by virtue of (5.2), (5.17) yields

$$g(h^\theta(U,V),Z) = -(Z\mu)g(U,V). \tag{5.18}$$

Also for any $U, V \in \Gamma(\mathcal{D}^\theta)$, we have

$$g(h^\theta(U,V),\xi) = g(\nabla_U V,\xi) = -g(V,\bar{\nabla}_U\xi) = -g(U,V).$$

Using (5.3), in above relation, we get

$$g(h^\theta(U,V),\xi) = -(\xi\mu)g(U,V). \tag{5.19}$$

From (5.16), (5.18) and (5.19) we conclude that

$$g(h^\theta(U,V),E) = -g(\boldsymbol{\nabla}\mu,E)g(X,Y) \tag{5.20}$$

for every $U, V \in \Gamma(\mathcal{D}^\theta)$ and $E \in \Gamma(\mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \{\xi\})$. Consequently, M_θ is totally umbilical in $\bar{M}$ with mean curvature vector $H^\theta = -\boldsymbol{\nabla}\mu$.

Finally we show that H^θ is parallel with respect to the normal connection D^N of M_θ in M. We take $E \in \Gamma(\mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \langle\xi\rangle)$ and $U \in \Gamma(\mathcal{D}^\theta)$, then we have

$$g(D^N_\theta\boldsymbol{\nabla}\mu,E) = g(\nabla_U\boldsymbol{\nabla}^T\mu,X) + g(\nabla_U\boldsymbol{\nabla}^\perp\mu,U) + g(\nabla_X\boldsymbol{\nabla}^\xi\mu,\xi),$$

where $\boldsymbol{\nabla}^T$, $\boldsymbol{\nabla}^\perp$, and $\boldsymbol{\nabla}^\xi$ are the gradient components of μ on M along $\mathcal{D}^T$, $\mathcal{D}^\perp$, $\mathcal{D}^\theta$ and $\langle\xi\rangle$ respectively. Then by the property of Riemannian metric, the above relation reduces to

$$\begin{aligned} g(D^N_U\boldsymbol{\nabla}\mu,E) &= Ug(\nabla^T\mu,X) - g(\boldsymbol{\nabla}^T\mu,\nabla_U X) + Ug(\nabla^\perp\mu,Z) \\ &\quad -g(\boldsymbol{\nabla}^\perp\mu,\nabla_U Z) + Ug(\nabla^\xi\mu,\xi) - g(\boldsymbol{\nabla}^\xi\mu,\nabla_U\xi) \\ &= U(X\mu) - g(\nabla^T\mu,[U,X]) - g(\boldsymbol{\nabla}^T\mu,\nabla_X U) \\ &\quad +U(Z\mu) - g(\nabla^\perp\mu,[U,Z]) - g(\boldsymbol{\nabla}^\perp\mu,\nabla_Z U) \\ &\quad +U(\xi\mu) - g(\nabla^\xi\mu,[U,\xi]) - g(\boldsymbol{\nabla}^\xi\mu,\nabla_\xi U) \\ &= X(U\mu) + g(\nabla_X\boldsymbol{\nabla}^T\mu,U) + Z(U\mu) \\ &\quad +g(\nabla_Z\boldsymbol{\nabla}^\perp\mu,U) + \xi(U\mu) + g(\nabla_\xi\boldsymbol{\nabla}^\xi\mu,U) \\ &= 0, \end{aligned}$$

since $(V\mu)=0$ for every $V\in\Gamma(\mathcal{D}^\theta)$ and $\nabla_X\boldsymbol{\nabla}^T\mu+\nabla_Z\boldsymbol{\nabla}^\perp\mu+\nabla_\xi\boldsymbol{\nabla}^\xi\mu=\nabla_E\boldsymbol{\nabla}\mu$ is orthogonal to $\mathcal{D}^\theta$ for any $E\in\Gamma(\mathcal{D}^T\oplus\mathcal{D}^\perp\oplus\langle\xi\rangle)$ as $\boldsymbol{\nabla}\mu$ is the gradient along M_3 and M_3 is totally geodesic in $\bar{M}$. Hence, the mean curvature vector H^θ of M_θ is parallel. Thus, M_θ is an extrinsic sphere in M. Hence, by Hiepko's Theorem (see, [12]), M is locally a warped product submanifold. Thus, the proof is complete. □

Theorem 5.1 is the generalization of the following results:

Corollary 5.2 *(*Theorem 3.3 of *[35]) Let M be a contact CR-submanifold of $\bar{M}$ such that ξ is orthogonal to $\mathcal{D}^T$. Then M is locally a warped product submanifold of the form $M_\perp\times_f M_T$ if and only if*

$$A_{\varphi Z}X=-\{(Z\mu)-\eta(Z)\}\varphi X$$

for any $X\in\Gamma(\mathcal{D}^T)$ and $Z\in\Gamma(\mathcal{D}^\perp\oplus\{\xi\})$, where μ is any smooth function on M such that $Y(\mu)=0$, $Y\in\Gamma(\mathcal{D}^T)$.

Proof The result follows from Theorem 5.1 by taking $\dim M_T=0$ and $\theta=0$. □

Corollary 5.3 *(*Theorem 2 of *[32]) Let M be a proper semislant submanifold of $\bar{M}$. Then M is locally a warped product submanifold of the form $M_T\times_f M_\theta$ if and only if*

$$A_{QV}\varphi X-A_{QPV}X=\sin^2\theta[X\mu-\eta(X)]V$$

for $X\in\Gamma(\mathcal{D}^T\oplus\{\xi\})$, $Z\in\Gamma(\mathcal{D}^\theta)$ and some smooth function μ on M such that $W(\mu)=0$ where $W\in\Gamma(\mathcal{D}^\theta)$.

Proof The result follows from Theorem 5.1 by taking $\dim M_\perp=0$ and $\theta=$ constant. □

Corollary 5.4 *Let M be a proper pointwise pseudoslant submanifold of a Kenmotsu manifold $\bar{M}$. Then M is locally a warped product submanifold of the form $M_\perp\times_f M_\theta$ if and only if*

$$A_{\varphi Z}PV-A_{QPV}Z=\cos^2\theta[\eta(Z)-Z\mu]V$$

for every $Z\in\Gamma(\mathcal{D}^\perp)$, $V\in\Gamma(\mathcal{D}^\theta)$ and for some smooth function μ on M such that $W(\mu)=0$, for some $W\in\Gamma(\mathcal{D}^\theta)$.

Proof The result follows from Theorem 5.1 by taking $\dim M_T=0$. □

6. Generalized inequalities

In this section, we establish two inequalities on a warped product submanifold $M=M_3\times_f M_\theta$ of $\bar{M}$ such that $M_3=M_T\times M_\perp$.

We consider $dimM_T=2p+1$, $dimM_\perp=q$, $dimM_\theta=2s$ and their corresponding tangent spaces are $\{\mathcal{D}^T\oplus\xi\}$, $\mathcal{D}^\perp$, and $\mathcal{D}^\theta$, respectively. Assume that $\{e_1,e_2,\cdots,e_p,\varphi e_1,\cdots,\varphi e_p,\xi\}$, $\{e_1^*,\cdots,e_q^*\}$ and $\{\hat{e}_1,\hat{e}_2,\cdots,\hat{e}_s,\sec\theta P\hat{e}_1,\sec\theta P\hat{e}_2,\cdots,\sec\theta P\hat{e}_s\}$ are local orthonormal frames of $\mathcal{D}^T\oplus\{\xi\}$, $D^\perp$ and D^θ respectively.

Then the local orthonormal frames for $\varphi D^\perp$, QD^θ and ν are $\{\varphi e_1^*=\tilde{e_1},\cdots,\varphi e_q^*=\tilde{e_q}\}$, $\{\tilde{e}_{q+1}=\csc\theta Q\hat{e}_1,\cdots,\tilde{e}_{q+s}=\csc\theta Q\hat{e}_s,\tilde{e}_{q+s+1}=\csc\theta\sec\theta QP\hat{e}_1,\cdots,\tilde{e}_{q+2s}=\csc\theta\sec\theta QP\hat{e}_s\}$ and $\{\tilde{e}_{q+2s+1},\cdots,\tilde{e}_{2m+1}\}$ of dimensions $q,2s$ and $(2m+1-q-2s)$ respectively.

Theorem 6.1 *Let $M = M_3 \times_f M_\theta$ be a proper warped product $\mathcal{D}^\perp - \mathcal{D}^\theta$ mixed geodesic submanifold of $\bar{M}$ such that ξ is tangent to M_T, then the squared norm of the second fundamental form satisfies*

$$\|h\|^2 \geq 2s[\cos^2\theta \parallel \boldsymbol{\nabla}^\perp \ln f \parallel^2 + 2(\csc^2\theta + \cot^2\theta)\{\parallel \boldsymbol{\nabla}^T \ln f \parallel^2 - 1\}], \tag{6.1}$$

where $\boldsymbol{\nabla}^\perp \ln f$ and $\boldsymbol{\nabla}^T \ln f$ are the gradients of warping function $\ln f$ along $M_\perp$ and M_T respectively and $2s$ is the dimension of M_θ.

If the equality sign of (6.1) holds, then M_3 is totally geodesic and M_θ is totally umbilical submanifold of $\bar{M}$.

Proof From the definition of h, we have

$$\|h\|^2 = \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), h(e_i, e_j)). \tag{6.2}$$

Now decomposing (6.2) in our constructed frame fields, we get

$$\begin{aligned} \|h\|^2 &= \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \tilde{e}_r)^2 + 2\sum_{r=n+1}^{2m+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{q} g(h(e_i, e_j^*), \tilde{e}_r)^2 \\ &+ \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{q} g(h(e_i^*, e_j^*), \tilde{e}_r)^2 + 2\sum_{r=n+1}^{2m+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{2s} g(h(e_i, \hat{e}_j), \tilde{e}_r)^2 \\ &+ 2\sum_{r=n+1}^{2m+1}\sum_{i=1}^{q}\sum_{j=1}^{2s} g(h(e_i^*, \hat{e}_j), \tilde{e}_r)^2 + \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{2s} g(h(\hat{e}_i, \hat{e}_j), \tilde{e}_r)^2. \end{aligned} \tag{6.3}$$

Neglecting the ν component terms of (6.3), we obtain

$$\begin{aligned}
\|h\|^2 \quad &\geq \sum_{r=1}^{q}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\varphi e_r^*)^2 + \sum_{r=1}^{s}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\csc\theta Q\hat{e}_r)^2 \qquad (6.4)\\
&+\sum_{r=1}^{s}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\csc\theta\sec\theta QP\hat{e}_r)^2 + 2\sum_{r=1}^{q}\sum_{i=1}^{2p+1}\sum_{j=1}^{q} g(h(e_i,e_j^*),\varphi e_r^*)^2\\
&+2\sum_{r=1}^{s}\sum_{i=1}^{2p+1}\sum_{j=1}^{q} g(h(e_i,e_j^*),\csc\theta Q\hat{e}_r)^2 + 2\sum_{r=1}^{s}\sum_{i=1}^{2p+1}\sum_{j=1}^{q} g(h(e_i,e_j^*),\csc\theta\sec\theta QP\hat{e}_r)^2\\
&+\sum_{r=1}^{q}\sum_{i,j=1}^{q} g(h(e_i^*,e_j^*),\varphi e_r^*)^2 + \sum_{r=1}^{s}\sum_{i,j=1}^{q} g(h(e_i^*,e_j^*),\csc\theta Q\hat{e}_r)^2\\
&+\sum_{r=1}^{s}\sum_{i,j=1}^{q} g(h(e_i^*,e_j^*),\csc\theta\sec\theta QP\hat{e}_r)^2 + 2\sum_{r=1}^{q}\sum_{i=1}^{2p+1}\sum_{j=1}^{2s} g(h(e_i,\hat{e}_j),\varphi e_r^*)^2\\
&+2\sum_{r=1}^{s}\sum_{i=1}^{2p+1}\sum_{j=1}^{2s} g(h(e_i,\hat{e}_j),\csc\theta Q\hat{e}_r)^2 + 2\sum_{r=1}^{s}\sum_{i=1}^{2p+1}\sum_{j=1}^{2s} g(h(e_i,\hat{e}_j),\csc\theta\sec\theta QP\hat{e}_r)^2\\
&+2\sum_{r=1}^{q}\sum_{i=1}^{q}\sum_{j=1}^{2s} g(h(e_i^*,\hat{e}_j),\varphi e_r^*)^2 + 2\sum_{r=1}^{s}\sum_{i=1}^{q}\sum_{j=1}^{2s} g(h(e_i^*,e_j^*),\csc\theta Q\hat{e}_r)^2\\
&+2\sum_{r=1}^{s}\sum_{i=1}^{q}\sum_{j=1}^{2s} g(h(e_i^*,\hat{e}_j),\csc\theta\sec\theta QP\hat{e}_r)^2 + \sum_{r=1}^{q}\sum_{i,j=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\varphi e_r^*)^2\\
&+\sum_{r=1}^{s}\sum_{i,j=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\csc\theta Q\hat{e}_r)^2 + \sum_{r=1}^{s}\sum_{i,j=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\csc\theta\sec\theta QP\hat{e}_r)^2.
\end{aligned}$$

In view of (4.2) and (4.3), the second, third, fifth, sixth, and tenth terms are equal to zero. For $\mathcal{D}^\perp - \mathcal{D}^\theta$ mixed geodesic condition, thirteenth, fourteenth, and fifteenth terms vanishes. Also we cannot find any relation for $g(h(X,Y),\varphi Z)$, $g(h(X,Z),\varphi W)$, $g(h(Z,W),\varphi Z)$, $g(h(Z,W),QU)$ and $g(h(U,V),QU)$, for $X,\ Y \in \Gamma(\mathcal{D}^T)$, $Z,\ W \in \Gamma(\mathcal{D}^\perp)$, $U,\ V \in \Gamma(\mathcal{D}^\theta)$. Thus, we neglect first, fourth, seventh, eighth, ninth, seventeenth, and eighteenth terms of (6.4) and obtain

$$\|h\|^2 \geq 2\csc^2\theta \sum_{r,j=1}^{s}\sum_{i=1}^{2p+1} g(h(e_i,\hat{e}_j),Q\hat{e}_r)^2 + 2\csc^2\theta\sec^2\theta \sum_{r,j=1}^{s}\sum_{i=1}^{2p+1} g(h(e_i,P\hat{e}_j),Q\hat{e}_r)^2$$
$$+2\csc^2\theta\sec^2\theta \sum_{r,j=1}^{s}\sum_{j=1}^{s} g(h(e_i,\hat{e}_j),QP\hat{e}_r)^2 + 2\csc^2\theta\sec^4\theta \sum_{r,j=1}^{s}\sum_{i=1}^{2p+1} g(h(e_i,P\hat{e}_j),QP\hat{e}_r)^2$$
$$+\sum_{r=1}^{q}\sum_{i,j=1}^{s} g(h(\hat{e}_i,\hat{e}_j),\varphi\overset{*}{e}_r)^2 + 2\sec^2\theta\sum_{r=1}^{q}\sum_{i,j=1}^{s} g(h(P\hat{e}_i,\hat{e}_j),\varphi\overset{*}{e}_r)^2$$
$$+\sec^4\theta\sum_{r=1}^{q}\sum_{i,j=1}^{s} g(h(P\hat{e}_i,\hat{e}_j),\varphi\overset{*}{e}_r)^2.$$

Using (4.11) and (4.13) in the above relation, we find after simplification

$$\begin{aligned} \|h\|^2 &\geq 4s(\csc^2\theta + \cot^2\theta)[\sum_{i=1}^{2p+1}(e_i lnf)^2 - (\xi lnf)^2] \\ &+ 2s\cos^2\theta\sum_{r=1}^{q}(e_r lnf)^2. \end{aligned} \tag{6.5}$$

Using (2.10) and (4.1) in (6.5), we get the inequality (6.1).

If equality of (6.1) holds, for omitting ν components terms of (6.3), we get
$h(\mathcal{D}^T,\mathcal{D}^T)\perp\nu,\ h(\mathcal{D}^T,\mathcal{D}^\perp)\perp\nu,\ h(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp\nu,\ h(\mathcal{D}^T,\mathcal{D}^\theta)\perp\nu,$
$h(\mathcal{D}^\perp,\mathcal{D}^\theta)\perp\nu$ and $h(\mathcal{D}^\theta,\mathcal{D}^\theta)\perp\nu$.
Next, for the neglecting terms of (6.4), we get
$h(\mathcal{D}^T,\mathcal{D}^T)\perp\varphi\mathcal{D}^\perp,\ h(\mathcal{D}^T,\mathcal{D}^\perp)\perp\varphi\mathcal{D}^\perp,\ h(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp\varphi\mathcal{D}^\perp,$
$h(\mathcal{D}^\perp,\mathcal{D}^\perp)\perp Q\mathcal{D}^\theta$ and $h(\mathcal{D}^\theta,\mathcal{D}^\theta)\perp Q\mathcal{D}^\theta$.
Moreover, from (4.2) and (4.3), we obtain
$h(\mathcal{D}^T,\mathcal{D}^T)\perp Q\mathcal{D}^\theta,\ h(\mathcal{D}^T,\mathcal{D}^\perp)\perp Q\mathcal{D}^\theta$ and $h(\mathcal{D}^T,\mathcal{D}^\theta)\perp\varphi\mathcal{D}^\perp$.
Thus, we get

$$h(\mathcal{D}^T,\mathcal{D}^T) = 0, \tag{6.6}$$
$$h(\mathcal{D}^\perp,\mathcal{D}^\perp) = 0, \tag{6.7}$$
$$h(\mathcal{D}^T,\mathcal{D}^\perp) = 0 \tag{6.8}$$

and

$$h(\mathcal{D}^\theta,\mathcal{D}^\theta) \subset \varphi\mathcal{D}^\perp. \tag{6.9}$$

Since M_3 is totally geodesic in M ([5, 7]), using this fact with (6.6), (6.7), and (6.8), we get M_3 is totally geodesic in $\bar{M}$. Moreover, since M_θ is totally umbilical in M ([5, 7]), using this fact with (5.20) and (6.9), we get M_θ is totally umbilical in $\bar{M}$. Hence, the theorem is proved completely. □

As an consequence of Theorem 6.1, we have the following:

Corollary 6.2 *(*Theorem 3 of *[32]) Let $M = M_T \times_f M_\theta$ be a proper warped product semislant submanifold of $\bar{M}$ such that $\xi \in \Gamma(M_T)$. Then the squared norm of the second fundamental form satisfies*

$$\|h\|^2 \geq 4s(\csc^2\theta + \cot^2\theta)(\|\boldsymbol{\nabla}^T lnf\|^2 - 1),$$

where $\boldsymbol{\nabla}^T \ln f$ is the gradient of lnf along M_T and $2s = dim(M_\theta)$.
If the equality sign holds then M_T is a totally geodesic submanifold and M_θ is a totally umbilical submanifold of $\bar{M}$.

Proof The results follows from Theorem 6.1 by taking $\dim M_\perp = 0$ and
$\theta =$ constant. □

Theorem 6.3 *Let $M = M_3 \times_f M_\theta$ be a proper warped product $\mathcal{D}^\perp - \mathcal{D}^\theta$ mixed geodesic submanifold of $\bar{M}$ such that ξ is tangent to $M_\perp$, then the squared norm of the second fundamental form satisfies*

$$\|h\|^2 \geq [\cos^2\theta(\|\boldsymbol{\nabla}^\perp lnf\|^2 - 1) + 2\|\boldsymbol{\nabla}^T lnf\|^2(\csc^2\theta + \cot^2\theta)], \tag{6.10}$$

where $\boldsymbol{\nabla}^\perp \ln f$ and $\boldsymbol{\nabla}^T \ln f$ are the gradients of warping function $\ln f$ along $M_\perp$ and M_T respectively and $2s$ is the dimension of M_θ.
If the equality sign of (6.1) holds, then M_3 is totally geodesic and M_θ is totally umbilical submanifold of $\bar{M}$.

Proof To prove this theorem we consider $dimM_T = 2p$ and $dimM_\perp = q+1$ and $dimM_\theta = 2s$. Thus, the orthonormal frames for $\mathcal{D}^T$ and $\mathcal{D}^\perp$ will be $\{e_1, e_2, \cdots, e_p, \varphi e_1 = e_{p+1}, \cdots, \varphi e_p = e_{2p}\}$ and $\{\overset{*}{e}_1, \cdots, \overset{*}{e}_q, \overset{*}{e}_{q+1} = \xi\}$. Now proceeding as the proof of Theorem 6.1 we can easily prove this theorem. □

As a consequence of Theorem 6.3, we have the following:

Corollary 6.4 *(*Theorem 3.4 of *[35]) Let $M = M_\perp \times_f M_T$ be a proper warped product contact CR-submanifold of $\bar{M}$ such that $\xi \in \Gamma(M_\perp)$. Then the squared norm of the second fundamental form satisfies*

$$\|h\|^2 \geq 2s(\|\boldsymbol{\nabla}^\perp lnf\|^2 - 1),$$

where $\boldsymbol{\nabla}^\perp \ln f$ is the gradient of lnf along $M_\perp$ and $2s = dim(M_T)$.
If the equality sign holds then $M_\perp$ is a totally geodesic submanifold and M_T is a totally umbilical submanifold of $\bar{M}$.

Proof The results follows from Theorem 6.2 by taking $\dim M_T = 0$ and $\theta = 0$. □

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