

Inverse problem for Sturm–Liouville differential operators with finite number of constant delays

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Abstract: In this manuscript, we study nonself-adjoint second-order differential operators with finite number of constant delays. We investigate the properties of the spectral characteristics and the inverse problem of recovering operators from their spectra. An inverse spectral problem is studied for recovering differential operator from the potential from spectra of two boundary value problems with one common boundary condition. The uniqueness theorem is proved for this inverse problem.

Key words: Sturm–Liouville differential operator, inverse problem, finite number of constant delays, asymptotic form of solution, eigenvalue

1. Introduction

The method of separation of variables for solving partial differential equations with finite number of constant delays naturally led to ordinary differential equation with finite number of constant delays inside of the interval which often appear in mathematics, physics, mechanics, geophysics, electronics, meteorology, etc.

The interest in differential equations with a constant delay has started intensively growing in the 20th century stimulated by the appearance of various applications in natural sciences and engineering, including the theory of automatic control, the theory of self-oscillating systems, long-term forecasting in the economy, biophysics, etc. For general background on functional differential equations. We refer, for example, to the monographs [9, 16, 25] and the references therein.

For Sturm–Liouville problems, we have three types of problems: direct problems, isospectral problems, and inverse problems. In direct problems, the eigenvalues, eigenfunctions, and some properties of the problem are estimated from the known coefficients. Different numerical methods for solving direct problem are applied in [7, 11]. In isospectral problems, for a given problem, we want to obtain different problems of the same form, which have the same eigenvalues of the initial problem. Isospectral Sturm–Liouville problems are studied in [8, 12, 13]. The third type of problems related to the Sturm–Liouville problems are inverse problems. The inverse spectral Sturm–Liouville problem can be regarded as three aspects: existence, uniqueness, and reconstruction of the coefficients with specific properties of eigenvalues and eigenfunctions, (see [1, 5, 10, 20, 21, 24, 26] and the references therein). The aim of this paper is to investigate the inverse problem of Sturm–Liouville differential equations with finite numbers of constant delays.

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There exist a number of results revealing spectral properties of differential operators with delay (see, e.g., [17] and the references therein). At the same time, concerning the inverse spectral theory, its classical methods do not work for such operators as well as for other classes of nonlocal operators; therefore, there are only few separate results in this direction, which do not form a comprehensive perspective. However, some aspects of inverse problems for differential operators with a constant delay were studied in [2, 6, 14, 15, 18, 19, 27, 30].

Freiling and Yurko [6] proved that if the spectra of the problems $L_j(q)$, $j = 0, 1$, coincide with the spectra of $L_j(0)$, $j = 0, 1$, respectively, then $q(x) = 0$ a.e. on $(0, \pi)$. Recently, Pikula et al. [19] and Vladicic and Pikula [27] studied the reconstruction of the potential function $q(x)$ and the delay point a , from the two spectra if $a \in (\pi/2, \pi)$. Buterin and Yurko [3] and Buterin et al. [2] studied the inverse Sturm–Liouville differential operator with constant delay. Moreover, the necessary and sufficient conditions for the solvability of the inverse problem in terms of asymptotic was proven. More recently, Shahriari et al. [23] studied the inverse delay Sturm–Liouville problems with a transmission conditions inside the interval. In addition, Shahriari [22], Vojvodic et al. [29], and Vojvodic and Vladicic [28] obtained the inverse problem in two constant delays inside the interval.

In the present paper, we study an inverse problem of Sturm–Liouville differential operators. We also discuss the uniqueness of spectral problem by developing the results in [6] and [22] for inverse Sturm–Liouville problem with finite number of constant delays inside the interval. For this purpose, we study the asymptotic form of solutions, eigenvalues, and eigenfunctions of the problem. Thus, we investigate the inverse spectral problem of recovering operators from their two spectra in the Dirichlet–Dirichlet and Dirichlet–Neumann boundary conditions with finite number of constant delays.

2. Asymptotic form of solutions and eigenvalues

Let us consider the delay boundary value problem

$$\ell y := -y''(x) + \sum_{j=1}^p q_j(x)y(x - a_j) = \lambda y(x), \quad x \in (0, \pi), \tag{2.1}$$

subject to the boundary conditions

$$y(0) = y^{(i)}(\pi) = 0, \quad i = 0, 1, \tag{2.2}$$

where $q_j(x) \in L(a_j, \pi)$ and $q_j(x) = 0$, for $x < a_j$. The coefficient $a_j \in (0, \pi)$ are real and assumed to be known, fixed, and $0 < a_1 < a_2 < \dots < a_p < \pi$. For simplicity, we use the notation $L_i := L_i(q_1(x); \dots; q_p(x); a_1; \dots; a_p)$, $i = 0, 1$, for the problems (2.1)–(2.2).

Let $\varphi(x, \lambda)$ be the solution of Eq. (2.1) under the initial conditions $\varphi(0, \lambda) = 0$, $\varphi'(0, \lambda) = 1$ with $a_j N_j < \pi \leq a_j(N_j + 1)$. Note that $1 \leq N_p \leq N_{p-1} \leq \dots \leq N_1$. The functions $\varphi^{(i)}(x, \lambda)$ for $i = 0, 1$ are entire in λ of order $1/2$. The function $\varphi(x, \lambda)$ is the unique solution of the integral equation

$$\varphi(x, \lambda) = \frac{\sin \rho x}{\rho} + \sum_{j=1}^p \int_0^x \frac{\sin \rho(x-t)}{\rho} q_j(t) \varphi(t - a_j, \lambda) dt, \tag{2.3}$$

with $\rho^2 = \lambda$ and $\rho = \sigma + i\tau$. Solving (2.3) by the method of successive approximations, we get

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \varphi_1(x, \lambda) + \dots + \varphi_{N_1}(x, \lambda). \tag{2.4}$$

Thus, we have

$$\varphi_0(x, \lambda) = \frac{\sin \rho x}{\rho}, \tag{2.5}$$

$$\varphi_k(x, \lambda) = \begin{cases} 0, & x \leq ka_1, \\ \int_{ka_1}^x \frac{\sin \rho(x-t)}{\rho} q_1(t) \varphi_{k-1}(t - a_1, \lambda) dt, & ka_1 < x \leq ka_2, \\ \sum_{j=1}^2 \int_{ka_j}^x \frac{\sin \rho(x-t)}{\rho} q_j(t) \varphi_{k-1}(t - a_j, \lambda) dt, & ka_2 < x \leq ka_3, \\ \vdots & \vdots \\ \sum_{j=1}^p \int_{ka_j}^x \frac{\sin \rho(x-t)}{\rho} q_j(t) \varphi_{k-1}(t - a_j, \lambda) dt, & x \geq ka_p, \end{cases} \tag{2.6}$$

and

$$\varphi'_k(x, \lambda) = \begin{cases} 0, & x \leq ka_1, \\ \int_{ka_1}^x \cos \rho(x-t) q_1(t) \varphi_{k-1}(t - a_1, \lambda) dt, & ka_1 < x \leq ka_2, \\ \sum_{j=1}^2 \int_{ka_j}^x \cos \rho(x-t) q_j(t) \varphi_{k-1}(t - a_j, \lambda) dt, & ka_2 < x \leq ka_3, \\ \vdots & \vdots \\ \sum_{j=1}^p \int_{ka_j}^x \cos \rho(x-t) q_j(t) \varphi_{k-1}(t - a_j, \lambda) dt, & x \geq ka_p. \end{cases} \tag{2.7}$$

Using the formulas (2.5)–(2.7) for $k \geq 1$, we calculate

$$\begin{aligned} \varphi_1(x, \lambda) &= \begin{cases} 0, & x \leq a_1, \\ \int_{a_1}^x \frac{\sin \rho(x-t)}{\rho} q_1(t) \frac{\sin \rho(t-a_1)}{\rho} dt, & a_1 < x \leq a_2, \\ \sum_{j=1}^2 \int_{a_j}^x \frac{\sin \rho(x-t)}{\rho} q_j(t) \frac{\sin \rho(t-a_j)}{\rho} dt, & a_2 < x \leq a_3, \\ \vdots & \vdots \\ \sum_{j=1}^p \int_{a_j}^x \frac{\sin \rho(x-t)}{\rho} q_j(t) \frac{\sin \rho(t-a_j)}{\rho} dt, & x \geq a_p, \end{cases} \\ &= \begin{cases} 0, & x \leq a_1, \\ \frac{1}{2\rho^2} \left(-\cos \rho(x - a_1) \int_{a_1}^x q_1(t) dt + \int_{a_1}^x \cos \rho(2t - x - a_1) q_1(t) dt \right), & a_1 < x \leq a_2, \\ \frac{1}{2\rho^2} \left(-\sum_{j=1}^2 \cos \rho(x - a_j) \int_{a_j}^x q_j(t) dt + \sum_{j=1}^2 \int_{a_j}^x \cos \rho(2t - x - a_j) q_j(t) dt \right), & a_2 < x \leq a_3, \\ \vdots & \vdots \\ \frac{1}{2\rho^2} \left(-\sum_{j=1}^p \cos \rho(x - a_j) \int_{a_j}^x q_j(t) dt + \sum_{j=1}^p \int_{a_j}^x \cos \rho(2t - x - a_j) q_j(t) dt \right), & x \geq a_p, \end{cases} \end{aligned} \tag{2.8}$$

and

$$\varphi'_1(x, \lambda) = \begin{cases} 0, & x \leq a_1, \\ \frac{1}{2\rho} \left(\sin \rho(x - a_1) \int_{a_1}^x q_1(t) dt + \int_{a_1}^x \sin \rho(2t - x - a_1) q_1(t) dt \right), & a_1 < x \leq a_2, \\ \frac{1}{2\rho} \left(\sum_{j=1}^2 \sin \rho(x - a_j) \int_{a_j}^x q_j(t) dt + \sum_{j=1}^2 \int_{a_j}^x \sin \rho(2t - x - a_j) q_j(t) dt \right), & a_2 < x \leq a_3, \\ \vdots & \vdots \\ \frac{1}{2\rho} \left(\sum_{j=1}^p \sin \rho(x - a_j) \int_{a_j}^x q_j(t) dt + \sum_{j=1}^p \int_{a_j}^x \sin \rho(2t - x - a_j) q_j(t) dt \right), & x \geq a_p. \end{cases} \quad (2.9)$$

Using Eqs. (2.5)–(2.9) by induction one can easily show that

$$\varphi_k^{(i)}(x, \lambda) = \begin{cases} 0, & x \leq ka_1, \\ O(\rho^{i-k-1} \exp(|\tau|(x - ka_1))), & x \geq ka_1, \end{cases} \quad |\rho| \rightarrow \infty, \quad (2.10)$$

where $\tau = \text{Im}\rho$.

Denote $\Delta_i(\lambda) := \varphi^{(i)}(\pi, \lambda)$, $i = 0, 1$. The functions $\Delta_i(\lambda)$ are entire functions in λ of order $\frac{1}{2}$ and the zeroes of $\Delta_i(\lambda)$ coincide with the eigenvalues λ_{ni} of L_i . Thus, the function $\Delta_i(\lambda)$ is called the characteristic function for L_i . From Eqs. (2.4)–(2.5) and (2.8)–(2.9), we obtain the following asymptotic formula for $|\rho| \rightarrow \infty$,

$$\Delta_0(\lambda) = \varphi(\pi, \lambda) \quad (2.11)$$

$$\begin{aligned} &= \frac{\sin \rho\pi}{\rho} + \frac{1}{2\rho^2} \left[- \sum_{j=1}^P \cos \rho(\pi - a_j) w_j + \sum_{j=1}^P \int_{a_j}^{\pi} \cos \rho(2t - \pi - a_j) q_j(t) dt \right] \\ &+ O\left(\frac{\exp(|\tau|(\pi - a_1))}{\rho^3}\right), \end{aligned}$$

and

$$\begin{aligned} \Delta_1(\lambda) &= \varphi'(\pi, \lambda) \\ &= \cos \rho\pi + \frac{1}{2\rho} \left[\sum_{j=1}^P \sin \rho(\pi - a_j) w_j + \sum_{j=1}^P \int_{a_j}^{\pi} \sin \rho(2t - \pi - a_j) q_j(t) dt \right] \\ &+ O\left(\frac{\exp(|\tau|(\pi - a_1))}{\rho^2}\right), \end{aligned} \quad (2.12)$$

where $w_j := \int_{a_j}^{\pi} q_j(t) dt$, $j = \overline{1, p}$.

Lemma 2.1 *The asymptotic formula for the eigenvalues $\lambda_{ni} = \rho_{ni}^2$ as $n \rightarrow \infty$ are the following forms:*

$$\rho_{n0} = n + \frac{1}{2\pi n} \sum_{j=1}^P w_j \cos na_j + o\left(\frac{1}{n}\right), \quad (2.13)$$

$$\rho_{n1} = n - \frac{1}{2} + \frac{1}{2\pi n} \left[\sum_{j=1}^P w_j \cos \left(n - \frac{1}{2}\right) a_j \right] + o\left(\frac{1}{n}\right). \quad (2.14)$$

Proof Denote

$$\Gamma_n = \{\lambda : |\lambda| = (n + 1/2)^2\}.$$

By virtue of (2.11)

$$\Delta_0(\lambda) = f(\lambda) + g(\lambda), \quad f(\lambda) = \frac{\sin \rho\pi}{\rho}, \quad |g(\lambda)| \leq C \frac{\exp(|\tau|(\pi - a_1))}{|\rho^2|}.$$

The functions $|f(\lambda)| > |g(\lambda)|$, $\lambda \in \Gamma_n$, for sufficiently large n ($n \geq n^*$). Then by Rouché's theorem [4, P. 125] the number of zeros of $\Delta_0(\lambda)$ inside Γ_n coincides with the number of zeros of $f(\lambda) = \frac{\sin \rho\pi}{\rho}$, i.e. it equals n . Thus, in the circle $|\lambda| < (n + 1/2)^2$, there exist exactly n eigenvalues of L_0 . Applying now Rouché's theorem, it follows that

$$\rho_{n0} = n + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \rightarrow \infty. \tag{2.15}$$

Substituting (2.15) into (2.11) we get

$$0 = \Delta_0(\rho_{n0}) = \frac{\sin(n + \varepsilon_n)\pi}{n + \varepsilon_n} + \frac{1}{2(n + \varepsilon_n)^2} \left[- \sum_{j=1}^P w_j \cos[(n + \varepsilon_n)(\pi - a_j)] \right] + \frac{k_n}{n^2} \tag{2.16}$$

where $\{k_n\}$ denotes various sequences from l_2 and consequently

$$n \sin(\varepsilon_n \pi) - \frac{1}{2} \sum_{j=1}^P w_j \cos(na_j + \varepsilon_n(\pi - a_j)) + k_n = 0. \tag{2.17}$$

Then $\sin(\varepsilon_n \pi) = O(\frac{1}{n})$, i.e. $\varepsilon_n = O(\frac{1}{n})$. Using (2.17) once more we obtain more precisely

$$\varepsilon_n = \frac{1}{2\pi n} \sum_{j=1}^P w_j \cos(na_j) - \frac{k_n}{n\pi}.$$

Thus, we get the asymptotic formula for the eigenvalues $\lambda_{n0} = \rho_{n0}^2$ as $n \rightarrow \infty$. Using the similar proof, we obtain the asymptotic form of ρ_{n1} in (2.14). □

Lemma 2.2 *The specification of the spectrum $\{\lambda_{ni}\}$, $n \geq 1$ and $i = 0, 1$, uniquely determines the characteristic functions $\Delta_i(\lambda)$ by the formulas:*

$$\Delta_0(\lambda) = \pi \prod_{n=1}^{\infty} \frac{\lambda_{n0} - \lambda}{n^2}, \quad \Delta_1(\lambda) = \prod_{n=1}^{\infty} \frac{\lambda_{n1} - \lambda}{(n - \frac{1}{2})^2}. \tag{2.18}$$

Proof By Hadamard's factorization theorem [4, P. 289], $\Delta_0(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta_0(\lambda) = C \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{n0}} \right), \tag{2.19}$$

(the case $\Delta_0(0) = 0$ requires minor modifications). Consider the function

$$\tilde{\Delta}_0(\lambda) := \frac{\sin \rho\pi}{\rho} = \pi \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{n^2} \right)$$

then

$$\frac{\Delta_0(\lambda)}{\tilde{\Delta}_0(\lambda)} = \frac{C}{\pi} \prod_{n=1}^{\infty} \frac{n^2}{\lambda_{n0}} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_{n0} - n^2}{n^2 - \lambda}\right).$$

Taking (2.11) and (2.13) into account, we calculate

$$\lim_{\lambda \rightarrow -\infty} \frac{\Delta_0(\lambda)}{\tilde{\Delta}_0(\lambda)} = 1, \quad \lim_{\lambda \rightarrow -\infty} \left(1 + \frac{\lambda_{n0} - n^2}{n^2 - \lambda}\right) = 1,$$

and hence

$$C = \pi \prod_{n=1}^{\infty} \frac{\lambda_{n0}}{n^2}.$$

Substituting this into (2.19), we arrive at (2.18). The proof of $\Delta_1(\lambda)$ is the same of $\Delta_0(\lambda)$. □

Define

$$\mathcal{L}(\rho) := \Delta_1(\lambda) + i\rho\Delta_0(\lambda).$$

The function $\mathcal{L}(\rho)$ is entire in ρ and $\mathcal{L}(\rho)$ is the characteristic function for the Regge-type boundary value problem $L := L(q_1(x); q_2(x); \dots; q_P(x); a_1; a_2; \dots; a_P)$ for Eq. (2.1) with the boundary conditions $y(0) = y'(\pi) + i\rho y(\pi) = 0$. It follows from (2.4) that

$$\mathcal{L}(\rho) = \mathcal{L}_0(\rho) + \mathcal{L}_1(\rho) + \mathcal{L}_2(\rho) + \dots + \mathcal{L}_{N_1}(\rho), \tag{2.20}$$

where $\mathcal{L}_k(\rho) = \varphi'_k(\pi, \lambda) + i\rho\varphi_k(\pi, \lambda)$. In particular, $\mathcal{L}_0(\rho) = \exp(i\rho\pi)$. Using (2.6) and (2.7), we get

$$\mathcal{L}_k(\rho) = \begin{cases} \sum_{j=1}^p \int_{ka_j}^{\pi} \exp(i\rho(\pi - t))q_j(t)\varphi_{k-1}(t - a_j, \lambda)dt, & \text{for } k = 1, 2, \dots, N_p, \\ \sum_{j=1}^{p-1} \int_{ka_j}^{\pi} \exp(i\rho(\pi - t))q_j(t)\varphi_{k-1}(t - a_j, \lambda)dt, & \text{for } k = N_p + 1, N_p + 2, \dots, N_{p-1}, \\ \vdots & \vdots \\ \int_{ka_1}^{\pi} \exp(i\rho(\pi - t))q_1(t)\varphi_{k-1}(t - a_1, \lambda)dt, & \text{for } k = N_2 + 1, N_2 + 2, \dots, N_1. \end{cases} \tag{2.21}$$

Moreover, it follows from (2.8) and (2.9) that

$$\mathcal{L}_1(\rho) = \sum_{j=1}^P \frac{\exp(i\rho(\pi - a_j))}{2i\rho} w_j - \sum_{j=1}^P \frac{\exp(i\rho(\pi + a_j))}{2i\rho} \int_{a_j}^{\pi} \exp(-2i\rho t)q_j(t)dt. \tag{2.22}$$

Taking (2.10) and (2.21) into account, we get

$$\mathcal{L}_k(\rho) = O\left(\frac{\exp i\rho(\pi + (k - 1)a_1)}{\rho^k} \sum_{j=1}^{N_P} \int_{ka_j}^{\pi} \exp(-i\rho(2t - a_j))q_j(t)dt\right), \tag{2.23}$$

for $\text{Im}\rho > 0$, $|\rho| \rightarrow \infty$, and $k \geq 1$.

3. The uniqueness theorem

Let $\{\tilde{\lambda}_{ni}\}_{n \geq 1}$, $i = 0, 1$, be the eigenvalues of the boundary value problems $\tilde{L}_i := L_i(\tilde{q}_1(x); \dots; \tilde{q}_p(x); a_1; \dots; a_p)$ with $\tilde{q}_j(x) = 0$. Then $\tilde{\lambda}_{n0} = n^2$ and $\tilde{\lambda}_{n1} = (n - \frac{1}{2})^2$. Denote by $\tilde{\mathcal{L}}(\rho)$ the characteristic function of $\tilde{L} := L(\tilde{q}_1; \tilde{q}_2; \dots; \tilde{q}_p; a_1; a_2; \dots; a_p)$. Clearly, $\tilde{\mathcal{L}}(\rho) = \exp(i\rho\pi)$.

Theorem 3.1 (Main theorem) *If $\tilde{\lambda}_{ni} = \lambda_{ni}$ for all $n_i \geq 1$ and $i = 0, 1$, then $q_j(x) = 0$ a.e. on $(a_j, \pi]$ for $j = \overline{1, p}$.*

Proof (1) Using Lemma 2.2, we have

$$\Delta_0(\lambda) = \frac{\sin \rho\pi}{\rho}, \quad \Delta_1(\lambda) = \cos \rho\pi,$$

and consequently $\mathcal{L}(\rho) = \exp(i\rho\pi)$. Using (2.20) we get

$$\mathcal{L}_1(\rho) = -\mathcal{L}^+(\rho), \tag{3.1}$$

where

$$\mathcal{L}^+(\rho) = \sum_{k=2}^{N_1} \mathcal{L}_k(\rho), \quad \text{for } k \geq 2 \quad \text{and } \mathcal{L}^+(\rho) = 0, \quad \text{for } k = 1.$$

It follows from (2.13) that $\sum_{j=1}^p w_j \cos na_j = 0$, consequently $w_j = 0$ for $j = \overline{1, p}$. Together with (2.22) this yields

$$\mathcal{L}_1(\rho) = - \sum_{j=1}^p \frac{\exp(i\rho(\pi + a_j))}{2i\rho} \int_{a_j}^{\pi} \exp(-2i\rho t) q_j(t) dt. \tag{3.2}$$

(2) Let $N_1 = 1$, i.e. $a_1 \in (\frac{\pi}{2}, \pi)$, then $\mathcal{L}^+(\rho) = 0$. From (3.1), we see that $\mathcal{L}_1(\rho) = 0$. Using (3.2), we get

$$\sum_{j=1}^p \frac{\exp(i\rho(\pi + a_j))}{2i\rho} \int_{a_j}^{\pi} \exp(-2i\rho t) q_j(t) dt = 0.$$

By rewriting the Eq. (3.2) and assumption of $q_j(x)$, $j = \overline{1, p}$, we obtain

$$\frac{\exp(i\rho\pi)}{2i\rho} \sum_{j=1}^p \int_{a_1}^{\pi} \exp(-2i\rho t) \exp(i\rho a_j) q_j(t) dt = 0,$$

or

$$\sum_{j=1}^p \int_{a_1}^{\pi} \exp(-2i\rho t) \exp(i\rho a_j) q_j(t) dt = 0.$$

By moving the order of sigma and integral and from the completeness of $\exp(-2i\rho t)$ on (a_1, π) , we get

$$\sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt = 0 \quad \text{a.e. on } (a_1, \pi).$$

The functions $\exp(i\rho a_j)$ are linear independent in ρ , so we get $q_j(x) = 0$ a.e. on (a_1, π) . Thus, Theorem 3.1 is proved for $N_1 = 1$.

Below, we will assume that $N_1 \geq 2$.

Lemma 3.2 *If $q_j(x) = 0$ a.e. on $(2a_1, \pi)$, then $q_j(x) = 0$ a.e. on (a_1, π) .*

Proof The proof of this lemma includes two cases:

- $2a_1 \leq a_2$. The proof is similar to [6, Lemma 2].
- $2a_1 > a_2$. By virtue of (2.23), for $q_j(x) = 0$ a.e. on $(2a_1, \pi)$, we get $\mathcal{L}_k(\rho) = 0$ for $k \geq 2$ and hence $\mathcal{L}^+(\rho) = 0$. From Eq. (3.1), we get $\mathcal{L}_1(\rho) = 0$, consequently $q_j(x) = 0$ a.e. on (a_1, π)

□

(3) For definiteness, we assume that $N_1 = 2S + 1$, $S \geq 1$, i.e. N_1 is odd. (The case $N_1 = 2S$, requires minor technical modifications).

Lemma 3.3 *Fix $\nu = \overline{0, 2S - 1}$. If $q_j(x) = 0$ a.e. on the interval $(\pi - \nu a_1/2, \pi)$ then $q_j(x) = 0$ a.e. on the interval $(\pi - (\nu + 1)a_1/2, \pi)$.*

Proof Since $\pi - \nu a_1/2 > 2a_1$, from (2.23) we have

$$\mathcal{L}_2(\rho) = \begin{cases} O\left(\frac{\exp(i\rho(\pi+a_1))}{\rho^2} \int_{2a_1}^{\pi-\nu a_1/2} \exp(-i\rho(2t-a_1))q_1(t)dt\right), & \pi - \nu a_1/2 < 2a_2; \\ O\left(\frac{\exp(i\rho(\pi+a_1))}{\rho^2} \sum_{j=1}^2 \int_{2a_j}^{\pi-\nu a_1/2} \exp(-i\rho(2t-a_j))q_j(t)dt\right), & 2a_2 < \pi - \nu a_1/2 < 2a_3; \\ \vdots & \vdots \\ O\left(\frac{\exp(i\rho(\pi+a_1))}{\rho^2} \sum_{j=1}^p \int_{2a_j}^{\pi-\nu a_1/2} \exp(-i\rho(2t-a_j))q_j(t)dt\right), & \pi - \nu a_1/2 > 2a_p. \end{cases} \tag{3.3}$$

In the integrals $2t - \pi - 2a_1 \in (2a_1 - \pi, \pi - (\nu + 2)a_1)$, where $\pi - (\nu + 2)a_1 \geq \pi - N_1 a_1 > 0$. This yields

$$\mathcal{L}_2(\rho) = O\left(\frac{1}{\rho^2} \exp(-i\rho(\pi - (\nu + 2)a_1))\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \tag{3.4}$$

For $k \geq 2$, the functions $\mathcal{L}_k(\rho)$ have less growth than the right-hand side in (3.4). This means that

$$\mathcal{L}^+(\rho) = O\left(\frac{1}{\rho^2} \exp(-i\rho(\pi - (\nu + 2)a_1))\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \tag{3.5}$$

It follows from (3.1), (3.2), and (3.5) that

$$\begin{aligned} \mathcal{L}_1(\rho) &= -\sum_{j=1}^p \frac{\exp(i\rho(\pi+a_j))}{2i\rho} \int_{a_j}^{\pi-\nu a_1/2} \exp(-2i\rho t)q_j(t)dt \\ &= O\left(\frac{1}{\rho} \exp(-i\rho(\pi - (\nu + 2)a_1))\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \end{aligned}$$

From the above equation we get

$$\begin{aligned} & \exp(i\rho(2\pi - (\nu + 1)a_1)) \sum_{j=1}^p \int_{a_j}^{\pi - \nu a_1/2} \exp(-2i\rho t) \exp(i\rho a_j) q_j(t) dt \\ &= O\left(\frac{1}{\rho}\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \sum_{j=1}^p \int_{a_j}^{\pi - \nu a_1/2} \exp(-2i\rho t) \exp(i\rho a_j) q_j(t) dt \\ &= O(\exp(i\rho(-2\pi + (\nu + 1)a_1))), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \end{aligned} \tag{3.6}$$

Let us define the function

$$F(\rho) := \exp(i\rho(2\pi - (\nu + 2)a_1)) \int_{\pi - (\nu + 1)a_1/2}^{\pi - \nu a_1/2} \exp(-2i\rho t) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt, \tag{3.7}$$

The function $F(\rho)$ is entire in ρ . Clearly, $F(\rho) = O(1)$ for $\text{Im}\rho \leq 0$. From (3.6) and (3.7) we obtain $F(\rho) = O(1)$ for $\text{Im}\rho \geq 0$. Using the Liouville’s theorem (see [4, P.77]), $F(\rho) = C$ -const. Since $F(\rho) = o(1)$ for real ρ , $|\rho| \rightarrow \infty$. Thus, we get $F(\rho) = 0$. From (3.7) we obtain

$$\int_{\pi - (\nu + 1)a_1/2}^{\pi - \nu a_1/2} \exp(-2i\rho t) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt = 0$$

From the completeness of $\exp(-2i\rho t)$ in the interval $(\pi - (\nu + 1)a_1/2, \pi - \nu a_1/2)$, we get $\sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt = 0$ a.e. on the interval. Thus, we get $q_j(x) = 0$ a.e. in $(\pi - (\nu + 1)a_1/2, \pi - \nu a_1/2)$ for $j = \overline{1, p}$. Lemma 3.3 is proved. \square

(4) Applying Lemma 3.3 successively for $\nu = \overline{0, 2S - 1}$, we obtain $q_j(x) = 0$ a.e. on the interval $(\pi - Sa_1, \pi)$. We note that it is not possible to use Lemma 3.3 for $\nu = 2S$, and we need the following Lemma for this fact.

Lemma 3.4 *If $q_j(x) = 0$, a.e. on the interval $(\pi - Sa_1, \pi)$, $j = \overline{1, p}$ then $q_j(x) = 0$ a.e. on the interval $((S + 2)a_1/2, \pi)$.*

Proof For $k = S + 2$, we have $\pi - Sa_1 - ka_1 \leq \pi - (N + 1)a_1 \leq 0$. Thus, $\mathcal{L}_k(\rho) = 0$ for $k \geq S + 2$. According to (2.23), for $k = \overline{2, S + 1}$, we get

$$\mathcal{L}_k(\rho) = O\left(\frac{\exp i\rho(\pi - (k - 1)a_1)}{\rho^k} \int_{ka_1}^{\pi - Sa_1} \sum_{j=1}^p \exp(-i\rho(2t - a_j)) q_j(t) dt\right). \tag{3.8}$$

Note that $2t - \pi - ka_1 \leq 0$, it follows that

$$\mathcal{L}_k(\rho) = O\left(\frac{1}{\rho^k} \exp(i\rho(\pi - ka_1))\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty, k = \overline{2, S + 1}, \tag{3.9}$$

and hence

$$\mathcal{L}^+(\rho) = O\left(\frac{1}{\rho^2} \exp(i\rho(\pi - (S + 1)a_1))\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \tag{3.10}$$

Applying (3.1), (3.2), and (3.10), we have

$$\begin{aligned} & \exp(i\rho(\pi + a_1)) \int_{ka_j}^{\pi - Sa_1} \sum_{j=1}^p \exp(-i\rho(2t - a_j)) q_j(t) dt \\ &= O\left(\frac{1}{\rho} \exp(i\rho(\pi - (S + 1)a_1))\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \end{aligned}$$

Rewriting the equation as follows, we have

$$\begin{aligned} & \exp(i\rho(S + 1)a_1) \int_{a_j}^{\pi - Sa_1} \exp(-2i\rho t) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt \\ &= O\left(\frac{1}{\rho}\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \end{aligned} \tag{3.11}$$

Thus, we get the equation

$$\begin{aligned} & \int_{a_j}^{\pi - Sa_1} \exp(-2i\rho t) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt \\ &= O\left(\frac{\exp(-i\rho(S + 1)a_1)}{\rho}\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \end{aligned} \tag{3.12}$$

Denote

$$F_1(\rho) := \exp(i\rho(S + 1)a_1) \int_{(S+2)a_1/2}^{\pi - Sa_1} \exp(-2i\rho t) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt.$$

The function $F_1(\rho)$ is entire in ρ , and $F_1(\rho) = O(1)$ for $\text{Im}\rho \leq 0$. By referring and reviewing (3.11) and (3.12), we get $F_1(\rho) = O(1)$ for $\text{Im}\rho \geq 0$. Therefore, $F_1(\rho) = C$. Since $F_1(\rho) = o(1)$ for real ρ , as $|\rho| \rightarrow \infty$, it follows that $F_1(\rho) = 0$, i.e.

$$\int_{(S+2)a_1/2}^{\pi - Sa_1} \exp(-2i\rho t) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt = 0.$$

The completeness of $\exp(-2i\rho t)$ on $((S + 2)a_1/2, \pi - Sa_1)$, implies

$$\sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt = 0$$

a.e. on the interval. Then $q_j(x) = 0$ a.e. on (a_1, π) . □

(5) If $S = 1$ or $S = 2$, then from Lemmas 3.3 and 3.4 we have proved that $q_j(x) = 0$ a.e. on $(2a_1, \pi)$. According to Lemma 3.2, we conclude that $q_j(x) = 0$ a.e. on (a_1, π) . Thus, Theorem 3.1 is proved for $S = 1$ and $S = 2$.

Let $S \geq 3$. Fix $\nu = \overline{5, S + 2}$. Denote $u := [(\nu + 1)/2]$. Clearly, $u < \nu$.

Lemma 3.5 *If $q_j(x) = 0$ a.e. on the interval $(\nu a_1/2, \pi)$, then $q_j(x) = 0$ a.e. on the interval $(ua_1/2, \pi)$.*

Proof Since $\nu/2 - k \leq \nu/2 - u \leq 0$ for $k > u$, it follows that $\mathcal{L}_k(\rho) = 0$ for $k > u$. From Eq. (2.23) and assumption of this Lemma

$$\mathcal{L}_k(\rho) = O\left(\frac{1}{\rho^k} \int_{ka_1}^{\nu a_1/2} \exp(-i\rho(2t - \pi - (k-1)a_1)) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt\right),$$

$$\text{Im}\rho \geq 0, |\rho| \rightarrow \infty.$$

Since $2t - \pi - ka_1 \leq 0$, we get

$$\mathcal{L}_k(\rho) = O\left(\frac{1}{\rho^k} \exp(i\rho(\pi - ka_1))\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty, k = \overline{2, u-1}.$$

and hence

$$\mathcal{L}^+(\rho) = O\left(\frac{1}{\rho^2} \exp(i\rho(\pi - (u-1)a_1))\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \tag{3.13}$$

From the Eqs. (3.1), (3.2), and (3.13), we obtain

$$\exp(i\rho(u-1)a_1) \int_{a_1}^{\nu a_1/2} \exp(-2i\rho t) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt$$

$$= O\left(\frac{1}{\rho}\right), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \tag{3.14}$$

Moreover,

$$\exp(i\rho(u-1)a_1) \int_{a_1}^{ua_1/2} \exp(-2i\rho t) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt$$

$$= O(\exp(-i\rho ua_1)), \quad \text{Im}\rho \geq 0, |\rho| \rightarrow \infty. \tag{3.15}$$

Denote

$$F_2(\rho) := \exp(i\rho(u-1)a_1) \int_{ua_1/2}^{\nu a_1/2} \exp(-2i\rho t) \sum_{j=1}^p \exp(i\rho a_j) q_j(t) dt.$$

The function $F_2(\rho)$ is entire in ρ and $F_2(\rho) = O(1)$ for $\text{Im}\rho \leq 0$. By referring and reviewing (3.14) and (3.15), we get $F_2(\rho) = O(1)$ for $\text{Im}\rho \geq 0$. Therefore, $F_2(\rho) = C$. Since $F_2(\rho) = o(1)$ for real ρ , as $|\rho| \rightarrow \infty$, it follows that $F_2(\rho) = 0$, i.e., and consequently $\sum_{j=1}^p \exp(i\rho a_j) q_j(t) = 0$ a.e. on the interval $(ua_1/2, \nu a_1/2)$. Thus, we get $q_j(x) = 0$ a.e. on the interval. □

Applying Lemma 3.5 several times successively starting from $\nu = S + 2$, we obtain the relation $q_j(x) = 0$ a.e. on $(2a_1, \pi)$. Then, by virtue of Lemma 3.2, $q_j(x) = 0$ a.e. on the interval (a_1, π) . Theorem 3.1 is proved. □

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