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Q-analogues of five difficult hypergeometric evaluations

Xiaojing CHEN[1], **Wenchang CHU**[2,*]
[1]School of Statistics, Qufu Normal University, Qufu (Shandong), P.R. China
[2]Department of Mathematics and Physics, University of Salento, Lecce, Italy

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Abstract: A nonterminating balanced q-series is examined by means of the modified Abel lemma on summation by parts that leads to q-analogues of five difficult identities for classical hypergeometric series, including three formulae conjectured by Gosper in 1977.

Key words: Abel's lemma on summation by parts, classical hypergeometric series, basic hypergeometric series, balanced series, bilateral series, the gamma function

1. Introduction and motivation

Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then for an indeterminate x and $n \in \mathbb{N}_0$, the shifted factorial is defined by the quotient

$$(x)_n = \Gamma(x+n)/\Gamma(x).$$

The Γ-function (see [12, §8] for example) is given by the beta integral

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \mathrm{d}u \quad \text{for} \quad \Re(x) > 0,$$

which satisfies Euler's reflection property

$$\Gamma(x) \times \Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Analogously for $0 < |q| < 1$, the infinite product below is well defined

$$(x;q)_\infty = \prod_{k=0}^{\infty} (1 - q^k x),$$

which can be used, in turn, to define the q-shifted factorial of order $n \in \mathbb{Z}$:

$$(x;q)_n = \frac{(x;q)_\infty}{(q^n x;q)_\infty} = \begin{cases} (1-x)(1-qx)\cdots(1-q^{n-1}x), & n \in \mathbb{N}; \\ 1, & n = 0; \\ \dfrac{1}{(1-q^{-1}x)(1-q^{-2}x)\cdots(1-q^n x)}, & n \in \mathbb{Z}\backslash\mathbb{N}_0. \end{cases}$$

*Correspondence: chu.wenchang@unisalento.it
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We have hence the q-gamma function [9, §1.10]

$$\Gamma_q(x) \;=\; (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} \quad\text{and}\quad \lim_{q\to 1^-}\Gamma_q(x) \;=\; \Gamma(x).$$

For the sake of brevity, we shall adopt the abbreviated notations for quotients

$$\begin{aligned}
\begin{bmatrix}\alpha,\beta,\cdots,\gamma\\ A,B,\cdots,C\end{bmatrix}_n &= \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n},\\
\left[\begin{matrix}\alpha,\beta,\cdots,\gamma\\ A,B,\cdots,C\end{matrix}\middle|q\right]_n &= \frac{(\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n\,(B;q)_n\cdots(C;q)_n},\\
\Gamma_q\begin{bmatrix}\alpha,\beta,\cdots,\gamma\\ A,B,\cdots,C\end{bmatrix} &= \frac{\Gamma_q(\alpha)\Gamma_q(\beta)\cdots\Gamma_q(\gamma)}{\Gamma_q(A)\Gamma_q(B)\cdots\Gamma_q(C)}.
\end{aligned}$$

According to Bailey [2] and Gasper–Rahman [9], the classical and basic hypergeometric series are defined, by

$$\begin{aligned}
{}_{1+\ell}F_\ell\left[\begin{matrix}a_0,a_1,\cdots,a_\ell\\ b_1,\cdots,b_\ell\end{matrix}\middle|z\right] &= \sum_{n=0}^{\infty}\begin{bmatrix}a_0,a_1,\cdots,a_\ell\\ 1,b_1,\cdots,b_\ell\end{bmatrix}_n z^n,\\
{}_{1+\ell}\varphi_\ell\left[\begin{matrix}a_0,a_1,\cdots,a_\ell\\ b_1,\cdots,b_\ell\end{matrix}\middle|q;z\right] &= \sum_{n=0}^{\infty}\left[\begin{matrix}a_0,a_1,\cdots,a_\ell\\ q,b_1,\cdots,b_\ell\end{matrix}\middle|q\right]_n z^n.
\end{aligned}$$

Inspired by the works of Andrews [1] and Gessel–Stanton [10], the authors [3, 8] recently evaluated, in closed forms, the terminating cases of the following balanced series

$$\Omega(\lambda,x,y) = \sum_{k=0}^{\infty}\left[\begin{matrix}x,y\\ q\lambda\end{matrix}\middle|q\right]_k \frac{(\lambda xy;q^3)_k}{(xy;q)_{2k}}q^k. \tag{1.1}$$

The aim of this paper is to investigate summation and transformation formulae when this series is nonterminating. They will lead, in particular, to q-analogues of difficult evaluations of three ${}_3F_2(\frac{3}{4})$-series and two ${}_5F_4(\frac{1}{9})$-series, with three of them having appeared in the conjectured list of hypergeometric series identities made by Bill Gosper in his private communication to Richard Askey (December 21, 1977).

Our approach will be the modified Abel lemma on summation by parts (cf. [4, 7]) which can be recorded as follows. For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ, respectively, by

$$\nabla\tau_k = \tau_k - \tau_{k-1} \quad\text{and}\quad \Delta\tau_k = \tau_k - \tau_{k+1}.$$

It should be pointed out that Δ is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign. Then the modified **Abel lemma** on summation by parts reads as

$$\sum_{k=0}^{\infty}B_k\nabla A_k \;=\; \big\{[AB]_+ - A_{-1}B_0\big\} + \sum_{k=0}^{\infty}A_k\,\Delta B_k, \tag{1.2}$$

provided that one of the two series converges and the following limit exists:

$$[AB]_+ = \lim_{n\to\infty}A_nB_{n+1}.$$

2. The first difference pair and implications

Define the difference pair $\{\mathcal{A}_k, \mathcal{B}_k\}$ by

$$\mathcal{A}_k = \left[\begin{matrix} qx,\ qy \\ q\lambda, qxy/\lambda \end{matrix}\Big| q\right]_k \quad \text{and} \quad \mathcal{B}_k = \frac{(qxy/\lambda;q)_k(\lambda xy;q^3)_k}{(xy;q)_{2k}}.$$

Then it is not hard to check the differences

$$\begin{aligned}\nabla\mathcal{A}_k =& q^k \left[\begin{matrix} x,\ y \\ q\lambda, qxy/\lambda \end{matrix}\Big| q\right]_k \frac{\lambda(1-x/\lambda)(1-y/\lambda)}{(1-x)(1-y)},\\ \Delta\mathcal{B}_k =& q^k \frac{(qxy/\lambda;q)_k(\lambda xy;q^3)_k}{(q^2xy;q)_{2k}} \frac{qxy/\lambda(1-q^k\lambda)(1-q^{k-1}\lambda)}{(1-xy)(1-qxy)};\end{aligned}$$

and to determine the boundary conditions

$$[\mathcal{A}\mathcal{B}]_+ = \left[\begin{matrix} qx,\ qy \\ q\lambda, xy \end{matrix}\Big| q\right]_\infty (\lambda xy;q^3)_\infty \quad \text{and} \quad \mathcal{A}_{-1}\mathcal{B}_0 = \frac{(1-\lambda)(1-xy/\lambda)}{(1-x)(1-y)}.$$

According to the modified Abel lemma on summation by parts, we can reformulate the $\Omega(\lambda,x,y)$-series as

$$\begin{aligned}\Omega(\lambda,x,y) =& \sum_{k=0}^{\infty} \left[\begin{matrix} x, y \\ q\lambda \end{matrix}\Big| q\right]_k \frac{(\lambda xy;q^3)_k}{(xy;q)_{2k}} q^k = \frac{(1-x)(1-y)}{\lambda(1-x/\lambda)(1-y/\lambda)} \sum_{k=0}^{\infty} \mathcal{B}_k \nabla\mathcal{A}_k\\ =& \frac{(1-x)(1-y)}{\lambda(1-x/\lambda)(1-y/\lambda)} \Big\{ [\mathcal{A}\mathcal{B}]_+ - \mathcal{A}_{-1}\mathcal{B}_0 + \sum_{k=0}^{\infty} \mathcal{A}_k \Delta\mathcal{B}_k \Big\},\end{aligned}$$

which can be expressed as the following recurrence relation

$$\begin{aligned}\Omega(\lambda,x,y) =& \Omega(q^{-2}\lambda,qx,qy) \frac{xy(1-x)(1-y)(1-1/\lambda)(1-q/\lambda)}{(1-xy)(1-qxy)(1-x/\lambda)(1-y/\lambda)}\\ &+ \frac{(1-1/\lambda)(1-xy/\lambda)}{(1-x/\lambda)(1-y/\lambda)} + \frac{\lambda^{-1}(\lambda xy;q^3)_\infty}{(1-x/\lambda)(1-y/\lambda)} \left[\begin{matrix} x,\ y \\ q\lambda, xy \end{matrix}\Big| q\right]_\infty.\end{aligned}$$

By iterating this relation m-times, we derive, after some simplification, the following transformation formula:

$$\begin{aligned}\Omega(\lambda,x,y) =& \Omega(q^{-2m}\lambda, q^m x, q^m y) q^{m^2-m}(xy)^m \frac{(x;q)_m\,(y;q)_m\,(1/\lambda;q)_{2m}}{(x/\lambda;q^3)_m\,(y/\lambda;q^3)_m\,(xy;q)_{2m}}\\ &+ \frac{\lambda^{-1}(\lambda xy;q^3)_\infty}{(1-x/\lambda)(1-y/\lambda)} \left[\begin{matrix} x,\ y \\ q\lambda, xy \end{matrix}\Big| q\right]_\infty \sum_{k=0}^{m-1} \frac{q^{3k^2}(xy/\lambda^2)^k}{(q^3x/\lambda;q^3)_k\,(q^3y/\lambda;q^3)_k}\\ &+ \frac{(1-1/\lambda)(1-xy/\lambda)}{(1-x/\lambda)(1-y/\lambda)} \sum_{k=0}^{m-1} \frac{1-q^{4k}xy/\lambda}{1-xy/\lambda} \frac{(x;q)_k\,(y;q)_k\,(q/\lambda;q)_{2k} q^{k^2-k}(xy)^k}{(q^3x/\lambda;q^3)_k\,(q^3y/\lambda;q^3)_k\,(xy;q)_{2k}}.\end{aligned}$$

Its limiting case as $m \to \infty$ is highlighted in the lemma below.

Lemma 2.1 (Transformation formula)

$$\Omega(\lambda,x,y)=\frac{\lambda^{-1}(\lambda xy;q^3)_\infty}{(1-x/\lambda)(1-y/\lambda)}\left[\begin{matrix}x,\ y\\ q\lambda,xy\end{matrix}\Big|q\right]_\infty\sum_{k=0}^{\infty}\frac{q^{3k^2}(xy/\lambda^2)^k}{(q^3x/\lambda;q^3)_k\,(q^3y/\lambda;q^3)_k}$$
$$+\frac{(1-1/\lambda)(1-xy/\lambda)}{(1-x/\lambda)(1-y/\lambda)}\sum_{k=0}^{\infty}\frac{1-q^{4k}xy/\lambda}{1-xy/\lambda}\frac{(x;q)_k\,(y;q)_k\,(q/\lambda;q)_{2k}q^{k^2-k}(xy)^k}{(q^3x/\lambda;q^3)_k\,(q^3y/\lambda;q^3)_k\,(xy;q)_{2k}}.$$

Recall the nonterminating transformation of ${}_3\varphi_2$-series (cf. [9, III-10])

$${}_3\varphi_2\left[\begin{matrix}a,\ b,\ c\\ d,\ e\end{matrix}\Big|q;\frac{de}{abc}\right]=\left[\begin{matrix}b,de/ab,de/bc\\ d,e,de/abc\end{matrix}\Big|q\right]_\infty{}_3\varphi_2\left[\begin{matrix}d/b,e/b,de/abc\\ de/ab,\ de/bc\end{matrix}\Big|q;b\right].$$

Then we have, in particular, the following reduced relation

$$\sum_{k=0}^{\infty}\frac{q^{3k^2}(xy/\lambda^2)^k}{(q^3x/\lambda;q^3)_k\,(q^3y/\lambda;q^3)_k}=\lim_{a,c\to\infty}{}_3\varphi_2\left[\begin{matrix}a,\ q^3,\ c\\ q^3x/\lambda,q^3y/\lambda\end{matrix}\Big|q^3;\frac{q^3xy}{\lambda^2ac}\right]$$
$$=\lim_{a,c\to\infty}\left[\begin{matrix}q^3,q^3xy/\lambda^2a,q^3xy/\lambda^2c\\ q^3x/\lambda,q^3y/\lambda,q^3xy/\lambda^2ac\end{matrix}\Big|q^3\right]_\infty{}_3\varphi_2\left[\begin{matrix}x/\lambda,y/\lambda,q^3xy/\lambda^2ac\\ q^3xy/\lambda^2a,q^3xy/\lambda^2c\end{matrix}\Big|q^3;q^3\right]$$
$$=\left[\begin{matrix}q^3\\ q^3x/\lambda,q^3y/\lambda\end{matrix}\Big|q^3\right]_\infty\sum_{k=0}^{\infty}\frac{(x/\lambda;q^3)_k\,(y/\lambda;q^3)_k}{(q^3;q^3)_k}q^{3k}.$$

Substituting this into Lemma 2.1, we get another expression.

Proposition 2.2 (Transformation formula)

$$\Omega(\lambda,x,y)=\frac{(1-1/\lambda)(1-xy/\lambda)}{(1-x/\lambda)(1-y/\lambda)}\sum_{k=0}^{\infty}\frac{1-q^{4k}xy/\lambda}{1-xy/\lambda}\frac{(x;q)_k\,(y;q)_k\,(q/\lambda;q)_{2k}q^{k^2-k}(xy)^k}{(q^3x/\lambda;q^3)_k\,(q^3y/\lambda;q^3)_k\,(xy;q)_{2k}}$$
$$+\frac{1}{\lambda}\left[\begin{matrix}x,\ y\\ q\lambda,xy\end{matrix}\Big|q\right]_\infty\left[\begin{matrix}q^3,\lambda xy\\ x/\lambda,y/\lambda\end{matrix}\Big|q^3\right]_\infty\sum_{k=0}^{\infty}\frac{(x/\lambda;q^3)_k\,(y/\lambda;q^3)_k}{(q^3;q^3)_k}q^{3k}.$$

When $\lambda=1$, the last formula becomes the following elegant formula.

Theorem 2.3 (Summation formula)

$$\sum_{k=0}^{\infty}\left[\begin{matrix}x,y\\ q\end{matrix}\Big|q\right]_k\frac{(xy;q^3)_k}{(xy;q)_{2k}}q^k=\left[\begin{matrix}qx,qy\\ q,xy\end{matrix}\Big|q\right]_\infty\left[\begin{matrix}q^3,xy\\ q^3x,q^3y\end{matrix}\Big|q^3\right]_\infty\sum_{k=0}^{\infty}q^{3k}\frac{(x;q^3)_k\,(y;q^3)_k}{(q^3;q^3)_k}.$$

This is a q-analogue of the formula due to Chu [7, Proposition 6], which has been utilized by Wang et al. [15, Lemma 1.1] to evaluate a nonterminating ${}_7F_6$-series:

$${}_3F_2\left[\begin{matrix}x,\ y,\ \frac{x+y}{3}\\ \frac{x+y}{2},\ \frac{1+x+y}{2}\end{matrix}\Big|\frac{3}{4}\right]=\Gamma\left[\begin{matrix}1+x+y,1+\frac{x}{3},1+\frac{y}{3}\\ 1+x,\ 1+y,\ 1+\frac{x+y}{3}\end{matrix}\right].$$

To evaluate the above series, a very tough and delicate limiting process had to be carried out (see Chu [7, §4], and [5] for a similar approach). However, we can easily deduce it from Theorem 2.3. Making the replacements $x \to q^x$ and $y \to q^y$ first, and then letting $q \to 1$ in Theorem 2.3, we can see that the series on the left becomes the above ${}_3F_2(\frac{3}{4})$-series, whereas the limit of the right member is determined as follows:

$$\begin{aligned}
&\lim_{q\to 1}\left[\begin{matrix} q^{1+x}, q^{1+y} \\ q, q^{x+y} \end{matrix}\middle| q\right]_\infty \left[\begin{matrix} q^3, q^{x+y} \\ q^{3+x}, q^{3+y} \end{matrix}\middle| q^3\right]_\infty \sum_{k=0}^{\infty} \frac{(q^x;q^3)_k\,(q^y;q^3)_k}{(q^3;q^3)_k} q^{3k} \\
=&\lim_{q\to 1} \frac{1-q^3}{1-q} \Gamma_q\left[\begin{matrix} 1,\ x+y \\ 1+x, 1+y \end{matrix}\right] \Gamma_{q^3}\left[\begin{matrix} 1+\frac{x}{3}, 1+\frac{y}{3} \\ 1,\ \frac{x+y}{3} \end{matrix}\right] \\
=&3\Gamma\left[\begin{matrix} 1+\frac{x}{3}, 1+\frac{y}{3}, x+y \\ 1+x, 1+y, \frac{x+y}{3} \end{matrix}\right].
\end{aligned}$$

□

3. The second difference pair and implications

For the difference pair $\{\mathrm{A}_k, \mathrm{B}_k\}$ given by

$$\mathrm{A}_k = \frac{(xy/q^2\lambda;q)_k(q^3\lambda xy;q^3)_k}{(xy;q)_{2k}} \quad \text{and} \quad \mathrm{B}_k = \left[\begin{matrix} x,\ y \\ q^3\lambda, xy/q^3\lambda \end{matrix}\middle| q\right]_k,$$

it is almost routine to compute the differences

$$\begin{aligned}
\nabla \mathrm{A}_k =& q^k \frac{(xy/q^3\lambda;q)_k(\lambda xy;q^3)_k}{(xy;q)_{2k}} \frac{(1-q^{k+1}\lambda)(1-q^{k+2}\lambda)}{(1-\lambda xy)(1-q^3\lambda/xy)}, \\
\Delta \mathrm{B}_k =& q^k \left[\begin{matrix} x,\ y \\ q^4\lambda, xy/q^2\lambda \end{matrix}\middle| q\right]_k \frac{(1-q^3\lambda/x)(1-q^3\lambda/y)}{(1-q^3\lambda)(1-q^3\lambda/xy)};
\end{aligned}$$

and to determine the boundary conditions

$$[\mathrm{AB}]_+ = \frac{(q^3\lambda xy;q^3)_\infty}{1-xy/q^3\lambda} \left[\begin{matrix} x,\ y \\ q^3\lambda, xy \end{matrix}\middle| q\right]_\infty \quad \text{and} \quad \mathrm{A}_{-1}\mathrm{B}_0 = \frac{(1-xy/q)(1-xy/q^2)}{(1-\lambda xy)(1-xy/q^3\lambda)}.$$

In view of the modified Abel lemma on summation by parts, we can manipulate the $\Omega(\lambda, x, y)$-series as follows:

$$\begin{aligned}
\Omega(\lambda, x, y) =& \sum_{k=0}^{\infty} \left[\begin{matrix} x, y \\ q\lambda \end{matrix}\middle| q\right]_k \frac{(\lambda xy;q^3)_k}{(xy;q)_{2k}} q^k = \frac{(1-\lambda xy)(1-q^3\lambda/xy)}{(1-q\lambda)(1-q^2\lambda)} \sum_{k=0}^{\infty} \mathrm{B}_k \nabla \mathrm{A}_k \\
=& \frac{(1-\lambda xy)(1-q^3\lambda/xy)}{(1-q\lambda)(1-q^2\lambda)} \Big\{ [\mathrm{AB}]_+ - \mathrm{A}_{-1}\mathrm{B}_0 + \sum_{k=0}^{\infty} \mathrm{A}_k \Delta \mathrm{B}_k \Big\}.
\end{aligned}$$

Writing this as the recurrence relation

$$\begin{aligned}
\Omega(\lambda, x, y) =& \Omega(q^3\lambda, x, y) \frac{(1-\lambda xy)(1-q^3\lambda/x)(1-q^3\lambda/y)}{(1-q\lambda)(1-q^2\lambda)(1-q^3\lambda)} \\
&+ (\lambda xy) \frac{(1-q/xy)(1-q^2/xy)}{(1-q\lambda)(1-q^2\lambda)} - (q^3\lambda/xy) \left[\begin{matrix} x,\ y \\ q\lambda, xy \end{matrix}\middle| q\right]_\infty (\lambda xy;q^3)_\infty
\end{aligned}$$

and then iterating this relation m-times, we find, after some simplification, the transformation formula:

$$\begin{aligned}\Omega(\lambda,x,y)=\Omega(q^{3m}\lambda,x,y)&\left[\begin{matrix}\lambda xy,q^3\lambda/x,q^3\lambda/y\\ q\lambda,q^2\lambda,q^3\lambda\end{matrix}\Big|q^3\right]_m\\&+\big(\lambda xy\big)\frac{(1-q/xy)(1-q^2/xy)}{(1-q\lambda)(1-q^2\lambda)}\sum_{k=0}^{m-1}q^{3k}\left[\begin{matrix}\lambda xy,q^3\lambda/x,q^3\lambda/y\\ q^3\lambda,q^4\lambda,q^5\lambda\end{matrix}\Big|q^3\right]_k\\&-(q^3\lambda/xy)\left[\begin{matrix}x,\ y\\ q\lambda,xy\end{matrix}\Big|q\right]_\infty(\lambda xy;q^3)_\infty\sum_{k=0}^{m-1}\big(q^3\lambda/x;q^3\big)_k\big(q^3\lambda/y;q^3\big)_k q^{3k}.\end{aligned}$$

Letting further $m\to\infty$, we arrive at a three term expression.

Proposition 3.1 (Transformation formula)

$$\begin{aligned}\Omega(\lambda,x,y)=&\left[\begin{matrix}\lambda xy,q^3\lambda/x,q^3\lambda/y\\ q\lambda,q^2\lambda,q^3\lambda\end{matrix}\Big|q^3\right]_\infty\sum_{k=0}^{\infty}\frac{(x;q)_k\,(y;q)_k}{(xy;q)_{2k}}q^k\\&+\big(\lambda xy\big)\frac{(1-q/xy)(1-q^2/xy)}{(1-q\lambda)(1-q^2\lambda)}\sum_{k=0}^{\infty}q^{3k}\left[\begin{matrix}\lambda xy,q^3\lambda/x,q^3\lambda/y\\ q^3\lambda,q^4\lambda,q^5\lambda\end{matrix}\Big|q^3\right]_k\\&-(q^3\lambda/xy)\left[\begin{matrix}x,\ y\\ q\lambda,xy\end{matrix}\Big|q\right]_\infty(\lambda xy;q^3)_\infty\sum_{k=0}^{\infty}\big(q^3\lambda/x;q^3\big)_k\big(q^3\lambda/y;q^3\big)_k q^{3k}.\end{aligned}$$

When $xy=q^{1+\delta}$ with $\delta=0,\ 1$, we deduce from Proposition 3.1 two identities.

Theorem 3.2 (Summation formulae)

$$\begin{aligned}\sum_{k=0}^{\infty}\left[\begin{matrix}x,q/x\\ q\lambda\end{matrix}\Big|q\right]_k\frac{(q\lambda;q^3)_k}{(q;q)_{2k}}q^k=&\left[\begin{matrix}q^2\lambda x,q^3\lambda/x\\ q^2\lambda,q^3\lambda\end{matrix}\Big|q^3\right]_\infty\sum_{k=0}^{\infty}\frac{(x;q)_k\,(q/x;q)_k}{(q;q)_{2k}}q^k\\&-(q^2\lambda)\left[\begin{matrix}x,\ q/x\\ q\lambda,q\end{matrix}\Big|q\right]_\infty(q\lambda;q^3)_\infty\sum_{k=0}^{\infty}\big(q^2\lambda x;q^3\big)_k\big(q^3\lambda/x;q^3\big)_k q^{3k},\end{aligned}\tag{3.1}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\left[\begin{matrix}x,q^2/x\\ q\lambda\end{matrix}\Big|q\right]_k\frac{(q^2\lambda;q^3)_k}{(q^2;q)_{2k}}q^k=&\left[\begin{matrix}q\lambda x,q^3\lambda/x\\ q\lambda,q^3\lambda\end{matrix}\Big|q^3\right]_\infty\sum_{k=0}^{\infty}\frac{(x;q)_k\,\big(q^2/x;q\big)_k}{(q^2;q)_{2k}}q^k\\&-(q\lambda)\left[\begin{matrix}x,\ q^2/x\\ q\lambda,q^2\end{matrix}\Big|q\right]_\infty(q^2\lambda;q^3)_\infty\sum_{k=0}^{\infty}\big(q\lambda x;q^3\big)_k\big(q^3\lambda/x;q^3\big)_k q^{3k}.\end{aligned}\tag{3.2}$$

They are q-analogues of the following two nonterminating series identities (cf. [13, Proposition 3] and [14, Proposition 4]), discovered by Gosper (1977) and rederived in [7, Propositions 3 & 4] by means of the modified Abel lemma on summation by parts:

$${}_3F_2\left[\begin{matrix}x,\ 1-x,\ \frac{1+\lambda}{3}\\ 1+\lambda,\ \frac{1}{2}\end{matrix}\Big|\frac{3}{4}\right]=\Gamma\left[\begin{matrix}\frac{2+\lambda}{3},\frac{3+\lambda}{3}\\ \frac{2+\lambda+x}{3},\frac{3+\lambda-x}{3}\end{matrix}\right]\frac{2\cos\pi(\frac{x}{3}-\frac{1}{6})}{\sqrt{3}},\tag{3.3}$$

$${}_3F_2\left[\begin{matrix}x,\ 2-x,\ \frac{2+\lambda}{3}\\ 1+\lambda,\ \frac{3}{2}\end{matrix}\Big|\frac{3}{4}\right]=\Gamma\left[\begin{matrix}\frac{1+\lambda}{3},\frac{3+\lambda}{3}\\ \frac{1+\lambda+x}{3},\frac{3+\lambda-x}{3}\end{matrix}\right]\frac{2\sin\pi(\frac{x-1}{3})}{\sqrt{3}(x-1)}.\tag{3.4}$$

In fact, letting $x \to q^x$, $\lambda \to q^\lambda$ and then $q \to 1$ in Theorem 3.2, we can see that both the second terms on the right-hand sides are annihilated. The two first terms on the right–hand sides can be expressed in a unified manner, as follows:

$$\begin{aligned}&\lim_{q\to 1}\left[\begin{matrix}q^{2-\delta+\lambda+x}, q^{3+\lambda-x}\\ q^{2-\delta+\lambda}, q^{3+\lambda}\end{matrix}\Big| q^3\right]_\infty \sum_{k=0}^{\infty}\frac{(q^x;q)_k\left(q^{1+\delta-x};q\right)_k}{(q^{1+\delta};q)_{2k}}q^k\\ &= \Gamma\left[\begin{matrix}\frac{2-\delta+\lambda}{3}, \frac{3+\lambda}{3}\\ \frac{2-\delta+\lambda+x}{3}, \frac{3+\lambda-x}{3}\end{matrix}\right] {}_2F_1\left[\begin{matrix}x, 1+\delta-x\\ \frac{1}{2}+\delta\end{matrix}\Big|\frac{1}{4}\right].\end{aligned}$$

Then we recover (3.3) and (3.4) by evaluating the last ${}_2F_1(\frac{1}{4})$-series for $\delta = 0,\ 1$, respectively, by the following known formulae (cf. Gradshteyn–Ryzhik [11, §9.12]):

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}+x, \frac{1}{2}-x\\ \frac{1}{2}\end{matrix}\Big| y^2\right] = \frac{\cos(2x\arcsin y)}{\sqrt{1-y^2}}, \tag{3.5}$$

$$ {}_2F_1\left[\begin{matrix}1+x, 1-x\\ \frac{3}{2}\end{matrix}\Big| y^2\right] = \frac{\sin(2x\arcsin y)}{2xy\sqrt{1-y^2}}. \tag{3.6}$$

4. Two further infinite series identities

By equating the two expressions displayed in Lemma 2.1 and Proposition 3.1, and then unifying two unilateral series to a bilateral one, we obtain the following strange transformation that expresses a bilateral series in terms of three unilateral series, including a well-poised one.

Proposition 4.1 (Transformation formula)

$$\begin{aligned}&(q^3\lambda/xy)\left[\begin{matrix}x,\ y\\ q\lambda, xy\end{matrix}\Big| q\right]_\infty (\lambda xy;q^3)_\infty \sum_{k=-\infty}^{\infty}\left(q^3\lambda/x;q^3\right)_k\left(q^3\lambda/y;q^3\right)_k q^{3k}\\ &= \left[\begin{matrix}\lambda xy, q^3\lambda/x, q^3\lambda/y\\ q\lambda, q^2\lambda, q^3\lambda\end{matrix}\Big| q^3\right]_\infty \sum_{k=0}^{\infty}\frac{(x;q)_k\,(y;q)_k}{(xy;q)_{2k}}q^k\\ &+(\lambda xy)\frac{(1-q/xy)(1-q^2/xy)}{(1-q\lambda)(1-q^2\lambda)}\sum_{k=0}^{\infty}q^{3k}\left[\begin{matrix}\lambda xy, q^3\lambda/x, q^3\lambda/y\\ q^3\lambda, q^4\lambda, q^5\lambda\end{matrix}\Big| q^3\right]_k\\ &-\frac{(1-1/\lambda)(1-xy/\lambda)}{(1-x/\lambda)(1-y/\lambda)}\sum_{k=0}^{\infty}\frac{1-q^{4k}xy/\lambda}{1-xy/\lambda}\frac{(x;q)_k\,(y;q)_k\,(q/\lambda;q)_{2k}q^{k^2-k}(xy)^k}{(q^3x/\lambda;q^3)_k\,(q^3y/\lambda;q^3)_k\,(xy;q)_{2k}}.\end{aligned}$$

By making use of the transformation

$$ {}_2\psi_2\left[\begin{matrix}a, c\\ 0, 0\end{matrix}\Big| q;q\right] = \sum_{n=-\infty}^{\infty}(a;q)_n(c;q)_n q^n = \left[\begin{matrix}qa\\ q, q/c\end{matrix}\Big| q\right]_\infty \sum_{k=-\infty}^{\infty}(-c)^k\frac{1-a}{1-q^k a}q^{\binom{k+1}{2}},$$

which can be deduced from a combination of Lemma 5, Theorem 6 and Equation 11 appearing in Chu [6], we can reformulate the bilateral series

$$\sum_{k=-\infty}^{\infty}\left(q^3\lambda/x;q^3\right)_k\left(q^3\lambda/y;q^3\right)_k q^{3k}$$

$$=\left[\begin{matrix}q^6\lambda/x\\ q^3,y/\lambda\end{matrix}\Big|q^3\right]_\infty\sum_{k=-\infty}^{\infty}\Big(-q^3\frac{\lambda}{y}\Big)^k\frac{1-q^3\lambda/x}{1-q^{3+3k}\lambda/x}q^{3\binom{k+1}{2}}$$

$$=\frac{q^{-3}xy/\lambda^2}{1-x/\lambda}\left[\begin{matrix}q^3\lambda/x\\ q^3,y/\lambda\end{matrix}\Big|q^3\right]_\infty\sum_{k=-\infty}^{\infty}\Big(-\frac{\lambda}{y}\Big)^k\frac{1-\lambda/x}{1-q^{3k}\lambda/x}q^{3\binom{k+1}{2}}.$$

Substituting this into the equality in Proposition 4.1 and then letting $xy=q^{1+\delta}$, we get the following reduced expression.

Theorem 4.2 (Summation formula: $\delta=0,\ 1$)

$$\sum_{k=0}^{\infty}\frac{1-q^{1+\delta+4k}/\lambda}{1-q^{1+\delta}/\lambda}\frac{(x;q)_k\left(q^{1+\delta}/x;q\right)_k(q/\lambda;q)_{2k}q^{k^2+k\delta}}{(q^3x/\lambda;q^3)_k\,(q^{4+\delta}/\lambda x;q^3)_k\,(q^{1+\delta};q)_{2k}}$$

$$=\left[\begin{matrix}\lambda/x,q^{-1-\delta}\lambda x\\ \lambda,q^{-1-\delta}\lambda\end{matrix}\Big|q^3\right]_\infty\sum_{k=0}^{\infty}\frac{(x;q)_k\left(q^{1+\delta}/x;q\right)_k}{(q^{1+\delta};q)_{2k}}q^k$$

$$+\frac{1-q^{1+\delta}\lambda}{1-q^{1+\delta}/\lambda}\left[\begin{matrix}x,q^{1+\delta}/x\\ \lambda,q^{1+\delta}\end{matrix}\Big|q\right]_\infty\left[\begin{matrix}q^3\lambda/x,q^{4+\delta}\lambda\\ q^3,q^{4+\delta}/\lambda x\end{matrix}\Big|q^3\right]_\infty$$

$$\times\sum_{k=-\infty}^{\infty}\big(-\lambda x\big)^k\frac{1-\lambda/x}{1-q^{3k}\lambda/x}q^{3\binom{k}{2}+2k-k\delta}.$$

The last formula results in q-analogues of the following two classical hypergeometric series identities (see Chu [7, Corollaries 8 & 9]) with the first one due to Gosper (1977):

$${}_5F_4\left[\begin{matrix}\frac{1-\lambda}{2},\frac{2-\lambda}{2},1+\frac{1-\lambda}{4},\ x,\ 1-x\\ \frac{1}{2},\frac{1-\lambda}{4},\frac{3+x-\lambda}{3},\frac{4-x-\lambda}{3}\end{matrix}\Big|\frac{1}{9}\right]=\Gamma\left[\begin{matrix}\frac{1}{3},\frac{2}{3},\frac{3+x-\lambda}{3},\frac{4-x-\lambda}{3}\\ \frac{1+x}{3},\frac{2-x}{3},\frac{3-\lambda}{3},\frac{4-\lambda}{3}\end{matrix}\right],\tag{4.1}$$

$${}_5F_4\left[\begin{matrix}\frac{1-\lambda}{2},\frac{2-\lambda}{2},1+\frac{2-\lambda}{4},\ x,\ 2-x\\ \frac{3}{2},\frac{2-\lambda}{4},\frac{3+x-\lambda}{3},\frac{5-x-\lambda}{3}\end{matrix}\Big|\frac{1}{9}\right]=\Gamma\left[\begin{matrix}\frac{2}{3},\frac{4}{3},\frac{3+x-\lambda}{3},\frac{5-x-\lambda}{3}\\ \frac{2+x}{3},\frac{4-x}{3},\frac{3-\lambda}{3},\frac{5-\lambda}{3}\end{matrix}\right].\tag{4.2}$$

We can justify this fact by examining the limiting case $q\to 1$ of Theorem 4.2 after having made the replacements $x\to q^x,\ y\to q^y$ and $\lambda\to q^\lambda$. The right hand side of the resulting equation reads as:

$$\Gamma\left[\begin{matrix}\frac{\lambda}{3},\frac{\lambda-1-\delta}{3}\\ \frac{\lambda-x}{3},\frac{\lambda-1-\delta+x}{3}\end{matrix}\right]{}_2F_1\left[\begin{matrix}x,1+\delta-x\\ \frac{1}{2}+\delta\end{matrix}\Big|\frac{1}{4}\right]+\frac{1+\delta+\lambda}{1+\delta-\lambda}\Gamma\left[\begin{matrix}1,\frac{4+\delta-\lambda-x}{3}\\ \frac{3+\lambda-x}{3},\frac{4+\delta+\lambda}{3}\end{matrix}\right]$$
$$\times\Gamma\left[\begin{matrix}\frac{1+\delta}{3},\frac{2+\delta}{3},\frac{3+\delta}{3},\frac{\lambda}{3},\frac{1+\lambda}{3},\frac{2+\lambda}{3}\\ \frac{x}{3},\frac{1+x}{3},\frac{2+x}{3},\frac{1+\delta-x}{3},\frac{2+\delta-x}{3},\frac{3+\delta-x}{3}\end{matrix}\right]\sum_{k=-\infty}^{\infty}(-1)^k\frac{\lambda-x}{3k+\lambda-x}.\tag{4.3}$$

Evaluating first the above ${}_2F_1$-series by (3.5) and (3.6)

$$ {}_2F_1\left[\begin{matrix} x, 1+\delta-x \\ \frac{1}{2}+\delta \end{matrix}\middle| \frac{1}{4}\right] = \begin{cases} \dfrac{2}{\sqrt{3}}\sin\dfrac{\pi}{3}(x+1), & \delta=0; \\ \dfrac{2}{\sqrt{3}}\dfrac{\sin\frac{\pi}{3}(x+2)}{1-x}, & \delta=1; \end{cases} $$

we can reformulate the first term of (4.3)

$$ \Gamma\left[\begin{matrix} \frac{\lambda}{3}, \frac{\lambda-1-\delta}{3} \\ \frac{\lambda-x}{3}, \frac{\lambda-1-\delta+x}{3} \end{matrix}\right] \frac{2}{\sqrt{3}} \frac{\sin\frac{\pi}{3}(1+\delta+x)}{(1-x)_\delta} = \frac{2}{\sqrt{3}} \frac{\sin\frac{\pi}{3}(1+\delta+x)}{(1-x)_\delta} \Gamma\left[\begin{matrix} \frac{3-\lambda+x}{3}, \frac{4+\delta-\lambda-x}{3} \\ \frac{3-\lambda}{3}, \frac{4+\delta-\lambda}{3} \end{matrix}\right] $$
$$ \times \frac{\sin\frac{\pi}{3}(\lambda-x)\sin\frac{\pi}{3}(\lambda-1-\delta+x)}{\sin\frac{\pi\lambda}{3}\sin\frac{\pi}{3}(\lambda-1-\delta)}. $$

Then by evaluating the bilateral series (cf. Gradshteyn–Ryzhik [11, §1.422]):

$$ \sum_{k=-\infty}^{\infty}(-1)^k\frac{\lambda-x}{3k+\lambda-x} = 1+\sum_{k=1}^{\infty}(-1)^k\frac{2(\lambda-x)^2}{(\lambda-x)^2-9k^2} $$
$$ = \frac{\frac{\pi}{3}(\lambda-x)}{\sin\frac{\pi}{3}(\lambda-x)} = \Gamma\Big(\frac{3+\lambda-x}{3}\Big)\Gamma\Big(\frac{3-\lambda+x}{3}\Big), $$

we can also manipulate the second term of (4.3)

$$ \Gamma\left[\begin{matrix} \frac{3-\lambda+x}{3}, \frac{4+\delta-\lambda-x}{3} \\ \frac{3-\lambda}{3}, \frac{4+\delta-\lambda}{3} \end{matrix}\right] \times \Gamma\left[\begin{matrix} \frac{1+\delta}{3}, \frac{2+\delta}{3}, \frac{3+\delta}{3}, \frac{\lambda}{3}, \frac{2-\delta+\lambda}{3}, \frac{3-\lambda}{3}, \frac{1+\delta-\lambda}{3} \\ \frac{x}{3}, \frac{1+x}{3}, \frac{2+x}{3}, \frac{1+\delta-x}{3}, \frac{2+\delta-x}{3}, \frac{3+\delta-x}{3} \end{matrix}\right] $$
$$ = \Gamma\left[\begin{matrix} \frac{3-\lambda+x}{3}, \frac{4+\delta-\lambda-x}{3} \\ \frac{3-\lambda}{3}, \frac{4+\delta-\lambda}{3} \end{matrix}\right] \frac{2\sin\frac{\pi x}{3}\sin\frac{\pi}{3}(1+x)\sin\frac{\pi}{3}(2+x)}{\sqrt{3}(1-x)_\delta\sin\frac{\pi\lambda}{3}\sin\frac{\pi}{3}(1+\delta-\lambda)}. $$

Now putting these two terms together, we can express (4.3) as

$$ \Gamma\left[\begin{matrix} \frac{3-\lambda+x}{3}, \frac{4+\delta-\lambda-x}{3} \\ \frac{3-\lambda}{3}, \frac{4+\delta-\lambda}{3} \end{matrix}\right] \frac{2\sin\frac{\pi}{3}(1+\delta+x)}{\sqrt{3}(1-x)_\delta\sin\frac{\pi\lambda}{3}\sin\frac{\pi}{3}(\lambda-1-\delta)} $$
$$ \times\Big\{\sin\frac{\pi}{3}(\lambda-x)\sin\frac{\pi}{3}(\lambda-1-\delta+x)-\sin\frac{\pi x}{3}\sin\frac{\pi}{3}(2-\delta+x)\Big\}. $$

After factorizing the trigonometric difference in the braces "$\{\cdots\}$", we get the following simplified expression

$$ \Gamma\left[\begin{matrix} \frac{3-\lambda+x}{3}, \frac{4+\delta-\lambda-x}{3} \\ \frac{3-\lambda}{3}, \frac{4+\delta-\lambda}{3} \end{matrix}\right] \frac{2\sin\frac{\pi}{3}(1+\delta+x)}{\sqrt{3}(1-x)_\delta} = \Gamma\left[\begin{matrix} \frac{2}{3}, \frac{1}{3}+\delta, \frac{3-\lambda+x}{3}, \frac{4+\delta-\lambda-x}{3} \\ \frac{1+\delta+x}{3}, \frac{2+2\delta-x}{3}, \frac{3-\lambda}{3}, \frac{4+\delta-\lambda}{3} \end{matrix}\right], $$

which coincides with those displayed in (4.1) and (4.2). □

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