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Stability in Commutative Rings

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Abstract: Let R be a commutative ring with zero-divisors and I an ideal of R. I is said to be *ES-stable* if $JI = I^2$ for some invertible ideal $J \subseteq I$, and I is said to be a *weakly ES-stable ideal* if there is an invertible fractional ideal J and an idempotent fractional ideal E of R such that $I = JE$. We prove useful facts for weakly ES-stability and investigate this stability in Noetherian-like settings. Moreover, we discuss a question of A. Mimouni on locally weakly ES-stable rings: is a locally weakly ES-stable domain of finite character weakly ES-stable?

Key words: Weakly ES-stable rings, Prüfer rings, H-local rings, local-global rings, Noetherian rings

1. Introduction

Let R be a commutative ring with zero divisors. We call an element of R *regular* if it is not a zero-divisor. Let $Reg(R)$ denote the monoid of regular elements of R and $Q(R) = Q$ denote the total ring of fractions R. We note that $Q = (Reg(R))^{-1}R$. Let $\tilde{R}$ denote the integral closure of R in $Q(R)$. We say that an ideal I of R is *regular* if I contains a regular element of R. We note that every invertible fractional ideal of R is finitely generated and regular. For a prime ideal P of R, we set $R_{(P)} = (Reg(R)\ P)^{-1}R \subseteq Q$.

We say that R is *of finite character* or *has finite character* if every $x \in Reg(R)$ is contained in at most finitely many maximal ideals of R. We call a ring R is *h-local* if R has finite character and every nonzero regular prime ideal of R is contained in a unique maximal ideal. We say that R is *local-global* if every polynomial over R in finitely many indeterminates which represents units locally, assumes a unit value when evaluated at properly chosen elements of R [10, V.4]. Rings of Krull dimension 0 and semilocal rings are local-global. A ring is *almost local-global* if every of its proper factor ring is local-global. We note that domains of finite character are almost local-global.

For the ideals I and J of R, the colon ideal $(I : J)$ is defined to be $\{q \in Q : qJ \subseteq I\}$. For the ideals I and J of the ring R, with J regular, the natural map from $(I : J)$ to $Hom_R(J, I)$ is an isomorphism [2, Lemma 1.1]. Thus, the endomorphism ring of a regular ideal I, $End_R(I) = (I : I)$. Furthermore, for a regular ideal I, the inverse of I in R, I^{-1} coincides with $(R : I)$.

For a nonzero ideal I of R, $(R : I) = I^{-1}$ and $(I^{-1})^{-1} = I_v$. I is a *v-ideal* if $I = I_v$. An ideal $I \neq 0$ is called a *t-ideal* if for nonzero $x_1, \ldots, x_n \in I, (x_1, \ldots, x_n)_v \subseteq I$. Thus, I is a t-ideal if and only if $I = \bigcup J_v$ where J runs over the set of nonzero finitely generated ideals of R contained in I. An ideal I of R is called

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a *w-ideal* if I is the set of all $x \in R$ such that $xJ \subseteq I$ for some nonzero finitely generated ideal J of R with $J_t = R$. Let $Max(R)$ be the set of maximal ideals of R.

In the literature, there are different types of stabilities described and facts relating these stabilities. Sally and Vasconcelos introduced the notion of SV-stability [21, 22]. An ideal I of R is called *SV-stable* if it is projective over its endomorphism ring, $End_R(I)$. Furthermore, we remark that an SV-stable ideal, over an integral domain, is invertible in $End_R(I)$. We note that, over a commutative ring, if I is a finitely generated *regular* SV-stable ideal then I is invertible in $End(I)$. R is called an *SV-stable* (*finitely SV-stable*, respectively) ring if every regular (finitely generated, respectively) ideal of R is SV-stable.

Another type of stability is introduced by Eakin and Sathaye: the notion of ES-stability [9, Section 7.4]. In a general commutative ring R, an ideal is called ES-stable if $I^2 = JI$ for some invertible ideal J of R such that $J \subseteq I$ [17]. We define R to be *ES-stable* (*finitely ES-stable*, respectively) if every regular ideal (finitely generated regular ideal, respectively) of R is ES-stable. We say that an ideal I of a ring is *ES-prestable* if some power of I is ES-stable. We define a ring R to be *ES-prestable* (*finitely ES-prestable*, respectively) if every regular ideal (finitely generated regular ideal, respectively) in R is ES-prestable.

In [17], a weak form of ES-stability for integral domains is defined. Here we need to modify its definition for commutative rings with zero-divisors. We call an ideal I of R is said to be a *weakly ES-stable ideal* if there is an invertible fractional ideal J and an idempotent fractional ideal E of R such that $I = JE$, and R is said to be a *weakly ES-stable ring* if every regular ideal of R is a weakly ES-stable ideal. R is said to be locally weakly ES-stable if R_M is weakly ES-stable for each maximal ideal M of R. We note that if R is weakly ES-stable, then R is locally weakly ES-stable. Moreover, R is said to be a *finitely weakly ES-stable ring* if every finitely generated regular ideal of R is weakly ES-stable. A nonzero ideal I of R is said to be an *almost weakly ES-stable ideal* if some power of I is a weakly ES-stable ideal, and R is said to be an *almost weakly ES-stable ring* if every regular ideal of R is almost weakly ES-stable. Moreover, R is said to be a *finitely almost weakly ES-stable ring* if every finitely generated regular ideal of R is almost weakly ES-stable.

In §2 we prove preliminary results for weakly ES-stable ideals over commutative rings with zero-divisors and later focus on finitely weakly ES-stable rings. In §3 we study finitely ES-stability in Prüfer rings and Notherian-like settings. §4 discusses a question of Mimouni on locally weakly ES-stable rings: is a locally weakly ES-stable domain of finite character weakly ES-stable? We show that in Krull domains, Prüfer h-local domains, and Noetherian local-global rings, these two notions coincide. We provide an example of a one dimensional Noetherian ring of finite character where there is a locally weakly ES-stable ideal which is not weakly ES-stable.

2. Some results on weakly ES-stability

In [17], many facts are stated and proved for weakly ES-stable ideals of an integral domain. We adapt these results to commutative rings with zero-divisors, and eventually, show that weakly finitely ES-stability coincides with finitely ES-stability.

Proposition 2.1 *Let R be a commutative ring and I a nonzero ideal of R.*

(i) I is a weakly ES-stable ideal if and only if $I^2 = JI$ for some invertible ideal J of R.

(ii) If I is a weakly ES-stable ideal and $I = JE$ where $JJ^{-1} = R$ and $E = E^2$, then $(I : I) = (E : E)$ and $E = I(I : I^2)$.

Proof

(i) If $I = JE$ with $JJ^{-1} = R$ and $E^2 = E$, then $I^2 = J^2E^2 = J^2E = JJE = JI$. Conversely, if $I^2 = JI$ for some invertible ideal J of R, then $I = J(J^{-1}I)$ and $J^{-1}I$ is idempotent.

(ii) Set $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. Let $x \in (I : I)$. Then $xJE = xI \subseteq I = JE$, and hence $xJJ^{-1}E \subseteq JJ^{-1}E = E$ implying that $x \in (E : E)$. If $x \in (E : E)$, then $xI = xJE \subseteq JE = I$. Thus, $(I : I) = (E : E)$. Next we claim that $E = I(I : I^2)$. By (i), $I^2 = JI$, so $J^{-1}I^2 = I$. Then $J^{-1} \subseteq (I : I^2)$ implying that $E = J^{-1}I \subseteq I(I : I^2)$. Conversely, let $x \in (I : I^2)$. Then $xJ^2E \subseteq JE$. Since J is invertible, $xJE \subseteq E$. So, $xJ \subseteq (E : E)$ implying that $xI = xJE \subseteq E(E : E) = E$. Thus, $I(I : I^2) \subseteq E$. Therefore, $E = I(I : I^2)$.

□

Lemma 2.2 *Let R be a commutative ring and I a regular ideal of R.*

(i) I is ES-stable if and only if $I = JE$ where J is invertible and $E = E^2$ and $J \subseteq I \subseteq E$.

(ii) If I is a finitely generated weakly ES-stable ideal, then $I_t \subsetneq R$.

(iii) If R is a weakly ES-stable ring, then $A_t \subsetneq R$ for every integral regular ideal A of R.

Proof

(i) If I is ES-stable, then $I^2 = JI$ for some invertible ideal $J \subseteq I$ of R. Let $E = J^{-1}I$. Since $JJ^{-1} = R$, $I = JE$ with $E^2 = E$. Since $J \subseteq I$ and I is regular, $I^{-1} \subseteq J^{-1}$, and hence $I \subseteq II^{-1} \subseteq IJ^{-1} = E$. Thus, $J \subseteq I \subseteq E$. The converse is clear.

(ii) Suppose that $I_t = R$. Since I is regular, $I^{-1} = (R : I)$, and since $I_v = I_t = R$, $(I : I) = I^{-1} = R$. Set $I = JE$ with $JJ^{-1} = R$ and $E^2 = E$. By Proposition 2.1, $(I : I) = (E : E)$, and hence $(E : E) = R$. Since $E^2 = E$, $E \subseteq (E : E) = R$ so that $I = JE \subseteq J$. Since I is regular, $J^{-1} \subseteq I^{-1} = R$ implying that $R = JJ^{-1} \subseteq J$. Then $I \subseteq IJ = I^2$ so that $I = I^2$. Since I is finitely generated, by Nakayama's Lemma, there exists $x \in R$ such that $xI = 0$, which is impossible because I is regular.

(iii) Suppose that R is a weakly ES-stable ring and $A_t = R$ for an integral regular ideal A of R. Then, by part (i), there exists a finitely generated sub-ideal J of A, and $J_t = J_v = R$, which is impossible by part (ii). Hence, $A_t \subsetneq R$.

□

Lemma 2.3 *If R is a finitely weakly ES-stable ring, then so is any overring R' of R, that is $R \subseteq R' \subseteq Q(R)$.*

Proof Let A be a finitely generated ideal of R'. Then $A = R's_1 + \ldots + R's_t$ for some $s_1, s_2, \ldots, s_t \in A$. So, there exists a regular element $c \in R$ such that $cs_i \in R$ for all i. Thus, $I = Rcs_1 + \ldots + Rcs_t$, which is isomorphic to A as an R-module, is a finitely generated regular ideal of R. If $I = JE$, where $JJ^{-1} = R$ and $E^2 = E$, then $JR'(R' : JR') = R'$ and $(ER')^2 = ER'$ with $A = (JR')(ER')$ implying that A is finitely weakly ES-stable. □

Lemma 2.4 *Let R be a commutative ring and I a finitely generated ideal of R. Then I is ES-stable if and only if I is weakly ES-stable. In particular, R is a finitely ES-stable ring if and only if R is finitely weakly ES-stable.*

Proof

One way is clear by Lemma 2.2. Conversely, let I be a finitely generated ideal. Then $I = JE$, where $JJ^{-1} = R$ and $E = E^2$. Set $T = (I : I)$. It follows from Lemma 2.3 that T is a finitely weakly ES-stable ring. Applying Proposition 2.1 to I, we have $E = I(T : I)$ is an idempotent integral ideal of T. We observe that $E = IJ^{-1}$ is a finitely generated fractional ideal of R, and hence of $T = (I : I) = (E : E) = (T : E)$. So, $E_{t_T} = E_{v_T} = T$ (t_T and v_T are the $t-$ and $v-$operations with respect to T), which is not possible by Lemma 2.2, and hence $E = T$ implying that $I = JT$. Thus, $J \subseteq I$, and hence I is ES-stable by Lemma 2.2. □

We recall that an integral domain R is said to be conducive if $(R : T) \neq (0)$ for each overring T of R with $T \subset Q(R)$. In [17, Corollary 4.4], it is proven that a conducive domain which is weakly ES-stable is semilocal. Here we observe that a conducive domain of finite character which is finitely weakly ES-stable must be semilocal.

Corollary 2.5 *Let R be a conducive domain which is finitely weakly ES-stable. If R has finite character, then R is semilocal.*

Proof Let $\tilde{R}$ be the integral closure of R. By [3, Lemma 3.4], $\tilde{R}$ is a Prüfer domain, and by [3, Lemma 3.6], $\tilde{R}$ is finitely ES-stable. Moreover, $\tilde{R}$ is a conducive domain. Since, for every $P \in Max(R)$, there is a $Q \in Max(\tilde{R})$ such that $P = Q \cap R$, it is enough to show that $\tilde{R}$ is semilocal. So, without loss of generality we assume that R is a conducive Prüfer domain which is finitely ES-stable.

Let $M \in Max(R)$, and set $P = (R : R_M)$. By assumption, $P \neq 0$. We may assume that R is not local, that is $R \neq R_M$, so P is a proper prime ideal of R. By [5, Lemma 2.10], P is a prime ideal of both R and R_M. Let $Q \in Max(R)$ with $Q \neq M$, and let $a \in Q - M$. Then, for each $x \in P$, $\frac{x}{a} \in PR_M = P$. So, $x \in aP \subseteq PQ \subseteq Q$. Thus, $P \subseteq Q$. Therefore, P is contained in all maximal ideals of R. Since R has finite character, R is semilocal. □

Next we prove a couple of helpful lemmas to show that, given a commutative ring R such that the endomorphism ring of each finitely generated regular ideal of R is local-global, R is ES-stable if and only if R is SV-stable.

Lemma 2.6 *Let R be a commutative ring and I a regular ideal of R. If I is an ES-stable ideal, then I is SV-stable.*

Proof Suppose that I is ES-stable. Then $JI = I^2$ for some invertible ideal $J \subseteq I$. Since I is regular, its endomorphism ring is $E = (I : I)$. So, $(J^{-1}I)I = I$, and hence $J^{-1}I \subseteq E$. Let $x \in E$. Then $xJ \subseteq I$. Hence, $x \in J^{-1}I$. Therefore, $J^{-1}I = E$, so that $J^{-1}E$ is the inverse of I in $End(I)$, that is I is SV-stable. □

Lemma 2.7 *Let R be an SV-stable ring and I a regular ideal of R. If the endomorphism ring of each finitely generated regular ideal of R is local-global, then I is ES-stable.*

Proof Let $E = (I : I)$, the endomorphism ring of I. If I is SV-stable, then $I = x_1E + \ldots + x_tE$ for some $x_1, \ldots, x_t \in I$. So, $I^2 = x_1I + \ldots + x_tI$. Let $J = x_1R + \ldots + x_tR$. We observe that $I^2 \subseteq J \subseteq I$ and $EJ = I$.

Since J is SV-stable and $E' = (J : J)$ is local-global, $J = jE'$ by [10, Proposition V.4.4]. Since $EJ = I$, $E' \subseteq E$, $I = EJ = Ej$ with $j \in J \subseteq I$. Since $E^2 = E$, $I^2 = j^2E = jI$, and hence I is ES-stable. □

Theorem 2.8 *Let R be a commutative ring such that the endomorphism ring of each finitely generated regular ideal of R is local-global, R is ES-stable if and only if R is SV-stable.*

Proof Follows immediately from Lemmas 2.6 and 2.7. □

Theorem 2.9 *Let R be a local-global ring. Then R is finitely ES-stable if and only if R is finitely SV-stable.*

Proof If I is a finitely generated regular ideal of R, then the endomorphism ring of I, $R \subseteq (I : I)$ is an integral extension. By [6, Corollary 2.3], $(I : I)$ is local-global. By Lemma 2.7 and Lemma 2.8, I is an ES-stable ideal if and only if I is SV-stable. □

Theorem 2.10 *Let R be a commutative ring. Then the following are equivalent.*

(i) R is finitely SV-stable.

(ii) R is locally finitely SV-stable.

(iii) R is locally finitely (weakly) ES-stable.

Proof $(i) \Rightarrow (ii)$ is trivial. $(ii) \Leftrightarrow (iii)$ holds by Theorem 2.8.

$(ii) \Rightarrow (i)$: Suppose R is locally finitely SV-stable. Let I be a finitely generated regular ideal of R. Since $(I : I)$ is contained in the integral closure of R, it is integral over R so that $M = N \cap R$ is a maximal ideal of R for each maximal ideal N of $(I : I)$. By assumption, I_M is invertible in $(I : I)_M$ so that I_N is invertible in $(I : I)_N$ for each maximal ideal N since $R \subseteq (I : I)_M \subseteq (I : I)_N$. Hence I is SV-stable. □

3. ES-stability in Prüfer rings and Noetherian-like settings

In this section, we study ES-stability and weakly ES-stability in Prüfer rings with zero-divisors and Noetherian-like settings, especially in Krull rings. We recall that R is a Prüfer ring if and only if every finitely generated (or two-generated) regular ideal is invertible.

In [3], an ideal I of a local ring is called ES-stable if $xI = I^2$ for some $x \in I$, and a commutative ring R is called ES-(pre)stable if any regular ideal I of R is locally ES-(pre)stable. This definition uses the terminology in [7]. It is proven that, for a commutative ring with zero-divisors, R is integrally closed and finitely ES-prestable (in the sense of [7]) if and only if R is a Prüfer ring [3, Theorem 4.1]. Also, by [3, Lemma 3.7], I is finitely ES-prestable (in the sense of [7]) if and only if I is invertible. Over an integrally closed ring R, if I is a regular finitely generated ideal of R, then $R = (I : I)$. Hence, if I is finitely ES-stable, then it is SV-stable so that I is invertible in R. Thus, R is a Prüfer ring if and only if R is integrally closed and it is finitely ES-stable (in the sense explained in Section 1).

Theorem 3.1 *Let R be a commutative ring with zero-divisors. The following are equivalent for R.*

(i) R is an integrally closed ring which is finitely (weakly) ES-stable,

(ii) R *is integrally closed and for each* $a, b \in R$ *with* a *regular there is a positive integer* n *such that* $(a, b)^n$ *can be generated by* n *elements.*

(iii) R *is a Prüfer ring.*

Proof

$(i) \Leftrightarrow (iii)$: From Lemma 2.4, finitely weakly ES-stability and ES-stability coincide. So, it follows immediately from the previous paragraph.

$(ii) \Leftrightarrow (iii)$: Follows from [3, Theorem 4.5].

□

In [17], the author shows that, for Noetherian domains, weakly ES-stability and ES-stability coincide [17, Theorem 3.1]. We show that this is true for Noetherian rings with zero-divisors.

Theorem 3.2 *Let* R *be a Noetherian ring with zero-divisors. Then* R *is weakly ES-stable if and only if* R *is ES-stable.*

Proof Follows immediately from Lemma 2.4 since each regular ideal of R is finitely generated. □

Since ES-stability implies SV-stability (Lemma 2.6) and an SV-stable Noetherian ring is at most one-dimensional [22, Proposition 2.1], a weakly ES-stable Noetherian ring has dimension at most 1.

Theorem 3.3 *A weakly ES-stable Noetherian ring with zero-divisors has dimension at most 1.*

We recall that an integral domain R is said to be a strong Mori domain if R satisfies the *acc* on *w-ideals*. We note that Noetherian domains are strong Mori domains. In [17, Corollary 3.2], it is proven that a strong Mori domain which is weakly ES-stable is Noetherian. Next we show that, for a strong Mori domain, being *finitely* weakly ES-stable is enough to be Noetherian.

Theorem 3.4 *Let* R *be a strong Mori domain which is finitely weakly ES-stable. Then* R *is Noetherian.*

Proof By [17, Lemma 2.4], each maximal ideal of R is a t-maximal ideal, and, hence by [16, Proposition 1.3], each ideal of R is a w-ideal. Thus, R is Noetherian. □

We recall that a commutative ring R is said to be a Krull ring if R is a completely integrally closed Mori ring. In the rest of this section, we study weakly ES-stability in Krull rings.

Theorem 3.5 *Let* R *be a Krull ring with zero-divisors and* I *an ideal of* R*. If* I *is weakly ES-stable ideal, then* I *is an invertible fractional ideal of* R*. Moreover,* I *is weakly ES-stable if and only if* I *is ES-stable.*

Proof Let I be a weakly ES-stable regular ideal of R. Then $I = JE$ with $JJ^{-1} = R$ and $E = E^2$. By [13, Theorem 8.4] and Proposition 2.1, $(E : E) = (I : I) = R$ is a Krull ring. Since $E^2 = E$, $E \subseteq (E : E) = R$, and hence E is an idempotent integral ideal of R. Since $(R : E) = ((E : E) : E) = (E : E^2) = (E : E) = R$, $E = R$, and hence $I = JR$ so that $I = J$, making I an invertible fractional ideal. Also, I is ES-stable. □

Lemma 3.6 *Let* R *be a completely integrally closed ring with zero-divisors which is finitely ES-stable. Then* R *is Prüfer.*

Proof Let I be a finitely generated regular ideal of R. Since I is an ES-stable ideal, and SV-stable by Lemma 2.6. So, I is invertible in its endomorphism ring $E = (I : I)$. Since R is integrally closed, $E = R$ so that I is an invertible ideal of R. □

Theorem 3.7 *Let R be a Krull domain which is finitely ES-stable. Then R is Dedekind.*

Proof This follows from Theorem 3.4 and Lemma 3.6. □

4. Some results on locally weakly ES-stability

In [17], Mimouni shows that a Prüfer domain that is locally weakly ES-stable need not be weakly ES-stable. Given the fact that an integral domain is SV-stable if and only if it is locally SV-stable with finite character ([19, Theorem 3.3]), Mimouni shows that a weakly ES-stable domain is a locally weakly ES-stable domain of finite character ([17, Remark 2.3(iii)], [17, Corollary 2.6]) and asks whether a locally weakly ES-stable domain of finite character is weakly ES-stable. We first show that this question has an affirmative answer for Krull domains. Then we show for Prüfer h-local domains and Noetherian local-global rings these two notions coincide.

We first discuss Mimouni's question for domains of finite character.

Lemma 4.1 *If R is finitely locally weakly ES-stable ring, then there exists a finitely generated ideal $J \subseteq I$ of R such that $I^2 = JI$.*

Proof R is finitely locally weakly ES-stable ring if and only if R is finitely SV-stable (Theorem 2.10). If I is a finitely generated regular ideal of R, then $I = J(I : I)$ for a finitely generated ideal J contained in I by assumption. So, $I^2 = IJ(I : I) = JI$.

□

Lemma 4.2 *Let R be an integral domain of finite character and I a nonzero ideal of R.*

(i) R is locally ES-stable if and only if R is SV-stable and R is locally weakly ES-stable.

(ii) If R is locally weakly ES-stable, then there exists a finitely generated ideal J of R such that $I^2 \subseteq JI$.

Proof

(i) If R is locally ES-stable, then R is locally SV-stable (Lemma 2.6) so that R is SV-stable by [19, Theorem 3.3]. If R is SV-stable, then it is locally SV-stable. So, the converse follows from [17, Corollary 2.5].

(ii) Let I be an ideal of R. Since R is of finite character, there are at most finitely many maximal ideals $M_1, M_2, \dots, M_t$ of R containing I. Since R is locally weakly ES-stable, $(I^2)_{M_i} = (J_i)_{M_i} I_{M_i}$ for some invertible ideal $(J_i)_{M_i}$ of R_{M_i}, by Proposition 2.1(i), for each $i \in \{1, 2, \dots, t\}$, and $(I^2)_M = R_M$ for each $M \in Max(R)$ such that $I \nsubseteq M$. Since $(J_i)_{M_i}$ is a principal ideal of R_{M_i}, we can write $(J_i)_{M_i} = x_i R_{M_i}$ for some $x_i \in J_i$. Let $J = (x_1, x_2, \dots, x_t)$. We observe that $(I^2)_{M_i} \subseteq (JI)_{M_i}$ for each i. Thus, $I^2 = \bigcap_{M \in Max(R)} (I^2)_M \subseteq \bigcap_{i=1}^{t} (I^2)_{M_i} \subseteq \bigcap_{i=1}^{t} (JI)_{M_i} = JI$. Therefore, $I^2 \subseteq JI$.

□

Lemma 4.3 *Let R be a completely integrally closed domain of finite character and I a nonzero ideal of R. If R is locally ES-stable, then R is SV-stable and I is invertible in R.*

Proof If R is locally ES-stable, then I is SV-stable and $I^2 = JI$ for some finitely generated ideal $J \subseteq I$ and $(E : J)$ is the inverse of I in $E(I) = (I : I)$ by Lemma 4.2. Since $E = R$ ([13, Theorem 2.4.8]) and $(R : J) = J^{-1}$, $IJ^{-1} = R$ so that I is invertible. □

Theorem 4.4 *Let R be a completely integrally closed domain of finite character and I a nonzero ideal of R. Then R is locally ES-stable if and only if R is ES-stable.*

Proof Suppose that R is locally ES-stable. Then, by Lemma 4.3, I is invertible so that I is finitely generated. Moreover, R is SV-stable. Hence, by [9, Prpoposition 7.4.4], R is ES-stable. □

Theorem 4.5 *Let R be a Krull domain and I a nonzero ideal of R. Then R is locally ES-stable if and only if R is ES-stable if and only if R is locally weakly ES-stable if and only if R is weakly ES-stable.*

Proof Follows immediately from Theorem 4.4 and Theorem 3.5. □

We recall that a Prüfer domain is strongly discrete if PR_P is a principal ideal for each prime ideal P of R. It is shown in [18, Theorem 4.6] that, for an integrally closed domain R, R is SV-stable if and only if it is a strongly discrete Prüfer domain of finite character.

Theorem 4.6 *Let R be a Prüfer domain of finite character and I a nonzero ideal of R.*

(i) R is locally ES-stable if and only if R is ES-stable if and only if R is strongly discrete.

(ii) If R is strongly discrete, then R is weakly ES-stable.

(iii) If R is locally weakly ES-stable, then there exists an invertible ideal J of R such that $I^2 \subseteq JI$ with $J = (x, y)$ for some $x \in J$ and $y \in I$.

Proof

(i) In a Prüfer domain, SV-stability and ES-stability coincide [9, Lemma 7.4.1]. So, by [19, Theorem 3.3], R is locally ES-stable of finite character if and only if R is ES-stable. From Theorem [18, Theorem 4.6], the latter holds if and only if R is strongly discrete.

(ii) If R is strongly discrete, then by part (i), R is ES-stable, and hence weakly ES-stable.

(iii) By Lemma 4.2(ii), $I^2 \subseteq JI$ for some finitely generated ideal J of R. Since R is Prüfer, J is invertible. Furthermore, since R is a Prüfer domain, J is $1\frac{1}{2}$-generated, so one of the generators of J can be chosen arbitrarily. Since $I^2 \subseteq J$, $J = (x, y)$ for some $x \in J$ and $y \in I^2$.

□

Remark 4.7 *A weakly ES-stable Prüfer domain (of finite character) R is not necessarily strongly discrete, and hence ES-stable, since the maximal ideal PR_P of the valuation domain R_P, for any prime ideal P of R, is either principal or idempotent.*

Proposition 4.8 *Let R be a Prüfer domain of finite character and I a locally weakly ES-stable ideal of R. Then there exixts an invertible fractional ideal B of $(I:I)$ such that $I^2 = BI$.*

Proof Let I be an ideal of R. Let $M_1, \ldots, M_t$ be the maximal ideals containing I. Then $(I^2)_{M_i} = (J_i)_{M_i} I_{M_i}$ for some invertible ideal of R_{M_i} for each i. We observe that these are the only maximal ideals which contain I^2, also. For all other maximal ideals $N \neq M_i$, for each i, $I_N^2 = R_N = I_N$. So, for each i, $(I^2)_{M_i} = j_i I_{M_i}$ for some $j_i \in J$. Thus, by [10, Lemma III.2.6], there exists a finitely generated ideal B of $(I:I)$ such that $I^2 = BI$. Since $(I:I)$ is a fractional overring of R, it is Prüfer, and hence B is an invertible fractional ideal of $(I:I)$. □

Theorem 4.9 *Let R be a completely integrally closed Prüfer domain of finite character. Then R is locally weakly ES-stable if and only if R is weakly ES-stable.*

Proof Follows immediately from Proposition 4.8 since $(I:I) = R$ for any ideal I of R. □

Theorem 4.10 *Let R be an h-local domain and I a flat ideal of R. Then I is locally weakly ES-stable if and only if I is weakly ES-stable.*

Proof If I is locally weakly ES-stable, then $(I^2)_M = J_M I_M$ for some invertible ideal J_M of R_M for each maximal ideal M of R. Since R has finite character, I is contained in at most finitely many maximal ideals, say $M_1, \ldots, M_t$. We have $I_N^2 = J_N I_N = R_N$ for each maximal ideal N of R not containing I, and $(I^2)_{M_i} = a_i I_{M_i}$ for some $a_i \in R$ for each $i \in \{1, 2, \ldots, t\}$. Let $A = R \cap a_1 R_{M_1} \cap \ldots \cap a_t R_{M_t}$. We observe that A is a fractional ideal of R. Then $AI = I \cap a_1 I_{M_1} \cap \ldots \cap a_t I_{M_t}$ by the flatness of I. Hence, we have $AI = \bigcap_{M \in Max(R)} I_M \cap I_{M_1}^2 \cap \ldots \cap I_{M_t}^2 = \bigcap_N I_N^2 \cap I_{M_1}^2 \cap \ldots \cap I_{M_t}^2 = I^2$, where $Max(R)$ is the set of all maximal ideals of R. Now, we claim that A is locally principal, and hence invertible. Since R is h-local, $(R_{M_i})_N = Q$ [10, Lemma IV.3.2], and hence $A_N = R_N \cap (a_1 R_{M_1})_N \cap \ldots \cap (a_t R_{M_t})_N = R_N$. Also, $(R_{M_j})_{M_i} = Q$ for $i \neq j$, we have $A_{M_i} = R_{M_i} \cap (a_1 R_{M_1})_{M_i} \cap \ldots \cap a_i R_{M_i} \cap \ldots \cap (a_t R_{M_t})_{M_i} = a_i R_{M_i}$. Thus, A is an invertible ideal of R so that I is weakly ES-stable. □

Theorem 4.11 *Let R be a Prüfer h-local domain. Then R is locally weakly ES-stable if and only if R is weakly ES-stable.*

Proof Since all ideals of a Prüfer domain are flat ([10, Theorem VI.9.10]), it follows from Theorem 4.10 immediately. □

Next we prove that Noetherian domains, which are locally (weakly) ES-stable, already have finite character, in deed, they are h-local.

Theorem 4.12 *Let R be a Noetherian domain. If R is locally (weakly) ES-stable, then R is*

(i) SV-stable,

(ii) one dimensional,

(iii) h-local,

Proof Suppose that R is locally (weakly) ES-stable.

(i) Since every ideal of a Noetherian domain is finitely generated, by Theorem 2.10, R is SV-stable.

(ii) Follows immediately from part (i) and [12, Lemma 2].

(iii) Follows immediately from part (i) and [12, Lemma 2].

□

Theorem 4.13 *Let R be a commutative ring such that the endomorphism ring of each finitely generated regular ideal of R is local-global. R is locally finitely (weakly) ES-stable if and only if R is finitely (weakly) ES-stable.*

Proof Suppose R is locally finitely (weakly) ES-stable. Let I be a finitely generated regular ideal of R. Then I is SV-stable, by Theorem 2.10. Since $(I : I)$ is local-global, I is ES-stable by Lemma 2.7. □

Since, for a Noetherian ring, the regular ideals are finitely generated, and ES-stability coincides with weakly ES-stability (Theorem 3.2), the following corollary immediately follows from Theorem 4.13.

Corollary 4.14 *Let R be a Noetherian ring such that the endomorphism ring of each ideal of R is local-global. Then R is locally (weakly) ES-stable if and only if R is (weakly) ES-stable.*

In [15], it is proven that a semilocal Noetherian one dimensional domain is SV-stable if and only if it is ES-stable. So, Corollary 4.14 generalizes this fact for one-dimensional local-global Noetherian rings.

We conclude that, over Noetherian local-global rings, locally (weakly) ES-stability, (weakly) ES-stability, locally SV-stability and SV-stability coincide. Moereover, these notions also coincide for one dimensional integrally closed Noetherian rings (Dedekind rings) [9, Proposition 7.4.4].

We observe that, at least for Noetherian rings, the finite character property does not seem to be useful to prove that locally (weakly) ES-stability implies (weakly) ES-stability. We provide an example of a one dimensional Noetherian ring of finite character in which there is an SV-stable (and hence locally ES-stable) ideal which is not (weakly) ES-stable. First we recall that an integral domain R has the trace property (or is a TP domain) if, for every ideal I of R, either $II^{-1} = R$ or II^{-1} is a prime ideal. An ideal I of R is strongly divisorial if I is divisorial, that is $(I^{-1})^{-1} = I$, and strong, that is $II^{-1} = I$.

Example 4.15 *Let R be a Noetherian TP domain which is not Dedekind. So, by [9, Theorem 4.2.48], R is one dimensional (so that R is h-local by [20, Example 3.1]), and it has a unique noninvertible maximal ideal M. In fact, M is stongly divisorial, and $M^{-1} = \bar{R}$, the integral closure of R. Hence*

$$MM^{-1} = M\bar{R} = M.$$

By [1, Proposition 2.4] and [9, Proposition 7.3.2], each nonzero prime ideal is SV-stable. So, M is SV-stable. Suppose M is (weakly) ES-stable. Then $M = JE$ for some invertible fractional ideal J of R and an idempotent fractional ideal E of R. Let $T = (M : M)$. By Proposition 2.1, $E = M(T : M)$, E is a trace (integral) ideal of T which is idempotent. Since T is Noetherian, $E = T$. So, $M = JT$. Since $MM^{-1} = M$, $(R : M) = M^{-1} \subseteq T$. Also, $T \subseteq (R : M) = M^{-1}$. Thus, $M^{-1} = T$. Hence,

$$M\bar{R} = M = JE = JM^{-1} = J\bar{R}.$$

So,

$$M\bar{R}J^{-1} = R\bar{R} = \bar{R}$$

which implies that J^{-1} *is the inverse of* $M\bar{R}$ *in* $\bar{R}$. *Since* $MM^{-1} = M$,

$$MJ^{-1} = MM^{-1}J^{-1} = M\bar{R}M^{-1}J^{-1} = \bar{R},$$

so $M^{-1}J^{-1}$ *the inverse of* $M\bar{R}$ *in* $\bar{R}$. *Therefore,* $J^{-1} = M^{-1}J^{-1}$ *which implies that* $M^{-1} = R$. *Since* $M^{-1} = \bar{R}$, $R = \bar{R}$ *so that* R *is integrally closed, but* R *is not Dedekind. Hence,* M *is SV-stable, but not ES-stable.*

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