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The Möbius transformation of continued fractions with bounded upper and lower partial quotients

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Abstract: Let $h\colon x \mapsto \frac{ax+b}{cx+d}$ be the nondegenerate Möbius transformation with integer entries. We get a bound of the continued fraction of $h(x)$ by upper and lower bounds of the continued fraction of x.

Key words: Möbius transformation, continued fraction expansion, partial quotient.

1. Introduction

A continued fraction representation of a number $x \in \mathbb{R}$ is an expansion of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \tag{1.1}$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}^+$, $i = 1, 2, \cdots$. A continued fraction may be finite or infinite. If (1.1) is a finite continued fraction, we denote it by $[a_0; a_1, a_2, \cdots, a_n]$; if (1.1) is infinite, then we denote it by $[a_0; a_1, a_2, \cdots]$. We call a_j the jth partial quotient. It is a well known fact that the continued fraction of x is infinite if and only if x is irrational.

Given a nondegenerate 2×2 matrix M with integer entries, that is $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$ and the determinant $ad - bc \neq 0$, we can define the associated Möbius transformation $h\colon x \mapsto \frac{ax+b}{cx+d}$. We also denote by

$$h(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}.$$

Irrational numbers with bounded quotients, usually referred as badly approximable numbers, are a subset of real numbers with zero Lebesgue measure. Those numbers play an important role in several topics in dynamical systems, number theory, and the spectral theory of quasiperiodic Schrödinger operators [1, 2, 4, 6, 8, 13].

It is an old result that a real number $\frac{ax+b}{cx+d}$ has bounded partial quotients if x does [5, 11, 12]. Thus, the quantitative bound becomes an interesting question. Based on Cusick-France [3], Lagarias-Shallit [7] obtained

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a quantitative bound, which stated that if x has bounded partial quotients with $a_j \leq K$ eventually, then the associated partial quotients $a_j^\star$ of $\frac{ax+b}{cx+d}$ satisfy $a_j^\star \leq D(K+2)$ eventually.

Using an algorithm developed by Liardet-Stambul [9] to calculate the partial quotients of $h(x)$, Stambul gave a better upper bound $a_j^\star \leq D-1+\left\lfloor D\frac{K+\sqrt{K^2+4K}}{2}\right\rfloor$ [14]. In this paper, we study partial quotients with lower and upper bounds at the same time. Denote by $\lfloor x \rfloor$ the integer part of x, namely, $\lfloor x \rfloor = \max\{j \in \mathbb{Z} : j \leq x\}$. Our main result is

Theorem 1.1 *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a nondegenerate matrix with entries in* $\mathbb{Z}$ *and* h *be the associated Möbius transformation. Let* $x = [a_0; a_1, a_2, \cdots]$ *be a real number such that* $B_1 \leq a_j \leq B_2$ *for* j *large enough. Let* $h(x) = [a_0^\star; a_1^\star, a_2^\star, \cdots]$*. Then* $a_j^\star \leq \left\lfloor \frac{D-1}{B_1} \right\rfloor + \left\lfloor D\frac{B_1B_2+\sqrt{B_1^2B_2^2+4B_1B_2}}{2B_1}\right\rfloor$ *for sufficiently large* j*, where* $D = |\det(M)|$*.*

Remark: If $B_1 = 1$, Theorem 1.1 gives the bound $D-1+\lfloor D\frac{K+\sqrt{K^2+4K}}{2}\rfloor$, which is exact the same bound as in Stambul [14].

This paper is entirely self-contained. Although our proof follows the scheme of [9, 14], the details are much more dedicate since we need to handle lower and upper bounds of partial quotients at the same.

Finally, we remark that the determinant of Möbius transformation can also be used to characterize upper and lower bounds of the ratio between the period of $h(x)$ and that of x [10].

2. Algorithm for partial quotients

In the following, we always assume $x = [a_0; a_1, a_2, \cdots]$ and $\frac{ax+b}{cx+d} = [a_0^\star; a_1^\star, a_2^\star, \cdots]$ with $D = |ad-bc| \geq 1$. Set $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $h(x) = \frac{ax+b}{cx+d}$.

At the beginning of this section, we will introduce some notations and the algorithm developed by Liardet-Stambul [9] and Stambul [14] to calculate the partial quotients of $h(x)$. Let $M_{2,\mathbb{N}}$ be the set of all matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(a,b,c,d \in \mathbb{N})$ such that $ad-bc \neq 0$. M is said to be in $\mathcal{D}_2$ when $a \geq c$ and $b \geq d$, in $\mathcal{D}_2'$ when $a \leq c$ and $b \leq d$, and in ε_2 when $(a-c)(b-d) < 0$. $\{\mathcal{D}_2, \mathcal{D}_2', \varepsilon_2\}$ is a partition of $M_{2,\mathbb{N}}$.

It is easy to see that $M \in \varepsilon_2$ satisfies

$$\max\{|a|+|b|, |c|+|d|\} \leq |\det M| = D. \tag{2.1}$$

For all matrices $M \in \mathcal{D}_2 \cup \mathcal{D}_2'$, there exists a unique factorization

$$M = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}M' \tag{2.2}$$

such that $c_0 \in \mathbb{N}$, $c_1, \cdots, c_n \in \mathbb{N}^+$ and $M' \in \varepsilon_2$ [9]. This factorization will be denoted by $M = \Pi_{c_0c_1,\cdots,c_n}M'$. Moreover, $[c_0; c_1, c_2, \cdots, c_{n-1}]$ is the common sequence of partial quotients of $\frac{a}{c}$ and $\frac{b}{d}$ if $n \neq 1$. c_n can be determined by the following several cases [9]:

Case 1 : If $\frac{a}{c} = [c_0; c_1, c_2, \cdots, c_{n-1}]$, then c_n is the nth partial quotient of $\frac{b}{d}$.

Case 2 : If $\frac{b}{d} = [c_0; c_1, c_2, \cdots, c_{n-1}]$, then c_n is the nth partial quotient of $\frac{a}{c}$.

Case 3 : Otherwise, c_n is the smaller one of nth partial quotients of $\frac{a}{c}$ and $\frac{b}{d}$.

Assume $M \in \varepsilon_2$ and h is the associated Möbius transformation. Let $x = [a_0; a_1, a_2, \cdots] > 1$. After the preparations, we are ready to recall the algorithm in [9, 14] to compute the partial quotients of $h(x)$.

Step 0: $M_0 = M \in \varepsilon_2, j = 0, n = 0$.

Let j_1 be the smallest positive integer (see [9] for the existence) such that $M_0\Pi_{a_0a_1\cdots a_{j_1-1}} \in \varepsilon_2$ and $M_0\Pi_{a_0a_1\cdots a_{j_1}} \in \mathcal{D}_2 \cup \mathcal{D}_2'$. Factorizing $M_0\Pi_{a_0a_1\cdots a_{j_1}}$ as (2.2), we get

$$M_0\Pi_{a_0a_1\cdots a_{j_1}} = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_{n_1} & 1 \\ 1 & 0 \end{pmatrix} M_1 \qquad \text{(Output-0)}$$

with $M_1 \in \varepsilon_2$.

Step 1: $M_1 \in \varepsilon_2, j = j_1 + 1, n = n_1 + 1$.

Let $j_2 \geq j_1+1$ be the smallest positive integer such that $M_1\Pi_{a_{j_1+1}a_{j_1+2}\cdots a_{j_2-1}} \in \varepsilon_2$ and $M_1\Pi_{a_{j_1+1}a_{j_1+2}\cdots a_{j_2}} \in \mathcal{D}_2 \cup \mathcal{D}_2'$. Factorizing $M_1\Pi_{a_{j_1+1}a_{j_1+2}\cdots a_{j_2}}$ as (2.2), we get

$$M_1\Pi_{a_{j_1+1}a_{j_1+2}\cdots a_{j_2}} = \begin{pmatrix} c_{n_1+1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_{n_1+2} & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_{n_2} & 1 \\ 1 & 0 \end{pmatrix} M_2 \qquad \text{(Output-1)}$$

with $M_2 \in \varepsilon_2$.

Step 2: $M_2 \in \varepsilon_2, j = j_2 + 1, n = n_2 + 1$.

Let $j_3 \geq j_2+1$ be the smallest positive integer such that $M_2\Pi_{a_{j_2+1}a_{j_2+2}\cdots a_{j_3-1}} \in \varepsilon_2$ and $M_2\Pi_{a_{j_2+1}a_{j_2+2}\cdots a_{j_3}} \in \mathcal{D}_2 \cup \mathcal{D}_2'$. Factorizing $M_2\Pi_{a_{j_2+1}a_{j_2+2}\cdots a_{j_3}}$ as (2.2), we get

$$M_2\Pi_{a_{j_2+1}a_{j_2+2}\cdots a_{j_3}} = \begin{pmatrix} c_{n_2+1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_{n_2+2} & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_{n_3} & 1 \\ 1 & 0 \end{pmatrix} M_3 \qquad \text{(Output-2)}$$

with $M_3 \in \varepsilon_2$.

$\cdots\cdots\cdots$

$\cdots\cdots\cdots$

Step k: $M_k \in \varepsilon_2, j = j_k + 1, n = n_k + 1$.

Let $j_{k+1} \geq j_k+1$ be the smallest positive integer such that $M_k\Pi_{a_{j_k+1}a_{j_k+2}\cdots a_{j_{k+1}-1}} \in \varepsilon_2$ and $M_k\Pi_{a_{j_k+1}a_{j_k+2}\cdots a_{j_{k+1}}} \in \mathcal{D}_2 \cup \mathcal{D}_2'$. Factorizing $M_k\Pi_{a_{j_k+1}a_{j_k+2}\cdots a_{j_{k+1}}}$ as (2.2), we get

$$M_k\Pi_{a_{j_k+1}a_{j_k+2}\cdots a_{j_{k+1}}} = \begin{pmatrix} c_{n_k+1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_{n_k+2} & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_{n_{k+1}} & 1 \\ 1 & 0 \end{pmatrix} M_{k+1} \qquad \text{(Output-k)}$$

with $M_{k+1} \in \varepsilon_2$.

Putting all the Output (Output-k) together, we get a sequence

$$c_0c_1c_2c_3\cdots c_{n_k}. \tag{2.3}$$

Unfortunately, many c_i maybe zero; thus, we must introduce the contraction map μ. For any word $c_0c_1c_2c_3\cdots c_n \in \mathbb{N}^n$, let μ be the contraction map which transforms a word into a word where all letters are positive integers (except perhaps the first one), replacing from left to right factors $a0b$ by the letter $a+b$.

By the fact

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 1 \\ 1 & 0 \end{pmatrix},$$

we have

$$\Pi_{\mu(c_0c_1c_2c_3\cdots c_n)} = \Pi_{c_0c_1c_2c_3\cdots c_n}. \tag{2.4}$$

Let μ act on (2.3), then we get

$$c_0^\star c_1^\star c_2^\star c_3^\star \cdots c_{n'_k}^\star = \mu(c_0c_1c_2c_3\cdots c_{n_k}). \tag{2.5}$$

By the arguments in [9], n'_k goes to infinity as k does, moreover,

$$\frac{ax+b}{cx+d} = [c_0^\star; c_1^\star, \cdots, c_{n'_k-1}^\star, \cdots] \tag{2.6}$$

and the n'_kth partial quotient following $c_{n'_k-1}^\star$ is no less than $c_{n'_k}^\star$.

Now, we give a quantitative estimate about c_i in (2.3).

Lemma 2.1 *Assume $M \in \varepsilon_2$ and $x = [a_0; a_1, a_2, \cdots] > 1$. Let h be the associated Möbius transformation and $D = |\det M| \geq 1$. Suppose $a_j \leq K$ for some $K \in \mathbb{N}^+$. We do the algorithm as above, then the following three claims hold,*

(i) *For any $n_k < j \leq n_{k+1}-1$, $c_j \leq D-1$.*

(ii) *For any k, $c_{n_{k+1}} \leq DK$.*

(iii) *If for some k, $c_{n_{k+1}} \geq D$, then the right upper entry of M_{k+1} must be zero, that is M_{k+1} has the form*

$$M_{k+1} = \begin{pmatrix} \star & 0 \\ \star & \star \end{pmatrix}. \tag{2.7}$$

Proof The three claims are from [14]. We rewrite the proof here to make the paper more readable. By the algorithm, we already have $M_k\Pi_{a_{j_k+1}a_{j_k+2}\cdots a_{j_{k+1}-1}} \in \varepsilon_2$ and $M_k\Pi_{a_{j_k+1}a_{j_k+2}\cdots a_{j_{k+1}}} \in \mathcal{D}_2 \cup \mathcal{D}'_2$.

For simplicity, let $M' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = M_k\Pi_{a_{j_k+1}a_{j_k+2}\cdots a_{j_{k+1}-1}} \in \varepsilon_2$ and $f = a_{j_{k+1}} \leq K$. Then $M'\Pi_f \in \mathcal{D}_2 \cup \mathcal{D}'_2$.

If $\gamma = 0$, then

$$M'\Pi_f = \begin{pmatrix} \alpha f+\beta & \alpha \\ \delta & 0 \end{pmatrix} \in \mathcal{D}_2 \cup \mathcal{D}'_2$$

and we must have $\alpha f+\beta \geq \delta$. Thus,

$$M'\Pi_f = \begin{pmatrix} \alpha f+\beta & \alpha \\ \delta & 0 \end{pmatrix} = \begin{pmatrix} \lfloor \frac{\alpha f+\beta}{\delta} \rfloor & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 \\ (\alpha f+\beta) \bmod \delta & \alpha \end{pmatrix}.$$

In this case, in order to prove the Lemma, it suffices to show that

$$\left\lfloor \frac{\alpha f+\beta}{\delta} \right\rfloor \leq DK. \tag{2.8}$$

Otherwise, one has

$$DK+1 \leq \left\lfloor \frac{\alpha f+\beta}{\delta} \right\rfloor = \left\lfloor \frac{\alpha f}{\delta} + \frac{\beta}{\delta} \right\rfloor \leq \left\lfloor \frac{\alpha K}{\delta} + \frac{\beta}{\delta} \right\rfloor, \tag{2.9}$$

since $f \leq K$.

By the fact $M' = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \varepsilon_2$, we have $\beta < \delta$, $|\alpha|+|\beta| \leq D$. This is contradicted to (2.9).

If $\alpha = 0$, then

$$M'\Pi_f = \begin{pmatrix} \beta & 0 \\ \gamma f+\delta & \gamma \end{pmatrix} \in \mathcal{D}_2 \cup \mathcal{D}_2'$$

and we must have $\gamma f+\delta \geq \beta$. Thus,

$$M'\Pi_f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lfloor \frac{\gamma f+\delta}{\beta} \rfloor & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ (\gamma f+\delta) \bmod \delta & \gamma \end{pmatrix}.$$

In this case, we can still prove the Lemma like the case $\gamma = 0$.

If $\alpha, \gamma \geq 1$, then

$$M'\Pi_f = \begin{pmatrix} \alpha f+\beta & \alpha \\ \gamma f+\delta & \gamma \end{pmatrix} \in \mathcal{D}_2 \cup \mathcal{D}_2'.$$

By the algorithm, $n_k \leq j \leq n_{k+1}-1$, c_j is the common partial quotient of $\frac{\alpha}{\gamma}$ and $\frac{\alpha f+\beta}{\gamma f+\delta}$.

We start with the proof of claim 1. Indeed, $\alpha \leq D$ and $\gamma \geq 1$. If $\alpha = D$ and $\gamma = 1$, we must have $\beta = 0$ and $\delta = 1$. This implies claim 1 when we consider the partial quotient of $\frac{\alpha f+\beta}{\gamma f+\delta}$. Otherwise ($\alpha = D$ and $\gamma = 1$ do not hold) claim 1 holds if we consider the partial quotient of $\frac{\alpha}{\gamma}$.

Suppose the last letter, i.e. $c_{n_{k+1}} \geq D$, then we must have

$$\frac{a}{c} = [c_{j_k+1}; c_{j_k+2}, c_{j_k+2}, \cdots, c_{j_{k+1}-1}]$$

by the (Case1-Case3) and $c_{n_{k+1}} \geq D$ is the $n_{k+1}-n_k+1$th partial quotient of $\frac{\alpha f+\beta}{\gamma f+\delta}$. This implies claims 2 and 3 if we can show

$$\frac{1}{DK} \leq \frac{\alpha f+\beta}{\gamma f+\delta} \leq DK.$$

We only prove the fact $\frac{\alpha f+\beta}{\gamma f+\delta} \leq DK$, the proof of lower bound $\frac{1}{DK} \leq \frac{\alpha f+\beta}{\gamma f+\delta}$ is the same.

If $\gamma f+\delta\geq 2$, then $\frac{\alpha f+\beta}{\gamma f+\delta}\leq\frac{DK+D}{2}\leq DK$. If $\gamma f+\delta\leq 1$, then we have $\delta=0$ and $\gamma=K=1$. This implies $\beta=D$ and $\alpha=0$. We still have $\frac{\alpha f+\beta}{\gamma f+\delta}\leq DK$.

□

3. Technical lemmas

We say a Möbius transformation $h(\cdot)=M\cdot$ cannot change the continued fraction eventually, if for any x, partial quotients of $h(x)$ and x are eventually equal.

Lemma 3.1 *The following forms of Möbius transformations cannot change the continued fraction eventually,*

$$S=\left\{\begin{pmatrix}1&k_1\\0&1\end{pmatrix},\begin{pmatrix}k_2&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\k_3&1\end{pmatrix},\begin{pmatrix}-1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\},\tag{3.1}$$

where $k_1,k_2,k_3\in\mathbb{Z}$.

Proof The proof is based on direct computation. □

Remark: The determinant of each matrix in S is ± 1.

Lemma 3.2 *Assume* $a,b,c,d\in\mathbb{Z}$ *and* $ad-bc\neq 0$, *then* $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ *can be rewritten in the following form*

$$M=S_1S_2\cdots S_nM'\tag{3.2}$$

with $M'\in\varepsilon_2$. *Moreover, if* $D=\det M=1$, *then* M' *can be* $\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

Proof Using Möbius transformations $\begin{pmatrix}-1&0\\0&1\end{pmatrix}\in S$ and $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\in S$, we can assume $a,c\geq 0$.

Using Möbius transformations $\begin{pmatrix}1&0\\k&1\end{pmatrix}\in S$ and $\begin{pmatrix}0&1\\1&0\end{pmatrix}\in S$, M can be changed to $M_1=\begin{pmatrix}a_1&b_1\\0&d_1\end{pmatrix}$ with $a_1\geq 1$.

Using Möbius transformations $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\in S$ and $\begin{pmatrix}1&k\\0&1\end{pmatrix}\in S$, M_1 can be changed to $M'=\begin{pmatrix}a_1&b_1 \bmod |d_1|\\0&|d_1|\end{pmatrix}\in\varepsilon_2$.

Moreover, if $D=1$, we must have $a_1=1,|b_1|=1$ and $b_1 \bmod |d_1|=0$.

□

Remark: If $|\det M|=1$, then the associated Möbius transformations cannot change the continued fraction eventually.

Lemma 3.3 *Let* $M\in\varepsilon_2$ *and* $D=|\det M|\geq 2$. *Let* $x=[a_0;a_1,a_2,\cdots]$ *such that* $B_1\leq a_j\leq B_2$ *for all* $j\geq 0$. *Using the Algorithm in Section 2, we get a sequence* $c_0^\star c_1^\star c_2^\star c_3^\star\cdots$ *by (2.5). If* $c_0^\star=0$, *then*

$$c_1^\star\leq\lfloor Dy_0\rfloor,\tag{3.3}$$

where $y_0 = [B_2; B_1, B_2, B_1, \cdots] \triangleq [\overline{B_2, B_1}] = \frac{B_1B_2+\sqrt{B_1^2B_2^2+4B_1B_2}}{2B_1}$. *Moreover, the equality in (3.3) holds iff* $a=0, b=1, c=D$ *and* $d=0$.

In addition, assume $M \neq \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}$, *then*

$$c_1^\star \leq \max\left\{\left\lfloor \frac{D}{4}y_0 + 1\right\rfloor, D-1\right\} \tag{3.4}$$

if $c_0^\star = 0$.

Proof Let

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \cdots, c_n],$$

then

$$\Pi_{a_0a_1\cdots a_n} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

Thus, we have the following simple facts

$$M\Pi_{a_0a_1\cdots a_n} = \begin{pmatrix} ap_n + bq_n & ap_{n-1} + bq_{n-1} \\ cp_n + dq_n & cp_{n-1} + dq_{n-1} \end{pmatrix}, \tag{3.5}$$

and

$$\lim_{n\to\infty} \frac{ap_n + bq_n}{cp_n + dq_n} = \frac{ap_{n-1} + bq_{n-1}}{cp_{n-1} + dq_{n-1}} = \frac{ax+b}{cx+d}.$$

If $c_0^\star = 0$, then $c_1^\star$ is the second common partial quotient of $\frac{ap_n+bq_n}{cp_n+dq_n}$ and $\frac{ap_{n-1}+bq_{n-1}}{cp_{n-1}+dq_{n-1}}$ for any large n. Combining with (3.5), we must have

$$c_1^\star = \left\lfloor \frac{cx+d}{ax+b} \right\rfloor. \tag{3.6}$$

Now we are in a position to prove the Lemma, based on (3.6).

Case 1: $a \geq 1$

Using $x > 1$, one has

$$\begin{aligned} \frac{cx+d}{ax+b} &\leq \frac{cx+d}{ax} \\ &< \frac{c+d}{a} \\ &\leq D, \end{aligned}$$

where the third inequality holds by (2.1). This implies $c_1^\star \leq D-1$.

Case 2: $a = 0$

In this case, we have $b > d$, $bc = D$ and $c + d \leq D$ by $M \in \varepsilon_2$, and

$$c_1^\star = \left\lfloor \frac{D}{b^2}x + \frac{d}{b} \right\rfloor. \tag{3.7}$$

If $b \geq 2$, by (3.7), one has

$$c_1^\star \leq \left\lfloor \frac{D}{4}x + 1 \right\rfloor.$$

Notice that if a real number with bounded partial quotients in $[B_1, B_2] \cap \mathbb{Z}$ is such that $x \leq y_0$, then

$$c_1^\star \leq \left\lfloor \frac{D}{4}y_0 + 1 \right\rfloor \leq \lfloor Dy_0 \rfloor - 1,$$

since $y_0 \geq \frac{\sqrt{5}+1}{2}$ and $D \geq 2$.

If $b = 1$, we must have $c = D$ and $d = 0$.

Putting all the cases together, we complete the proof. $\square$

Lemma 3.4 *Let $M \in \varepsilon_2$ with the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ and $D = |\det M| \geq 1$. Let $x = [a_0; a_1, a_2, \cdots]$ such that $B_1 \leq a_j \leq B_2$ for all $j \geq 0$. Applying the Algorithm in Section 2 to $M \cdot x$, we get a sequence $c_0^\star c_1^\star c_2^\star c_3^\star \cdots$ by (2.5). If $c_0^\star = 0$, we must have*

$$c_1^\star \leq \left\lfloor \frac{D}{x_0} \right\rfloor,$$

where $x_0 = [B_1; B_2, B_1, B_2, \cdots] \triangleq [\overline{B_1, B_2}] = \frac{B_2B_1 + \sqrt{B_1^2B_2^2 + 4B_1B_2}}{2B_2}$.

Proof Let $b = 0$ in (3.6), then we get

$$c_1^\star = \left\lfloor \frac{cx + d}{ax} \right\rfloor. \tag{3.8}$$

Notice that if a real number with bounded partial quotients in $[B_1, B_2] \cap \mathbb{Z}$ is such that $x \geq x_0$, then

$$c_1^\star \leq \left\lfloor \frac{cx_0 + d}{ax_0} \right\rfloor. \tag{3.9}$$

Thus, in order to prove this Lemma, it suffices to show

$$\frac{cx_0 + d}{ax_0} \leq \frac{D}{x_0}. \tag{3.10}$$

.

If $a = 1$, we must have $c = 0$ and $d = D$, this implies (3.10).

If $a \geq 2$, we already have $ad = D$ and $c \leq a - 1$.

Case 1: $D \geq 2x_0 > 2$

One has

$$\begin{aligned} cx_0 + d &\leq (a-1)x_0 + \frac{D}{2} \\ &\leq \frac{D(a-1)}{2} + \frac{D}{2} \\ &\leq Da. \end{aligned}$$

This implies (3.10).

Case 2: $x_0 \leq D < 2x_0$

It suffices to show

$$\frac{cx_0 + d}{ax_0} < 2. \tag{3.11}$$

This is obvious by the following computation,

$$\begin{aligned} cx_0 + d &\leq (a-1)x_0 + D \\ &< ax_0 + 2x_0 \\ &\leq 2ax_0. \end{aligned}$$

This implies (4.4).

Case 3: $D < x_0$

By direct computation,

$$\begin{aligned} \frac{cx_0 + d}{ax_0} &= \frac{c}{a} + \frac{D}{a^2 x_0} \\ &< \frac{a-1}{a} + \frac{1}{a^2} \\ &< 1. \end{aligned}$$

This also implies (3.10).

□

4. Proof of Theorem 1.1

Proof Suppose $x = [a_0; a_1, a_2, \cdots]$ is such that $B_1 \leq a_j \leq B_2$ for $j \geq j_0$, and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is such that $D = |\det M| \geq 1$. By Lemmas 3.1 and 3.2, we may assume $M \in \varepsilon_2$. By the fact

$$h(x) = M \cdot x = M\Pi_{a_0 a_1 \cdots a_{j_0}} \cdot [a_{j_0+1}; a_{j_0+2}, \cdots] \tag{4.1}$$

and (2.2), in order to prove Theorem 1.1, we only need to prove the case when all the partial quotients of x satisfy $B_1 \leq a_i \leq B_2$.

By the Algorithm, it suffices to show that for any word $k_1 0 k_2 0 \cdots 0 k_p$ in (2.3) with $k_i \in \mathbb{N}^+, i = 1, 2, \cdots, p$, we have

$$k_1 + k_2 + \cdots + k_p \leq \left\lfloor \frac{D-1}{B_1} \right\rfloor + \left\lfloor D \frac{B_1 B_2 + \sqrt{B_1^2 B_2^2 + 4B_1 B_2}}{2B_1} \right\rfloor. \tag{4.2}$$

Assume k_1 is the last letter of kth step (2.3). Then the output of $k+1$th step is $0k_2$, $k+2$th step is $0k_3$, $\cdots$.

Case 1: $k_1 \geq D$

By (iii) of Lemma 2.1, M_{k+1} has the form

$$M_{k+1} = \begin{pmatrix} a_k & 0 \\ c_k & d_k \end{pmatrix} \in \varepsilon_2.$$

By Lemma 3.4, we have

$$\sum_{j=2}^{p} k_j \leq \left\lfloor \frac{D}{x_0} \right\rfloor .$$

By (ii) of Lemma 2.1, $k_1 \leq DB_2$, then

$$\begin{aligned}
\sum_{j=1}^{p} k_j &\leq \left\lfloor \frac{D}{x_0} \right\rfloor + DB_2 \\
&\leq \left\lfloor D\frac{B_2B_1 + \sqrt{B_1^2B_2^2 + 4B_1B_2}}{2B_1} \right\rfloor \\
&\leq \left\lfloor \frac{D-1}{B_1} \right\rfloor + \left\lfloor D\frac{B_1B_2 + \sqrt{B_1^2B_2^2 + 4B_1B_2}}{2B_1} \right\rfloor .
\end{aligned}$$

This implies Theorem 1.1 in this case.

By the Remark following Lemma 3.2, we can assume $D \geq 2$.

Case 2: $k_1 \leq D-1$

If $M_{k+1} \neq \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}$, by (3.5) one has

$$\sum_{j=2}^{p} k_j \leq \max\left\{ \left\lfloor \frac{D}{4} y_0 + 1 \right\rfloor, D-1 \right\}.$$

Direct computation (spliting the computation into $B_1 = 1$ or $B_1 \geq 2$),

$$\begin{aligned}
\sum_{j=1}^{p} k_j &\leq D - 1 + \max\left\{ \left\lfloor \frac{D}{4} y_0 + 1 \right\rfloor, D-1 \right\} \\
&\leq \left\lfloor \frac{D-1}{B_1} \right\rfloor + \left\lfloor D\frac{B_1B_2 + \sqrt{B_1^2B_2^2 + 4B_1B_2}}{2B_1} \right\rfloor .
\end{aligned}$$

This implies Theorem 1.1 in this case.

If $M_{k+1} = \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}$, by (4.2) one has

$$c_1^{\star} \leq \lfloor Dy_0 \rfloor .$$

Thus, in order to prove Theorem 1.1 in this case, it suffices to show

$$k_1 \leq \frac{D-1}{B_1}. \tag{4.3}$$

By the Algorithm of kth step, we have

$$M_k \Pi_{a_1 a_2 \cdots a_N} = \Pi_{c_1 c_2 \cdots c_{N'-1}} \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix} \in \mathcal{D}_2 \cup \mathcal{D}_2', \tag{4.4}$$

and $M_k\Pi_{a_1a_2\cdots a_{N-1}} \in \varepsilon_2$.

This implies

$$M_k\Pi_{a_1a_2\cdots a_{N-1}} = \Pi_{c_1c_2\cdots c_{N'-1}} \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix} \begin{pmatrix} a_N & 1 \\ 1 & 0 \end{pmatrix}^{-1}. \tag{4.5}$$

By direct computation, one has

$$M_k\Pi_{a_1a_2\cdots a_{N-1}} = \Pi_{c_1c_2\cdots c_{N'-1}} \begin{pmatrix} k_1 & -k_1a_N + D \\ 1 & -a_N \end{pmatrix}. \tag{4.6}$$

Since all entries of $M_k\Pi_{a_1a_2\cdots a_{N-1}}$ are nonnegative, we must have

$$-k_1a_N + D \geq 1. \tag{4.7}$$

This implies

$$k_1 \leq \left\lfloor \frac{D-1}{B_1} \right\rfloor,$$

since $a_N \geq B_1$. We complete the proof.

□

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