

## Existence and uniqueness of solutions for nonlinear Caputo fractional difference equations

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**Abstract:** We study two cases of nabla fractional Caputo difference equations. Our main tool used is a Banach fixed point theorem, which allows us to give some existence and uniqueness theorems of solutions for discrete fractional Caputo equations. In addition, we develop the existence results for delta fractional Caputo difference equations, which correct ones obtained in Chen and Zhou. We present two examples to illustrate our main results.

**Key words:** Nabla difference equation, existence and uniqueness, Caputo operator

### 1. Introduction

Discrete fractional calculus, allowing difference operators to have noninteger orders, can be regarded as a general concept of difference calculus. Since the early article [2] considering two-point boundary value problem (BVP) for finite fractional difference equations, a gradually increasing amount of researchers has started studying BVPs for discrete fractional equations. Goodrich [6], for example, investigated a similar case with right-focal boundary conditions. Then, after one year, Holm [7] explored  $(N - 1, 1)$  fractional BVPs in his Ph.D. Dissertation. Xie et al. [10] studied multiple solutions for a fractional difference BVP by a variational approach. Sitthiwirattam et al. [9] and Reunsumrit et al. [8] considered BVP for fractional difference equations with three-point and four-point fractional sum boundary conditions, respectively. Recently, Chen and Zhou [3] have obtained the existence and Ulam stability of solutions for

$$\begin{cases} \Delta_{(\nu-2)^*}^\nu x(t) = f(t + \nu - 1, x(t + \nu - 1)), & t \in \mathbb{N}_0^b, \\ x(\nu - 1) + x(b + \nu) = 0, \Delta x(\nu - 1) + \Delta x(b + \nu - 1) = 0 \end{cases}$$

and

$$\begin{cases} \Delta_{(\nu-2)^*}^\nu x(t) = f(t + \nu - 1, x(t + \nu - 1)), & t \in \mathbb{N}_0^b, \\ x(\nu - 1) = y(\nu - 1), x(b + \nu) = y(b + \nu), \end{cases}$$

respectively, where  $1 < \nu < 2$ ,  $x(t) \in \mathbb{R}$ ,  $b \in \mathbb{N}_1$  and  $f : \mathbb{N}_{\nu-1}^{b+\nu-1} \rightarrow \mathbb{R}$ . However, there are some inconsistencies in the proof of [3, Lemma 6, Theorem 15], which lead to the invalidity of their uniqueness theorems of solutions and the Ulam stability results. As a complement, we correct the relevant results (see Section 5 for details).

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In Section 2, we first start with some basics and fundamental properties of discrete fractional calculus. Two cases of BVPs are considered in Section 3 and Section 4, respectively. We point out the inconsistencies in the calculations arising in [3] and provide the corrections in Section 5. Finally, in Section 6, two numerical examples are presented.

## 2. Preliminaries

We now give some definitions and preliminary results for Caputo operators. Denote  $\mathbb{N}_{a+1}^b := \{a + 1, a + 2, \dots, b\}$  for any  $a \in \mathbb{R}$ .

**Definition 2.1** (See [5, Definition 3.56]) *For  $\nu \notin -\mathbb{N}$ , put*

$$H_\nu(t, a) := \frac{(t - a)^{\bar{\nu}}}{\Gamma(\nu + 1)}, \quad t \in \mathbb{N}_a, \tag{2.1}$$

where

$$t^{\bar{\nu}} = \frac{\Gamma(t + \nu)}{\Gamma(t)}.$$

**Definition 2.2** (See [5, Definition 3.58]) *Let  $f : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ . For  $\nu > 0$ , put*

$$\nabla_a^{-\nu} f(t) := \int_a^t H_{\nu-1}(t, \rho(s)) f(s) \nabla s, \quad t \in \mathbb{N}_{a+1}, \tag{2.2}$$

where  $\rho(t) := t - 1$  and  $\nabla_a^{-\nu} f(a) := 0$ .

**Definition 2.3** (See [5, Definition 3.117]) *Let  $f : \mathbb{N}_{a-N+1} \rightarrow \mathbb{R}$ . For  $\nu > 0$  and  $N := \lceil \nu \rceil$ , put*

$$\nabla_{a^*}^\nu f(t) := \nabla_a^{-(N-\nu)} \nabla^N f(t), \quad t \in \mathbb{N}_{a-N+1}, \tag{2.3}$$

and by convention  $\nabla_{a^*}^\nu f(a) = 0$ .

**Lemma 2.4** (See [5, Theorem 3.119]) *Let  $f : \mathbb{N}_{a-N+1} \rightarrow \mathbb{R}$ . If  $\nu > 0$  and  $N := \lceil \nu \rceil$ , then*

$$f(t) = \sum_{k=0}^{N-1} H_k(t, a) \nabla^k f(a) + \nabla_a^{-\nu} \nabla_{a^*}^\nu f(t), \quad t \in \mathbb{N}_{a-N+1}. \tag{2.4}$$

## 3. Nabla case, $0 < \nu \leq 1$

Having established the preliminary results for Caputo operators, we now consider existence and uniqueness of solutions for nonlinear Caputo fractional difference equations of the form

$$\begin{cases} \nabla_{a^*}^\nu x(t) = f(t, x(t-1)), & t \in \mathbb{N}_{a+1}^b, \\ mx(a) + nx(b) = 0, \end{cases} \tag{3.1}$$

where  $0 < \nu \leq 1$ ,  $x(t) \in \mathbb{R}$ ,  $f : \mathbb{N}_{a+1}^b \times \mathbb{R} \rightarrow \mathbb{R}$ , and  $m, n \in \mathbb{R}$  with  $m + n \neq 0$ .

To begin with, we consider the linear version of (3.1) of the form

$$\begin{cases} \nabla_{a^*}^\nu x(t) = f(t), & t \in \mathbb{N}_{a+1}^b, \\ mx(a) + nx(b) = 0, \end{cases} \quad (3.2)$$

where  $0 < \nu \leq 1$ ,  $x(t) \in \mathbb{R}$ ,  $f : \mathbb{N}_{a+1}^b \rightarrow \mathbb{R}$ , and  $m, n \in \mathbb{R}$  with  $m + n \neq 0$ .

**Theorem 3.1** *x is a solution of*

$$x(t) = -\frac{n}{m+n} \nabla_a^{-\nu} f(b) + \nabla_a^{-\nu} f(t), \quad t \in \mathbb{N}_a^b \quad (3.3)$$

*iff x is a solution of (3.2).*

**Proof** If  $x$  satisfies (3.3), then

$$\begin{aligned} mx(a) + nx(b) &= -\frac{mn}{m+n} \nabla_a^{-\nu} f(b) + n \left( -\frac{n}{m+n} \nabla_a^{-\nu} f(b) + \nabla_a^{-\nu} f(b) \right) \\ &= -\frac{mn}{m+n} \nabla_a^{-\nu} f(b) - \frac{n^2}{m+n} \nabla_a^{-\nu} f(b) + n \nabla_a^{-\nu} f(b) = \left( -\frac{mn}{m+n} - \frac{n^2}{m+n} + n \right) \nabla_a^{-\nu} f(b) = 0. \end{aligned}$$

Moreover, for  $t \in \mathbb{N}_{a+1}^b$ ,

$$\begin{aligned} \nabla_{a^*}^\nu x(t) &= \nabla_{a^*}^\nu \left( -\frac{n}{m+n} \nabla_a^{-\nu} f(b) + \nabla_a^{-\nu} f(t) \right) \\ &\stackrel{(3.4)}{=} \nabla_{a^*}^\nu \nabla_a^{-\nu} f(t) \stackrel{(2.3)}{=} \nabla_a^{-(1-\nu)} \nabla \nabla_a^{-\nu} f(t) \stackrel{(3.5)}{=} \nabla_a^{-(1-\nu)} \nabla_a^{1-\nu} f(t) = f(t), \end{aligned}$$

where we used the nice property of the discrete Caputo operator that for constant  $C$ ,

$$\nabla_{a^*}^\nu C = \nabla_a^{-(N-\nu)} \nabla^N C = \nabla_a^{-(N-\nu)} 0 = 0 \quad (3.4)$$

and the composition rule in [4, Lemma 4.9]

$$\nabla_a^{-\nu} \nabla_a^\nu f(t) = f(t), \quad \nu > 0, \quad t \in \mathbb{N}_a. \quad (3.5)$$

Hence,  $x$  solves (3.2).

On the other hand, assume  $x$  satisfies (3.2). From Lemma 2.4, setting  $N = 1$ , we immediately obtain that

$$x(t) = H_0(t, a)x(a) + \nabla_a^{-\nu} \nabla_{a^*}^\nu x(t) = x(a) + \nabla_a^{-\nu} f(t), \quad t \in \mathbb{N}_a^b. \quad (3.6)$$

We plug (3.6) into the boundary condition of (3.2) and derive

$$mx(a) + nx(b) = mx(a) + n(x(a) + \nabla_a^{-\nu} f(b)) = (m+n)x(a) + n \nabla_a^{-\nu} f(b) = 0,$$

which implies that

$$x(a) = -\frac{n}{m+n} \nabla_a^{-\nu} f(b). \quad (3.7)$$

Substituting (3.7) into (3.6), we have (3.3). □

Due to Theorem 3.1, we obtain the next result.

**Theorem 3.2**  $x$  is a solution of

$$x(t) = -\frac{n}{m+n} \nabla_a^{-\nu} f(b, x(b-1)) + \nabla_a^{-\nu} f(t, x(t-1)), \quad t \in \mathbb{N}_a^b$$

iff  $x$  is a solution of (3.1).

We make the following assumption.

( $H_1$ )  $f : \mathbb{N}_a^b \times \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous, i.e., there exists  $L > 0$  such that

$$|f(t, x) - f(t, y)| \leq L|x - y| \text{ for all } x, y \in \mathbb{R} \text{ and } t \in \mathbb{N}_a^b. \quad (3.8)$$

**Theorem 3.3** Assume ( $H_1$ ). Then (3.1) has a unique solution on  $\mathbb{N}_a^b$  provided

$$\lambda := \left( \left| \frac{n}{m+n} \right| + 1 \right) LH_\nu(b, a) < 1. \quad (3.9)$$

**Proof** Let  $(X, \|\cdot\|)$  be the Banach space of all real-valued functions defined on  $\mathbb{N}_a^b$  with norm  $\|x\| = \sup_{t \in \mathbb{N}_a^b} |x(t)|$ . We next define the operator  $T : X \rightarrow X$  by

$$(Tx)(t) := -\frac{n}{m+n} \nabla_a^{-\nu} f(b, x(b-1)) + \nabla_a^{-\nu} f(t, x(t-1)). \quad (3.10)$$

For the existence of solutions of (3.1), one needs to prove that  $T$  has a fixed point. Notice that for all  $x, y \in X$  and  $t \in \mathbb{N}_a^b$ , we have

$$\begin{aligned} & |(Tx)(t) - (Ty)(t)| \\ &= \left| -\frac{n}{m+n} \nabla_a^{-\nu} f(b, x(b-1)) + \nabla_a^{-\nu} f(t, x(t-1)) - \left( -\frac{n}{m+n} \nabla_a^{-\nu} f(b, y(b-1)) + \nabla_a^{-\nu} f(t, y(t-1)) \right) \right| \\ &= \left| \frac{n}{m+n} (\nabla_a^{-\nu} f(b, y(b-1)) - \nabla_a^{-\nu} f(b, x(b-1))) + \nabla_a^{-\nu} f(t, x(t-1)) - \nabla_a^{-\nu} f(t, y(t-1)) \right| \\ &= \left| \frac{n}{m+n} \nabla_a^{-\nu} (f(b, y(b-1)) - f(b, x(b-1))) + \nabla_a^{-\nu} (f(t, x(t-1)) - f(t, y(t-1))) \right| \\ &\stackrel{(2.2)}{\leq} \left| \frac{n}{m+n} \right| \nabla_a^{-\nu} |(f(b, y(b-1)) - f(b, x(b-1)))| + \nabla_a^{-\nu} |(f(t, x(t-1)) - f(t, y(t-1)))| \\ &\stackrel{(H_1)}{\leq} \left| \frac{n}{m+n} \right| L \nabla_a^{-\nu} |y(b-1) - x(b-1)| + L \nabla_a^{-\nu} |x(t-1) - y(t-1)| \\ &\leq \left| \frac{n}{m+n} \right| L \nabla_a^{-\nu} \|x - y\| + L \nabla_a^{-\nu} \|x - y\| \\ &= \left( \left| \frac{n}{m+n} \right| + 1 \right) L \|x - y\| \nabla_a^{-\nu} 1 \\ &= \left( \left| \frac{n}{m+n} \right| + 1 \right) LH_\nu(t, a) \|x - y\| \\ &\leq \left( \left| \frac{n}{m+n} \right| + 1 \right) LH_\nu(b, a) \|x - y\| \end{aligned}$$

$$\stackrel{(3.9)}{=} \lambda \|x - y\|,$$

where we use the fact that  $H_\nu(\cdot, a)$  is a nondecreasing function. That is,

$$|(Tx)(t) - (Ty)(t)| \leq \lambda \|x - y\|, \quad t \in \mathbb{N}_a^b.$$

Then

$$\|Tx - Ty\| \leq \lambda \|x - y\|, \text{ with } \lambda \in (0, 1).$$

Therefore,  $T$  is a contraction mapping and from the Banach fixed point theorem, see [11, Theorem 1.A], we know that there exists a unique  $\bar{x}$  such that  $T\bar{x} = \bar{x}$ , which implies by Theorem 3.2 that  $\bar{x}$  is a solution of (3.1).

We next show that  $\bar{x}$  is the unique solution of (3.1). Let  $y$  be any solution of (3.1). From Theorem 3.2 and using the fact that  $T$  has a unique fixed point  $\bar{x}$ , we have

$$\begin{aligned} y(t) &= -\frac{n}{m+n} \nabla_a^{-\nu} f(b, y(b-1)) + \nabla_a^{-\nu} f(t, y(t-1)) \\ &= (Ty)(t) = \bar{x}(t). \end{aligned}$$

Thus, (3.1) has a unique solution on  $\mathbb{N}_a^b$ . □

#### 4. Nabla case, $1 < \nu \leq 2$

We now investigate existence and uniqueness of solutions for nonlinear Caputo fractional difference equations of the form

$$\begin{cases} \nabla_{a^*}^\nu x(t) = f(t, x(t-1)), & t \in \mathbb{N}_{a+1}^b, \\ x(a-1) = g(x), x(b) = x_b, \end{cases} \quad (4.1)$$

where  $1 < \nu \leq 2$ ,  $x_b, x(t) \in \mathbb{R}$ ,  $f : \mathbb{N}_{a+1}^b \times \mathbb{R} \rightarrow \mathbb{R}$ , and  $g : X \rightarrow \mathbb{R}$ , where  $X$  is the space of all real-valued functions defined on  $\mathbb{N}_{a-1}^b$ .

First, let us consider the linear version of (4.1) of the form

$$\begin{cases} \nabla_{a^*}^\nu x(t) = f(t), & t \in \mathbb{N}_{a+1}^b, \\ x(a-1) = g(x), x(b) = x_b, \end{cases} \quad (4.2)$$

where  $1 < \nu \leq 2$ ,  $x_b, x(t) \in \mathbb{R}$ ,  $f : \mathbb{N}_{a+1}^b \rightarrow \mathbb{R}$ , and  $g : X \rightarrow \mathbb{R}$ , where  $X$  is the space of all real-valued functions defined on  $\mathbb{N}_{a-1}^b$ .

**Theorem 4.1**  *$x$  is a solution of*

$$x(t) = \frac{(t-a+1)(x_b - \nabla_a^{-\nu} f(b)) + (b-t)g(x)}{b-a+1} + \nabla_a^{-\nu} f(t), \quad t \in \mathbb{N}_{a-1}^b \quad (4.3)$$

*iff  $x$  is a solution of (4.2).*

**Proof** Assume  $x$  solves (4.3). It is not difficult to see that (4.3) satisfies the boundary conditions of (4.2). In addition, for  $t \in \mathbb{N}_{a+1}^b$ ,

$$\nabla_{a^*}^\nu x(t) = \nabla_{a^*}^\nu \left( \frac{(t-a+1)(x_b - \nabla_a^{-\nu} f(b)) + (b-t)g(x)}{b-a+1} + \nabla_a^{-\nu} f(t) \right)$$

$$\begin{aligned}
 &= \frac{x_b - \nabla_a^{-\nu} f(b)}{b - a + 1} \nabla_{a^*}^{\nu} (t - a + 1) + \frac{g(x)}{b - a + 1} \nabla_{a^*}^{\nu} (b - t) + \nabla_{a^*}^{\nu} \nabla_a^{-\nu} f(t) \\
 &\stackrel{(2.3)}{=} \nabla_a^{-(2-\nu)} \nabla^2 \nabla_a^{-\nu} f(t) \stackrel{(3.5)}{=} \nabla_a^{-(2-\nu)} \nabla_a^{2-\nu} f(t) = f(t).
 \end{aligned}$$

Hence,  $x$  solves (4.2).

For the opposite direction, let  $x$  be a solution of (4.2). From Lemma 2.4, setting  $N = 2$ , we have

$$\begin{aligned}
 x(t) &= H_0(t, a)x(a) + H_1(t, a)\nabla x(a) + \nabla_a^{-\nu} \nabla_{a^*}^{\nu} x(t) \\
 &= x(a) + \frac{(t - a)^{\bar{1}}}{\Gamma(2)} \nabla x(a) + \nabla_a^{-\nu} f(t) \\
 &= x(a) + (t - a)\nabla x(a) + \nabla_a^{-\nu} f(t), \quad t \in \mathbb{N}_{a-1}^b.
 \end{aligned} \tag{4.4}$$

Combining (4.4) and the boundary conditions of (4.2), we derive

$$\begin{aligned}
 x_b &= x(a) + (b - a)\nabla x(a) + \nabla_a^{-\nu} f(b) \\
 &= x(a) + (b - a)(x(a) - x(a - 1)) + \nabla_a^{-\nu} f(b) \\
 &= (b - a + 1)x(a) - (b - a)x(a - 1) + \nabla_a^{-\nu} f(b) \\
 &= (b - a + 1)x(a) - (b - a)g(x) + \nabla_a^{-\nu} f(b),
 \end{aligned}$$

which implies that

$$x(a) = \frac{x_b + (b - a)g(x) - \nabla_a^{-\nu} f(b)}{b - a + 1}. \tag{4.5}$$

Then

$$\nabla x(a) = x(a) - x(a - 1) = \frac{x_b + (b - a)g(x) - \nabla_a^{-\nu} f(b)}{b - a + 1} - g(x) = \frac{x_b - g(x) - \nabla_a^{-\nu} f(b)}{b - a + 1}. \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), collecting like terms, we have (4.3). □

**Theorem 4.2**  $x$  is a solution of

$$x(t) = \frac{(t - a + 1)(x_b - \nabla_a^{-\nu} f(b, x(b - 1))) + (b - t)g(x)}{b - a + 1} + \nabla_a^{-\nu} f(t, x(t - 1)), \quad t \in \mathbb{N}_{a-1}^b$$

iff  $x$  is a solution of (4.1).

We make an assumption about the function  $g$ .

(H<sub>2</sub>) There exists  $M > 0$  such that

$$|g(x) - g(y)| \leq M|x - y| \text{ for all } x, y \in X.$$

**Theorem 4.3** Assume (H<sub>1</sub>) and (H<sub>2</sub>). Then (3.2) has a unique solution on  $\mathbb{N}_{a-1}^b$  provided

$$\mu := 2LH_{\nu}(b, a) + M < 1. \tag{4.7}$$

**Proof** This result can be proved in exactly the same way as that of Theorem 3.3. Therefore, it is not surprising for us to find that the proof here is analogous as well. Let  $X$  be the space of all real-valued functions defined on  $\mathbb{N}_{a-1}^b$  with norm  $\|x\| = \sup_{t \in \mathbb{N}_{a-1}^b} |x(t)|$ . Hence,  $(X, \|\cdot\|)$  is a Banach space. We next define the operator  $T : X \rightarrow X$  by

$$(Tx)(t) = \frac{(t-a+1)(x_b - \nabla_a^{-\nu} f(b, x(b-1))) + (b-t)g(x)}{b-a+1} + \nabla_a^{-\nu} f(t, x(t-1)), \quad t \in \mathbb{N}_{a-1}^b.$$

We claim that  $T$  is a contraction mapping. Indeed,

$$\begin{aligned} & |(Tx)(t) - (Ty)(t)| \\ &= \left| \frac{(t-a+1)(x_b - \nabla_a^{-\nu} f(b, x(b-1))) + (b-t)g(x)}{b-a+1} + \nabla_a^{-\nu} f(t, x(t-1)) \right. \\ &\quad \left. - \frac{(t-a+1)(x_b - \nabla_a^{-\nu} f(b, y(b-1))) + (b-t)g(y)}{b-a+1} - \nabla_a^{-\nu} f(t, y(t-1)) \right| \\ &= \left| \frac{(t-a+1)\nabla_a^{-\nu}(f(b, y(b-1)) - f(b, x(b-1))) + (b-t)(g(x) - g(y))}{b-a+1} \right. \\ &\quad \left. + \nabla_a^{-\nu}(f(t, x(t-1)) - f(t, y(t-1))) \right| \\ &\leq \frac{t-a+1}{b-a+1} |\nabla_a^{-\nu}(f(b, y(b-1)) - f(b, x(b-1)))| + \frac{b-t}{b-a+1} |g(x) - g(y)| \\ &\quad + |\nabla_a^{-\nu}(f(t, x(t-1)) - f(t, y(t-1)))| \\ &\stackrel{(2.2)}{\leq} \frac{t-a+1}{b-a+1} \nabla_a^{-\nu} |f(b, y(b-1)) - f(b, x(b-1))| + \frac{b-t}{b-a+1} |g(x) - g(y)| \\ &\quad + \nabla_a^{-\nu} |f(t, x(t-1)) - f(t, y(t-1))| \\ &\stackrel{(H_1)}{\leq} \frac{t-a+1}{b-a+1} L \nabla_a^{-\nu} |y(b-1) - x(b-1)| + \frac{b-t}{b-a+1} M \|x - y\| + L \nabla_a^{-\nu} |x(t-1) - y(t-1)| \\ &\stackrel{(H_2)}{\leq} \frac{t-a+1}{b-a+1} L \nabla_a^{-\nu} \|x - y\| + \frac{b-t}{b-a+1} M \|x - y\| + L \nabla_a^{-\nu} \|x - y\| \\ &= \left( \frac{t-a+1}{b-a+1} + 1 \right) L \|x - y\| \nabla_a^{-\nu} 1 + \frac{b-t}{b-a+1} M \|x - y\| \\ &= \left( \frac{t-a+1}{b-a+1} + 1 \right) L H_\nu(t, a) \|x - y\| + \frac{b-t}{b-a+1} M \|x - y\| \\ &\leq (2LH_\nu(b, a) + M) \|x - y\| \\ &\stackrel{(4.7)}{=} \mu \|x - y\|, \quad t \in \mathbb{N}_{a-1}^b, \end{aligned}$$

which implies

$$\|Tx - Ty\| \leq \mu \|x - y\|, \text{ with } \mu \in (0, 1).$$

From the Banach fixed point theorem, see [11, Theorem 1.A], we know that there exists a unique  $\bar{x}$  such that  $T\bar{x} = \bar{x}$ , which implies by Theorem 4.2 that  $\bar{x}$  is a solution of (4.1).

We next show that  $\bar{x}$  is the unique solution of (4.1). Let  $y$  be any solution of (4.1). From Theorem 4.2 and using the fact that  $T$  has a unique fixed point  $\bar{x}$ , we have

$$y(t) = \frac{(t - a + 1)(x_b - \nabla_a^{-\nu} f(b, y(b - 1))) + (b - t)g(y)}{b - a + 1} + \nabla_a^{-\nu} f(t, y(t - 1))$$

$$=(Ty)(t) = \bar{x}(t).$$

Thus, (4.1) has a unique solution on  $\mathbb{N}_{a-1}^b$ . □

**5. Delta case,  $1 < \nu < 2$**

Chen and Zhou [3, Lemma 6] investigated existence of solutions for discrete fractional boundary value problems of the form

$$\begin{cases} \Delta_{(\nu-2)^+}^\nu x(t) = f(t + \nu - 1), & t \in \mathbb{N}_0^b, \\ x(\nu - 1) + x(b + \nu) = 0, \Delta x(\nu - 1) + \Delta x(b + \nu - 1) = 0, \end{cases} \tag{5.1}$$

where  $1 < \nu < 2$ ,  $x(t) \in \mathbb{R}$ ,  $b \in \mathbb{N}_1$  and  $f : \mathbb{N}_{\nu-1}^{b+\nu-1} \rightarrow \mathbb{R}$ . They obtained the following result.

**Theorem 5.1** (See [3, Lemma 6])  *$x$  is a solution of (5.1) iff  $x$  is a solution of*

$$x(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - s - 1)^{\nu-1} f(s + \nu - 1) - \frac{1}{2\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1)$$

$$+ \frac{b + 2\nu - 1 - t}{2\Gamma(\nu - 1)} \sum_{s=0}^{b-1} (b + \nu - s - 2)^{\nu-2} f(s + \nu - 1), \quad t \in \mathbb{N}_{\nu-1}^{b+\nu}. \tag{5.2}$$

However, we have to point out that this result is incorrect. We only need to verify that (5.2) does not satisfy the first boundary condition of (5.1). Let  $x(t)$  have the representation (5.2). Then

$$x(\nu - 1) = -\frac{1}{2\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1) + \frac{b + \nu}{2\Gamma(\nu - 1)} \sum_{s=0}^{b-1} (b + \nu - s - 2)^{\nu-2} f(s + \nu - 1)$$

and

$$x(b + \nu) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1) - \frac{1}{2\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1)$$

$$+ \frac{\nu - 1}{2\Gamma(\nu - 1)} \sum_{s=0}^{b-1} (b + \nu - s - 2)^{\nu-2} f(s + \nu - 1).$$

Hence,

$$x(\nu - 1) + x(b + \nu) = \frac{b + 2\nu - 1}{2\Gamma(\nu - 1)} \sum_{s=0}^{b-1} (b + \nu - s - 2)^{\nu-2} f(s + \nu - 1).$$

Since  $b \in \mathbb{N}_1$ , we derive that  $x(\nu - 1) + x(b + \nu) \neq 0$ , which contradicts the boundary condition of (5.1). Hence, in this way, we see that Theorem 5.1 is incorrect.



In addition, they also considered the Ulam stability (see [3, Section 4] for details) for

$$\begin{cases} \Delta_{(\nu-2)^*}^\nu x(t) = f(t + \nu - 1, x(t + \nu - 1)), & t \in \mathbb{N}_0^b, \\ x(\nu - 1) = y(\nu - 1), x(b + \nu) = y(b + \nu), \end{cases} \quad (5.3)$$

where  $1 < \nu < 2$ ,  $x(t) \in \mathbb{R}$ ,  $b \in \mathbb{N}_1$ , and  $f : \mathbb{N}_{\nu-1}^{b+\nu-1} \times \mathbb{R} \rightarrow \mathbb{R}$ . At the beginning of the proof of [3, Theorem 15], they gave the solution for (5.3) as

$$\begin{aligned} x(t) &= y(\nu - 1) + \frac{t}{b + \nu} (y(b + \nu) - y(\nu - 1)) \\ &\quad - \frac{t}{(b + \nu)\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1, x(s + \nu - 1)) \\ &\quad + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - s - 1)^{\nu-1} f(s + \nu - 1, x(s + \nu - 1)), \quad t \in \mathbb{N}_{\nu-1}^{b+\nu}. \end{aligned} \quad (5.4)$$

Unfortunately, it seems that (5.4) is not a solution for (5.3) either. From (5.4), we have

$$\begin{aligned} x(\nu - 1) &= y(\nu - 1) + \frac{\nu - 1}{b + \nu} (y(b + \nu) - y(\nu - 1)) \\ &\quad - \frac{\nu - 1}{(b + \nu)\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1, x(s + \nu - 1)). \end{aligned} \quad (5.5)$$

If (5.5) satisfies the first boundary condition of (5.3), then

$$0 = \frac{\nu - 1}{b + \nu} (y(b + \nu) - y(\nu - 1)) - \frac{\nu - 1}{(b + \nu)\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1, x(s + \nu - 1)).$$

That is,

$$\begin{aligned} y(b + \nu) - y(\nu - 1) &= \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1, x(s + \nu - 1)) \\ &= \Delta_0^{-\nu} f(t + \nu - 1, x(t + \nu - 1)) \Big|_{t=b+\nu}. \end{aligned} \quad (5.6)$$

However, so far, the validity for (5.6) is still a question since we do not know the exact representation of  $f(t + \nu - 1, x(t + \nu - 1))$ . In fact, the validity of (5.6) is negative. One can see Remark 5.5 for details.

Notice that the results in the rest of [3] are strongly based on the solutions of (5.1) and (5.3). The problems need to be addressed. For completeness, the correct solutions of (5.1) and (5.3) are provided as follows.

**Lemma 5.2 (See [1, Proposition 15])** *Let  $f : \mathbb{N}_a \rightarrow \mathbb{R}$ . For  $\nu > 0$  and  $N := \lceil \nu \rceil$ , we have*

$$\Delta_{a+N-\nu}^{-\nu} \Delta_a^\nu f(t) = f(t) - \sum_{k=0}^{N-1} \frac{(t-a)^k}{k!} \Delta^k f(a) = f(t) + c_0 + c_1 t + \dots + c_{N-1} t^{N-1},$$

where  $c_i$  are constants,  $i \in \mathbb{N}_0^{N-1}$ .

**Theorem 5.3**  $x$  is a solution of (5.1) iff  $x$  is a solution of

$$x(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\nu-1} f(s+\nu-1) - c_0 - c_1 t, \quad t \in \mathbb{N}_{\nu-1}^{b+\nu},$$

where

$$c_1 = \frac{1}{2} \left( f(\nu-1) + \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^b (b+\nu-s-2)^{\nu-2} f(s+\nu-1) \right)$$

and

$$c_0 = \frac{1}{2} \left( \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b+\nu-s-1)^{\nu-1} f(s+\nu-1) - c_1(b+2\nu-1) \right).$$

**Proof** Applying  $\Delta_0^{-\nu}$  to each side of (5.1), from Lemma 5.2, we have

$$x(t) = \Delta_0^{-\nu} f(t+\nu-1) - c_0 - c_1 t. \tag{5.7}$$

Then

$$\Delta x(t) = \Delta \Delta_0^{-\nu} f(t+\nu-1) - c_1 = \Delta_0^{1-\nu} f(t+\nu-1) - c_1. \tag{5.8}$$

Combining (5.8) and the second boundary condition of (5.1), we have

$$\begin{aligned} \Delta x(\nu-1) + \Delta x(b+\nu-1) &= \Delta_0^{1-\nu} f(t+\nu-1)|_{t=\nu-1} - c_1 + \Delta_0^{1-\nu} f(t+\nu-1)|_{t=b+\nu-1} - c_1 \\ &= f(\nu-1) + \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^b (b+\nu-s-2)^{\nu-2} f(s+\nu-1) - 2c_1 = 0. \end{aligned}$$

That is,

$$c_1 = \frac{1}{2} \left( f(\nu-1) + \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^b (b+\nu-s-2)^{\nu-2} f(s+\nu-1) \right).$$

Besides, from (5.7) and the first boundary condition of (5.1), we have

$$\begin{aligned} x(\nu-1) + x(b+\nu) &= \Delta_0^{-\nu} f(t+\nu-1)|_{t=\nu-1} - c_0 - c_1(\nu-1) + \Delta_0^{-\nu} f(t+\nu-1)|_{t=b+\nu} - c_0 - c_1(b+\nu) \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b+\nu-s-1)^{\nu-1} f(s+\nu-1) - 2c_0 - c_1(b+2\nu-1) = 0. \end{aligned}$$

That is,

$$c_0 = \frac{1}{2} \left( \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b+\nu-s-1)^{\nu-1} f(s+\nu-1) - c_1(b+2\nu-1) \right).$$

Thus, we obtain our required result. □

**Theorem 5.4**  $x$  is a solution for (5.3) iff  $x$  is a solution of

$$x(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\nu-1} f(s+\nu-1, x(s+\nu-1)) - c_0 - c_1 t, \quad t \in \mathbb{N}_{\nu-1}^{b+\nu},$$

where

$$c_1 = \frac{1}{1+b} \left( \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b+\nu-s-1)^{\nu-1} f(s+\nu-1, x(s+\nu-1)) + y(\nu-1) - y(b+\nu) \right)$$

and

$$c_0 = -c_1(\nu-1) - y(\nu-1).$$

**Proof** Similar to the proof of Theorem 5.3, we have

$$x(t) = \Delta_0^{-\nu} f(t+\nu-1, x(t+\nu-1)) - c_0 - c_1 t.$$

Then from the boundary conditions of (5.3), we know

$$x(\nu-1) = \Delta_0^{-\nu} f(t+\nu-1, x(t+\nu-1)) \Big|_{t=\nu-1} - c_0 - c_1(\nu-1) = -c_0 - c_1(\nu-1) = y(\nu-1). \quad (5.9)$$

and

$$\begin{aligned} x(b+\nu) &= \Delta_0^{-\nu} f(t+\nu-1, x(t+\nu-1)) \Big|_{t=b+\nu} - c_0 - c_1(b+\nu) \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b+\nu-s-1)^{\nu-1} f(s+\nu-1, x(s+\nu-1)) - c_0 - c_1(b+\nu) \\ &= y(b+\nu). \end{aligned} \quad (5.10)$$

Combining (5.9) and (5.10), we have that

$$c_1 = \frac{1}{1+b} \left( \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b+\nu-s-1)^{\nu-1} f(s+\nu-1, x(s+\nu-1)) + y(\nu-1) - y(b+\nu) \right)$$

and

$$c_0 = -c_1(\nu-1) - y(\nu-1).$$

Thus, we obtain our required result. □

**Remark 5.5** Using the boundary conditions of (5.3) and Theorem 5.4, we can prove that the validity of (5.6) is negative. We have

$$\begin{aligned} y(b+\nu) - y(\nu-1) &= x(b+\nu) - x(\nu-1) \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b+\nu-s-1)^{\nu-1} f(s+\nu-1, x(s+\nu-1)) - c_0 - c_1(b+\nu) - (-c_0 - c_1(\nu-1)) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1, x(s + \nu - 1)) - c_1(b + 1) \\
 &\neq \frac{1}{\Gamma(\nu)} \sum_{s=0}^b (b + \nu - s - 1)^{\nu-1} f(s + \nu - 1, x(s + \nu - 1)).
 \end{aligned}$$

This implies that (5.4) is not a solution of (5.3).

### 6. Examples

We now give some examples in order to illustrate our results.

**Example 6.1** Let us consider the discrete Caputo BVP

$$\begin{cases} \nabla_0^{0.5} x(t) = 0.25 \sin x(t - 1), & t \in \mathbb{N}_1^5, \\ x(0) + x(5) = 0. \end{cases} \tag{6.1}$$

Using the parameters

$$\nu = 0.5, a = 0, b = 5, m = n = 1,$$

from Theorem 3.2, we obtain that a solution of (6.1) satisfies

$$\begin{aligned}
 x(t) &= -\frac{1}{2} \nabla_0^{-0.5} 0.25 \sin x(4) + \nabla_0^{-0.5} 0.25 \sin x(t - 1) \\
 &= -0.125 \nabla_0^{-0.5} \sin x(4) + 0.25 \nabla_0^{-0.5} \sin x(t - 1), \quad t \in \mathbb{N}_0^5.
 \end{aligned}$$

We next show that (6.1) has a unique solution. To begin with, we need to make sure  $f$  satisfies  $(H_1)$ . Indeed,

$$\begin{aligned}
 |f(t, x) - f(t, y)| &= |0.25 \sin x - 0.25 \sin y| = 0.25 |\sin x - \sin y| \\
 &= 0.25 |(x - y) \cos(\xi)| \leq 0.25 |x - y|, \quad t \in \mathbb{N}_0^5,
 \end{aligned}$$

where we used the Lagrange mean value theorem. Hence,  $L = 0.25$  and  $f$  is Lipschitz continuous for  $t \in \mathbb{N}_0^5$ . Notice that

$$\lambda = \left( \left| \frac{1}{1 + 1} \right| + 1 \right) \cdot 0.25 \cdot H_{0.5}(5, 0) \approx 0.9229 < 1.$$

Applying Theorem 3.3, we come to the conclusion that (6.1) has a unique solution on  $\mathbb{N}_0^5$ .

**Example 6.2** Let us consider the discrete Caputo BVP

$$\begin{cases} \nabla_0^{0.25} x(t) = 0.25 \sin x(t - 1), & t \in \mathbb{N}_1^5, \\ x(-1) = \frac{\|x\|}{6}, x(5) = 1, \end{cases} \tag{6.2}$$

where  $\|x\| = \sup_{t \in \mathbb{N}_{-1}^5} |x(t)|$ . Using

$$\nu = 0.25, a = 0, b = 5, g(x) = \frac{\|x\|}{6}, x_b = 1, L = 0.25, M = \frac{1}{6},$$

from Theorem 4.2, we obtain that a solution of (6.2) satisfies

$$x(t) = \frac{(t+1)(1 - 0.25\nabla_0^{-0.25} \sin x(4)) + \frac{(5-t)\|x\|}{6}}{6} + 0.25\nabla_0^{-0.25} \sin x(t-1), \quad t \in \mathbb{N}_{-1}^5.$$

Since

$$\mu = 2 \cdot 0.25 \cdot H_{0.25}(5, 0) + \frac{1}{6} \approx 0.9760 < 1,$$

from Theorem 4.3, we obtain that (6.2) has a unique solution on  $\mathbb{N}_{-1}^5$ .

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### References

- [1] Abdeljawad T. On Riemann and Caputo fractional differences. *Computers and Mathematics with Applications* 2011; 62 (3): 1602-1611.
- [2] Atici FM, Eloe PW. Two-point boundary value problems for finite fractional difference equations. *Journal of Difference Equations and Applications* 2011; 17 (4): 445-456.
- [3] Chen FL, Zhou Y. Existence and Ulam stability of solutions for discrete fractional boundary value problem. *Discrete Dynamics in Nature Society* 2013; Article ID 459161: 7 pages.
- [4] Du FF, Erbe L, Jia BG, Peterson AC. Two asymptotic results of solutions for nabla fractional  $(q, h)$ -difference equations. *Turkish Journal of Mathematics* 2018; 42 (5): 2214-2242.
- [5] Goodrich CS, Peterson AC. *Discrete fractional calculus*. Switzerland: Springer International Publishing, 2015.
- [6] Goodrich CS. Solutions to a discrete right-focal fractional boundary value problem. *International Journal of Difference Equations* 2010; 5 (2): 195-216.
- [7] Holm MT. *The theory of discrete fractional calculus: development and application*. PhD, University of Nebraska - Lincoln, Lincoln, NE, USA, 2011.
- [8] Reunsumrit J, Sitthiwirattam T. A new class of four-point fractional sum boundary value problems for nonlinear sequential fractional difference equations involving shift operators. *Kragujevac Journal of Mathematics* 2018; 42 (3): 371-387.
- [9] Sitthiwirattam T, Tariboon J, Ntouyas SK. Boundary value problems for fractional difference equations with three-point fractional sum boundary conditions. *Advances in Difference Equations* 2013; 296 (2013).
- [10] Xie ZS, Jin YF, Hou CM. Multiple solutions for a fractional difference boundary value problem via variational approach. *Abstract and Applied Analysis* 2012; Article ID 143914: 16 pages.
- [11] Zeidler E. *Nonlinear functional analysis and its applications I: Fixed-point theorems*. Translated from the German by Peter R. Wadsack. New York, NY, USA: Springer-Verlag New York, 1986.