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Bertrand and Mannheim curves of framed curves in the 3-dimensional Euclidean space

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Abstract: A Bertrand curve is a space curve whose principal normal line is the same as the principal normal line of another curve. On the other hand, a Mannheim curve is a space curve whose principal normal line is the same as the binormal line of another curve. By definitions, another curve is a parallel curve with respect to the direction of the principal normal vector. Even if that is the regular case, the existence conditions of the Bertrand and Mannheim curves seem to be wrong in some previous research. Moreover, parallel curves may have singular points. As smooth curves with singular points, we consider framed curves in the Euclidean space. Then we define and investigate Bertrand and Mannheim curves of framed curves. We clarify that the Bertrand and Mannheim curves of framed curves are dependent on the moving frame.

Key words: Bertrand curve, Mannheim curve, framed curve, singularity

1. Introduction

Bertrand and Mannheim curves are classical objects in differential geometry ([1–4, 7, 15–17, 20]). A Bertrand curve is a space curve whose principal normal line is the same as the principal normal line of another curve. On the other hand, a Mannheim curve is a space curve whose principal normal line is the same as the binormal line of another curve. Bertrand curves have been applied in computer-aided geometric design (cf. [19]). Moreover, there are many papers of Bertrand curves in the other spaces (for instance, [8, 14, 18]). By definitions, another curve is a parallel curve with respect to the direction of the principal normal vector. Even if that is the regular case, the existence conditions of the Bertrand and Mannheim curves seem to be wrong in some previous research. In order to define principal normal vector, the nondegenerate condition is needed. In general, the parallel curve does not satisfy the nondegenerate condition. We clarify correct existence conditions of Bertrand and Mannheim curves of regular space curves in §2. Moreover, parallel curves may have singular points. The locus of the singular points of parallel curves is the evolute of the original curve, see [6, 9, 10, 13]. We consider smooth curves with singular points. As smooth curves with singular points, we introduced the notion of framed curves in the Euclidean space in [12]. Then we define Bertrand and Mannheim curves of framed curves in Sections 3 and 4, respectively. We give existence conditions of the Bertrand and Mannheim curves of framed curves, respectively (Theorems 3.3 and 4.3). Moreover, we clarify that the Bertrand and Mannheim curves of framed curves are dependent on the moving frame (Remarks 3.7 and 4.8). We also give a difference between nondegenerate regular space curves and framed curves (Theorem 4.5).

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All maps and manifolds considered in this paper are differentiable of class C^∞.

2. Preliminaries

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$, where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$ and the vector product is given by

$$\boldsymbol{a}\times\boldsymbol{b} = \det\begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. Let S^2 be the unit sphere in $\mathbb{R}^3$, that is, $S^2 = \{\boldsymbol{a}\in\mathbb{R}^3 | |\boldsymbol{a}| = 1\}$. We denote the 3-dimensional smooth manifold $\{(\boldsymbol{a}, \boldsymbol{b}) \in S^2\times S^2 | \boldsymbol{a}\cdot\boldsymbol{b} = 0\}$ by Δ.

We quickly review the theories of Bertrand curves, Mannheim curves of regular curves, and framed curves.

Even if that is the regular case, the existence conditions of the Bertrand and Mannheim curves seem to be wrong in classical (and recent) books [1–3, 7, 16, 20]. We clarify existence conditions of Bertrand and Mannheim curves.

2.1. Regular space curves

Let I be an interval of $\mathbb{R}$ and let $\gamma : I \to \mathbb{R}^3$ be a regular space curve, that is, $\dot\gamma(t) \neq 0$ for all $t \in I$, where $\dot\gamma(t) = (d\gamma/dt)(t)$. We say that γ is nondegenerate, or γ satisfies the nondegenerate condition if $\dot\gamma(t)\times\ddot\gamma(t) \neq 0$ for all $t \in I$.

If we take the arc-length parameter s, that is, $|\gamma'(s)| = 1$ for all s, then the tangent vector, the principal normal vector, and the binormal vector are given by

$$\boldsymbol{t}(s) = \gamma'(s),\ \boldsymbol{n}(s) = \frac{\gamma''(s)}{|\gamma''(s)|},\ \boldsymbol{b}(s) = \boldsymbol{t}(s)\times\boldsymbol{n}(s),$$

where $\gamma'(s) = (d\gamma/ds)(s)$. Then $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ is a moving frame of $\gamma(s)$ and we have the Frenet–Serret formula:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \\ \boldsymbol{b}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{pmatrix},$$

where

$$\kappa(s) = |\gamma''(s)|,\ \tau(s) = \frac{\det(\gamma'(s), \gamma''(s), \gamma'''(s))}{\kappa^2(s)}.$$

If we take a general parameter t, then the tangent vector, the principal normal vector and the binormal vector are given by

$$\boldsymbol{t}(t) = \frac{\dot\gamma(t)}{|\dot\gamma(t)|},\ \boldsymbol{n}(s) = \boldsymbol{b}(t)\times\boldsymbol{t}(t),\ \boldsymbol{b}(t) = \frac{\dot\gamma(t)\times\ddot\gamma(t)}{|\dot\gamma(t)\times\ddot\gamma(t)|}.$$

Then $\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\}$ is a moving frame of $\gamma(t)$ and we have the Frenet–Serret formula:

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \\ \dot{\boldsymbol{b}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) & 0 \\ -|\dot{\gamma}(t)|\kappa(t) & 0 & |\dot{\gamma}(t)|\tau(t) \\ 0 & -|\dot{\gamma}(t)|\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \\ \boldsymbol{b}(t) \end{pmatrix},$$

where

$$\kappa(t) = \frac{|\dot{\gamma}(t) \times \ddot{\gamma}(t)|}{|\dot{\gamma}(t)|^3}, \ \tau(t) = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t))}{|\dot{\gamma}(t) \times \ddot{\gamma}(t)|^2}.$$

Note that in order to define $\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t), \kappa(t)$, and $\tau(t)$, we assume that γ is not only regular, but also nondegenerate.

2.2. Bertrand curves of regular space curves

Definition 2.1 *Let γ and $\overline{\gamma} : I \to \mathbb{R}^3$ be different nondegenerate curves. We say that γ and $\overline{\gamma}$ are Bertrand mates if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\boldsymbol{n}(t)$ and $\boldsymbol{n}(t) = \pm\overline{\boldsymbol{n}}(t)$ for all $t \in I$.*

We also say that $\gamma : I \to \mathbb{R}^3$ is a Bertrand curve if there exists another nondegenerate curve $\overline{\gamma} : I \to \mathbb{R}^3$ such that γ and $\overline{\gamma}$ are Bertrand mates.

If γ and $\overline{\gamma}$ are Bertrand mates, then the principal normal line of γ and the principal normal line of $\overline{\gamma}$ are the same for each points. Note that if we take $-\lambda$ instead of λ, then we may assume that $\boldsymbol{n}(t) = \overline{\boldsymbol{n}}(t)$.

By a parameter change, we may assume that s is the arc-length parameter of γ.

Lemma 2.2 *Let $\gamma : I \to \mathbb{R}^3$ be nondegenerate with the arc-length parameter. Under the notation in Definition 2.1, if γ and $\overline{\gamma}$ are Bertrand mates, then λ is a nonzero constant.*

Proof By differentiating $\overline{\gamma}(s) = \gamma(s) + \lambda(s)\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - \lambda(s)\kappa(s))\boldsymbol{t}(s) + \lambda'(s)\boldsymbol{n}(s) + \lambda(s)\tau(s)\boldsymbol{b}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{n}}(s)$, we have $\lambda'(s) = 0$ for all $s \in I$. Therefore λ is a constant. If $\lambda = 0$, then $\overline{\gamma}(s) = \gamma(s)$ for all $s \in I$. Hence, λ is a nonzero constant. □

Theorem 2.3 *Let $\gamma : I \to \mathbb{R}^3$ be nondegenerate with the arc-length parameter. Suppose that $\tau(s) \neq 0$ for all $s \in I$ and A is a nonzero constant. Then γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ if and only if there exists a constant B such that $A\kappa(s) + B\tau(s) = 1$ and $B\kappa(s) - A\tau(s) \neq 0$ for all $s \in I$.*

Proof Suppose that $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ and $\boldsymbol{n}(s) = \overline{\boldsymbol{n}}(s)$ for all $s \in I$. Note that s is not the arc-length parameter of $\overline{\gamma}$. By differentiating $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{n}}(s)$, there exists a smooth function $\theta : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\boldsymbol{b}}(s) \\ \overline{\boldsymbol{t}}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \boldsymbol{b}(s) \\ \boldsymbol{t}(s) \end{pmatrix}.$$

Then $|\dot{\overline{\gamma}}(s)|\sin\theta(s) = A\tau(s)$ and $|\dot{\overline{\gamma}}(s)|\cos\theta(s) = 1 - A\kappa(s)$. It follows that

$$-A\cos\theta(s)\tau(s) + (1 - A\kappa(s))\sin\theta(s) = 0. \tag{2.1}$$

By differentiating $\overline{\boldsymbol{t}}(s) = \sin\theta(s)\boldsymbol{b}(s) + \cos\theta(s)\boldsymbol{t}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\overline{\boldsymbol{n}}(s) = \theta'(s)\cos\theta(s)\boldsymbol{b}(s) - \theta'(s)\sin\theta(s)\boldsymbol{t}(s) + (-\sin\theta(s)\tau(s) + \cos\theta(s)\kappa(s))\boldsymbol{n}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{n}}(s)$, $\theta'(s) = 0$ for all $s \in I$. Therefore, θ is a constant. By $\tau(s) \neq 0$ and $|\dot{\overline{\gamma}}(s)|\sin\theta = A\tau(s)$, we have $\sin\theta \neq 0$. By the equation (2.1), we have $A\kappa(s) + A(\cos\theta/\sin\theta)\tau(s) = 1$. Hence, if we put $B = A\cos\theta/\sin\theta$, then $A\kappa(s) + B\tau(s) = 1$ for all $s \in I$. Moreover,

$$|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s) = -\sin\theta\tau(s) + \cos\theta\kappa(s) = \frac{\sin\theta}{A}(-A\tau(s) + B\kappa(s)).$$

Since $\overline{\kappa}(s) \neq 0$, we have $B\kappa(s) - A\tau(s) \neq 0$ for all $s \in I$.

Conversely, suppose that there exists a constant B such that $A\kappa(s) + B\tau(s) = 1$, $B\kappa(s) - A\tau(s) \neq 0$ and $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ for all $s \in I$. It follows that

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s) = \tau(s)(B\boldsymbol{t}(s) + A\boldsymbol{b}(s)).$$

Since $|\dot{\overline{\gamma}}(s)| = \sqrt{A^2 + B^2}|\tau(s)|$, we have $\overline{\boldsymbol{t}}(s) = \mathrm{sgn}(\tau(s))(1/\sqrt{A^2 + B^2})(B\boldsymbol{t}(s) + A\boldsymbol{b}(s))$, where $\mathrm{sgn}(\tau(s)) = 1$ if $\tau(s) > 0$ and $\mathrm{sgn}(\tau(s)) = -1$ if $\tau(s) < 0$. By differentiating $\overline{\boldsymbol{t}}(s)$, we have $|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\overline{\boldsymbol{n}}(s) = \mathrm{sgn}(\tau(s))(1/\sqrt{A^2 + B^2})(B\kappa(s) - A\tau(s))\boldsymbol{n}(s)$. By the condition, we have $\boldsymbol{n}(s) = \pm\overline{\boldsymbol{n}}(s)$ for all $s \in I$. □

By a direct calculation and the proof of Theorem 2.3, we have the curvature and the torsion of $\overline{\gamma}$

Proposition 2.4 *Let γ and $\overline{\gamma} : I \to \mathbb{R}^3$ be different nondegenerate curves. Under the same assumptions in Theorem 2.3, suppose that γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ and $A\kappa(s) + B\tau(s) = 1$ for all $s \in I$, where B is a constant. Then the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of $\overline{\gamma}$ are given by*

$$\overline{\kappa}(s) = \frac{|B\kappa(s) - A\tau(s)|}{(A^2 + B^2)|\tau(s)|}, \quad \overline{\tau}(s) = \frac{1}{(A^2 + B^2)\tau(s)}.$$

Proof Since $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$, we have

$$\dot{\overline{\gamma}}(s) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s) = \tau(s)(B\boldsymbol{t}(s) + A\boldsymbol{b}(s)).$$

Therefore,

$$\begin{aligned} \ddot{\overline{\gamma}}(s) &= \tau'(s)(B\boldsymbol{t}(s) + A\boldsymbol{b}(s)) + \tau(s)(B\kappa(s) - A\tau(s))\boldsymbol{n}(s), \\ \dddot{\overline{\gamma}}(s) &= \tau''(s)(B\boldsymbol{t}(s) + A\boldsymbol{b}(s)) + 2\tau'(s)(B\kappa(s) - A\tau(s))\boldsymbol{n}(s) \\ &\quad + \tau(s)(B\kappa'(s) - A\tau'(s))\boldsymbol{n}(s) + \tau(s)(B\kappa(s) - A\tau(s))(-\kappa(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s)). \end{aligned}$$

Since

$$\begin{aligned} |\dot{\overline{\gamma}}(s)| &= |\tau(s)|(A^2 + B^2)^{\frac{1}{2}}, \\ |\dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)| &= \tau(s)^2|B\kappa(s) - A\tau(s)|(A^2 + B^2)^{\frac{1}{2}}, \\ \det(\dot{\overline{\gamma}}(s), \ddot{\overline{\gamma}}(s), \dddot{\overline{\gamma}}(s)) &= \tau(s)^3(B\kappa(s) - A\tau(s))^2, \end{aligned}$$

we have the curvature and the torsion as

$$\overline{\kappa}(s) = \frac{|\dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)|}{|\dot{\overline{\gamma}}(s)|^3} = \frac{|B\kappa(s) - A\tau(s)|}{(A^2+B^2)|\tau(s)|},$$

$$\overline{\tau}(s) = \frac{\det(\dot{\overline{\gamma}}(s), \ddot{\overline{\gamma}}(s), \dddot{\overline{\gamma}}(s))}{|\dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)|^2} = \frac{1}{(A^2+B^2)\tau(s)}.$$

□

As a corollary of Proposition 2.4, we have a well-known result that $\tau(s)\overline{\tau}(s)$ is a positive constant.

On the other hand, $A\kappa(s) + B\tau(s) = 1$ and $B\kappa(s) - A\tau(s) = 0$ for all $s \in I$ if and only if $\kappa(s) = A/(A^2+B^2)$ and $\tau(s) = B/(A^2+B^2)$. It follows that γ is a helix up to congruence, that is, $\gamma(s)$ is given by

$$\gamma(s) = \left(A\cos\frac{s}{\sqrt{A^2+B^2}}, A\sin\frac{s}{\sqrt{A^2+B^2}}, \frac{Bs}{\sqrt{A^2+B^2}}\right).$$

By a direct calculation, we have $\boldsymbol{n}(s) = (-\cos(s/\sqrt{A^2+B^2}), -\sin(s/\sqrt{A^2+B^2}), 0)$. Hence,

$$\overline{\gamma}(s) = \gamma(s) + \lambda\boldsymbol{n}(s) = \left((A-\lambda)\cos\frac{s}{\sqrt{A^2+B^2}}, (A-\lambda)\sin\frac{s}{\sqrt{A^2+B^2}}, \frac{Bs}{\sqrt{A^2+B^2}}\right),$$

where λ is a constant. If $\lambda = A$, then $\overline{\gamma}(s) = (0, 0, Bs/\sqrt{A^2+B^2})$. Then $\overline{\gamma}$ is degenerate, that is, $\overline{\kappa}(s) = 0$ for all $s \in I$. In this case, if $\lambda \neq A$, then $\overline{\gamma}$ is nondegenerate and γ and $\overline{\gamma}$ are Bertrand mates, since

$$\overline{\boldsymbol{n}}(s) = \mathrm{sgn}(A-\lambda)\left(-\cos\frac{s}{\sqrt{A^2+B^2}}, -\sin\frac{s}{\sqrt{A^2+B^2}}, 0\right),$$

where $\mathrm{sgn}(A-\lambda) = 1$ if $A > \lambda$ and $\mathrm{sgn}(A-\lambda) = -1$ if $A < \lambda$.

2.3. Mannheim curves of regular space curves

Definition 2.5 *Let γ and $\overline{\gamma} : I \to \mathbb{R}^3$ be different nondegenerate curves. We say that γ and $\overline{\gamma}$ are Mannheim mates if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\boldsymbol{n}(t)$ and $\boldsymbol{n}(t) = \pm\overline{\boldsymbol{b}}(t)$ for all $t \in I$.*

We also say that $\gamma : I \to \mathbb{R}^3$ is a Mannheim curve if there exists another nondegenerate curve $\overline{\gamma} : I \to \mathbb{R}^3$ such that γ and $\overline{\gamma}$ are Mannheim mates.

If γ and $\overline{\gamma}$ are Mannheim mates, then the principal normal line of γ and the binormal line of $\overline{\gamma}$ are the same for each points. Note that if we take $-\lambda$ instead of λ, then we may assume that $\boldsymbol{n}(t) = \overline{\boldsymbol{b}}(t)$.

By a parameter change, we may assume that s is the arc-length parameter of γ.

Lemma 2.6 *Let $\gamma : I \to \mathbb{R}^3$ be nondegenerate with the arc-length parameter. Under the notation in Definition 2.5, if γ and $\overline{\gamma}$ are Mannheim mates, then λ is a nonzero constant.*

Proof By differentiating $\overline{\gamma}(s) = \gamma(s) + \lambda(s)\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - \lambda(s)\kappa(s))\boldsymbol{t}(s) + \lambda'(s)\boldsymbol{n}(s) + \lambda(s)\tau(s)\boldsymbol{b}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{b}}(s)$, we have $\lambda'(s) = 0$ for all $s \in I$. Therefore, λ is a constant. If $\lambda = 0$, then $\overline{\gamma}(s) = \gamma(s)$ for all $s \in I$. Hence, λ is a nonzero constant. □

Theorem 2.7 *Let $\gamma : I \to \mathbb{R}^3$ be nondegenerate with the arc-length parameter. Suppose that $\tau(s) \neq 0$ for all $s \in I$ and A is a nonzero constant. Then γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ if and only if $A(\kappa^2(s) + \tau^2(s)) = \kappa(s)$ and $\kappa(s)\tau'(s) - \kappa'(s)\tau(s) > 0$ for all $s \in I$.*

Proof Suppose that $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ and $\boldsymbol{n}(s) = \overline{\boldsymbol{b}}(s)$ for all $s \in I$. Note that s is not the arc-length parameter of $\overline{\gamma}$. By differentiating $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{b}}(s)$, there exists a smooth function $\theta : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\boldsymbol{t}}(s) \\ \overline{\boldsymbol{n}}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \boldsymbol{b}(s) \\ \boldsymbol{t}(s) \end{pmatrix}.$$

Then $|\dot{\overline{\gamma}}(s)|\cos\theta(s) = A\tau(s)$ and $-|\dot{\overline{\gamma}}(s)|\sin\theta(s) = 1 - A\kappa(s)$. It follows that

$$A\tau(s)\sin\theta(s) + (1 - A\kappa(s))\cos\theta(s) = 0. \tag{2.2}$$

By differentiating $\overline{\boldsymbol{t}}(s) = \cos\theta(s)\boldsymbol{b}(s) - \sin\theta(s)\boldsymbol{t}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\overline{\boldsymbol{n}}(s) = -\theta'(s)\sin\theta(s)\boldsymbol{b}(s) - \theta'(s)\cos\theta(s)\boldsymbol{t}(s) - (\cos\theta(s)\tau(s) + \sin\theta(s)\kappa(s))\boldsymbol{n}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{b}}(s)$,

$$\cos\theta(s)\tau(s) + \sin\theta(s)\kappa(s) = 0 \tag{2.3}$$

for all $s \in I$. By $\tau(s) \neq 0$ and $|\dot{\overline{\gamma}}(s)|\cos\theta(s) = A\tau(s)$, we have $\cos\theta(s) \neq 0$. Hence, $\sin\theta(s) \neq 0$. Since $\overline{\boldsymbol{n}}(s) = \sin\theta(s)\boldsymbol{b}(s) + \cos\theta(s)\boldsymbol{t}(s)$, we have $|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s) = -\theta'(s)$. By the equations (2.2) and (2.3), we have

$$A(\kappa^2(s) + \tau^2(s)) = \kappa(s).$$

By differentiating (2.3), we have

$$-\theta'(s)\sin\theta(s)\tau(s) + \cos\theta(s)\tau'(s) + \theta'(s)\cos\theta(s)\kappa(s) + \sin\theta(s)\kappa'(s) = 0.$$

Hence, $\theta'(s) = (-\kappa(s)\tau'(s) + \kappa'(s)\tau(s))/(\kappa^2(s) + \tau^2(s))$. Since $|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s) > 0$, we have $\kappa(s)\tau'(s) - \kappa'(s)\tau(s) > 0$ for all $s \in I$.

Conversely, suppose that $A(\kappa^2(s) + \tau^2(s)) = \kappa(s)$, $\kappa(s)\tau'(s) - \kappa'(s)\tau(s) > 0$, and $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ for all $s \in I$. By differentiating $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$, we have

$$\begin{aligned}
\dot{\overline{\gamma}}(s) &= |\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(t) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s) = A\frac{\tau(s)}{\kappa(s)}(\tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s)), \\
\ddot{\overline{\gamma}}(s) &= \frac{d}{ds}(|\dot{\overline{\gamma}}(s)|)\overline{\boldsymbol{t}}(s) + |\dot{\overline{\gamma}}(s)|^2\overline{\kappa}(s)\overline{\boldsymbol{n}}(s) \\
&= A\left(\frac{\tau(s)}{\kappa(s)}\right)'(\tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s)) + A\frac{\tau(s)}{\kappa(s)}(\tau'(s)\boldsymbol{t}(s) + \kappa'(s)\boldsymbol{b}(s)).
\end{aligned}$$

Therefore, $|\dot{\overline{\gamma}}(s)|^3\overline{\kappa}(s)\overline{\boldsymbol{b}}(s) = A^2(\tau(s)/\kappa(s))^2(\kappa(s)\tau'(s) - \kappa'(s)\tau(s))\boldsymbol{n}(s)$. By the condition, we have $\boldsymbol{n}(s) = \overline{\boldsymbol{b}}(s)$. It follows that γ and $\overline{\gamma}$ are Mannheim mates. □

By a direct calculation and the proof of Theorem 2.7, we have the curvature and the torsion of $\overline{\gamma}$.

Proposition 2.8 *Let γ and $\overline{\gamma}: I \to \mathbb{R}^3$ be different nondegenerate curves. Under the same assumptions in Theorem 2.7, suppose that γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$. Then the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of $\overline{\gamma}$ are given by*

$$\overline{\kappa}(s) = \frac{\kappa(s)(\kappa(s)\tau'(s) - \kappa'(s)\tau(s))}{|A\tau(s)|(\kappa^2(s)+\tau^2(s))^{\frac{3}{2}}}, \quad \overline{\tau}(s) = \frac{\kappa(s)}{A\tau(s)} = \frac{\kappa^2(s)+\tau^2(s)}{\tau(s)}.$$

Proof Since $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$, we have

$$\dot{\overline{\gamma}}(s) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s) = A\frac{\tau(s)}{\kappa(s)}(\tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s)).$$

Therefore,

$$\begin{aligned}
\ddot{\overline{\gamma}}(s) &= A\left(\frac{\tau(s)}{\kappa(s)}\right)'(\tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s)) + A\frac{\tau(s)}{\kappa(s)}(\tau'(s)\boldsymbol{t}(s) + \kappa'(s)\boldsymbol{b}(s)),\\
\dddot{\overline{\gamma}}(s) &= A\left(\frac{\tau(s)}{\kappa(s)}\right)''(\tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s)) + 2A\left(\frac{\tau(s)}{\kappa(s)}\right)'(\tau'(s)\boldsymbol{t}(s) + \kappa'(s)\boldsymbol{b}(s))\\
&\quad + A\frac{\tau(s)}{\kappa(s)}(\tau'(s)\kappa(s) - \kappa'(s)\tau(s))\boldsymbol{n}(s).
\end{aligned}$$

By the proof of Theorem 2.7, we have $\kappa(s)\tau'(s) - \kappa'(s)\tau(s) > 0$ for all $s \in I$. Since

$$\begin{aligned}
|\dot{\overline{\gamma}}(s)| &= \frac{|A\tau(s)|}{\kappa(s)}(\kappa^2(s)+\tau^2(s))^{\frac{1}{2}},\\
|\dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)| &= A^2\left(\frac{\tau(s)}{\kappa(s)}\right)^2(\kappa(s)\tau'(s) - \kappa'(s)\tau(s)),\\
\det(\dot{\overline{\gamma}}(s), \ddot{\overline{\gamma}}(s), \dddot{\overline{\gamma}}(s)) &= A^3\left(\frac{\tau(s)}{\kappa(s)}\right)^3(\kappa(s)\tau'(s) - \kappa'(s)\tau(s))^2,
\end{aligned}$$

we have the curvature and the torsion as

$$\begin{aligned}
\overline{\kappa}(s) &= \frac{|\dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)|}{|\dot{\overline{\gamma}}(s)|^3} = \frac{\kappa(s)(\kappa(s)\tau'(s) - \kappa'(s)\tau(s))}{|A\tau(s)|(\kappa^2(s)+\tau^2(s))^{\frac{3}{2}}},\\
\overline{\tau}(s) &= \frac{\det(\dot{\overline{\gamma}}(s), \ddot{\overline{\gamma}}(s), \dddot{\overline{\gamma}}(s))}{|\dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)|^2} = \frac{\kappa(s)}{A\tau(s)} = \frac{\kappa^2(s)+\tau^2(s)}{\tau(s)}.
\end{aligned}$$

□

Note that $A(\kappa^2(s)+\tau^2(s)) = \kappa(s)$ and $\kappa(s)\tau'(s) - \kappa'(s)\tau(s) = 0$ for all $s \in I$ if and only if there exists a constant B such that $\kappa(s) = 1/(A(1+B^2))$ and $\tau(s) = B/(A(1+B^2))$. It follows that γ is a helix up to congruence, that is, $\gamma(s)$ is given by

$$\gamma(s) = \left(A\cos\frac{s}{A\sqrt{1+B^2}}, A\sin\frac{s}{A\sqrt{1+B^2}}, \frac{Bs}{\sqrt{1+B^2}}\right).$$

By a direct calculation, we have $\boldsymbol{n}(s) = (-\cos(s/A\sqrt{1+B^2}), -\sin(s/A\sqrt{1+B^2}), 0)$. Hence,

$$\overline{\gamma}(s) = \gamma(s) + \lambda \boldsymbol{n}(s) = \left((A-\lambda)\cos\frac{s}{A\sqrt{1+B^2}}, (A-\lambda)\sin\frac{s}{A\sqrt{1+B^2}}, \frac{Bs}{\sqrt{1+B^2}}\right),$$

where λ is a constant. If $\lambda = A$, then $\overline{\gamma}(s) = (0, 0, Bs/\sqrt{1+B^2})$. Then $\overline{\gamma}$ is degenerate, that is, $\overline{\kappa}(s) = 0$ for all $s \in I$. If $\lambda \neq A$, then $\overline{\gamma}$ is nondegenerate. However, in this case, γ and $\overline{\gamma}$ are not Mannheim mates, since

$$\overline{\boldsymbol{b}}(s) = \frac{1}{\sqrt{B^2 + (\frac{A-\lambda}{A})^2}} \left(B\sin\frac{s}{A\sqrt{1+B^2}}, B\cos\frac{s}{A\sqrt{1+B^2}}, \frac{A-\lambda}{A}\right).$$

Remark 2.9 *If there exist nonzero constants A, C and a constant B such that $A\kappa(s) + B\tau(s) = 1$ and $C(\kappa^2(s) + \tau^2(s)) = \kappa(s)$, then $\kappa(s) = (1 - B\tau(s))/A$ and*

$$(A^2 + B^2)C\tau^2(s) + (-2BC + AB)\tau(s) + C - A = 0.$$

If there exists a solution, then $\tau(s)$ and $\kappa(s)$ are constants. Hence, $\kappa(s)\tau'(s) - \kappa'(s)\tau(s) = 0$ for all $s \in I$. It follows that there are no Bertrand and Mannheim curves of regular space curves (cf. Theorem 4.5 and Remark 4.8).

2.4. Framed curves in the 3-dimensional Euclidean space

A framed curve in the 3-dimensional Euclidean space is a smooth space curve with a moving frame, in detail see [12].

Definition 2.10 *We say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve if $\dot{\gamma}(t) \cdot \nu_1(t) = 0$ and $\dot{\gamma}(t) \cdot \nu_2(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^3$ is a framed base curve if there exists $(\nu_1, \nu_2) : I \to \Delta$ such that (γ, ν_1, ν_2) is a framed curve.*

We denote $\boldsymbol{\mu}(t) = \nu_1(t) \times \nu_2(t)$. Then $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$ is a moving frame along the framed base curve $\gamma(t)$ in $\mathbb{R}^3$ and we have the Frenet type formula,

$$\begin{pmatrix} \dot{\nu_1}(t) \\ \dot{\nu_2}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t),$$

where $\ell(t) = \dot{\nu_1}(t) \cdot \nu_2(t)$, $m(t) = \dot{\nu_1}(t) \cdot \boldsymbol{\mu}(t), n(t) = \dot{\nu_2}(t) \cdot \boldsymbol{\mu}(t)$ and $\alpha(t) = \dot{\gamma}(t) \cdot \boldsymbol{\mu}(t)$. We call the mapping (ℓ, m, n, α) the curvature of the framed curve (γ, ν_1, ν_2). Note that t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$.

Definition 2.11 *Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. We say that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are congruent as framed curves if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + \boldsymbol{a}$, $\widetilde{\nu_1}(t) = A(\nu_1(t))$ and $\widetilde{\nu_2}(t) = A(\nu_2(t))$ for all $t \in I$.*

We gave the existence and uniqueness theorems for framed curves in terms of the curvatures in [12], also see [11].

Theorem 2.12 (Existence Theorem for framed curves) *Let* $(\ell, m, n, \alpha) : I \to \mathbb{R}^4$ *be a smooth mapping. Then, there exists a framed curve* $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ *whose curvature is given by* (ℓ, m, n, α).

Theorem 2.13 (Uniqueness Theorem for framed curves) *Let* (γ, ν_1, ν_2) *and* $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ *be framed curves with curvatures* (ℓ, m, n, α) *and* $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$, *respectively. Then* (γ, ν_1, ν_2) *and* $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ *are congruent as framed curves if and only if the curvatures* (ℓ, m, n, α) *and* $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$ *coincide.*

Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). For the normal plane of $\gamma(t)$, spanned by $\nu_1(t)$ and $\nu_2(t)$, there is some ambient of framed curves similarly to the case of the Bishop frame of a regular space curve (cf. [5]). We define $(\widetilde{\nu}_1(t), \widetilde{\nu}_2(t)) \in \Delta_2$ by

$$\begin{pmatrix} \widetilde{\nu}_1(t) \\ \widetilde{\nu}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix},$$

where $\theta(t)$ is a smooth function. Then $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ is also a framed curve and $\widetilde{\boldsymbol{\mu}}(t) = \boldsymbol{\mu}(t)$. By a direct calculation, we have

$$\begin{aligned} \dot{\widetilde{\nu}}_1(t) &= (\ell(t) - \dot{\theta}(t)) \sin\theta(t)\nu_1(t) + (\ell(t) - \dot{\theta}(t)) \cos\theta(t)\nu_2(t) + (m(t)\cos\theta(t) - n(t)\sin\theta(t))\boldsymbol{\mu}(t), \\ \dot{\widetilde{\nu}}_2(t) &= -(\ell(t) - \dot{\theta}(t)) \cos\theta(t)\nu_1(t) + (\ell(t) - \dot{\theta}(t)) \sin\theta(t)\nu_2(t) + (m(t)\sin\theta(t) + n(t)\cos\theta(t))\boldsymbol{\mu}(t). \end{aligned}$$

If we take a smooth function $\theta : I \to \mathbb{R}$ which satisfies $\dot{\theta}(t) = \ell(t)$, then we call the frame $\{\widetilde{\nu}_1(t), \widetilde{\nu}_2(t), \boldsymbol{\mu}(t)\}$ an adapted frame along $\gamma(t)$. It follows that the Frenet–Serret type formula is given by

$$\begin{pmatrix} \dot{\widetilde{\nu}}_1(t) \\ \dot{\widetilde{\nu}}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \widetilde{m}(t) \\ 0 & 0 & \widetilde{n}(t) \\ -\widetilde{m}(t) & -\widetilde{n}(t) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\nu}_1(t) \\ \widetilde{\nu}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{2.4}$$

where $\widetilde{m}(t)$ and $\widetilde{n}(t)$ are given by

$$\begin{pmatrix} \widetilde{m}(t) \\ \widetilde{n}(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} m(t) \\ n(t) \end{pmatrix}.$$

We also consider a special moving frame along a framed base curve under a condition. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with $m^2(t) + n^2(t) \neq 0$. Then we define $(\boldsymbol{n}_1(t), \boldsymbol{n}_2(t)) \in \Delta$ by

$$\boldsymbol{n}_1(t) = \frac{m(t)\nu_1(t) + n(t)\nu_2(t)}{\sqrt{m^2(t) + n^2(t)}}, \ \boldsymbol{n}_2(t) = \frac{-n(t)\nu_1(t) + m(t)\nu_2(t)}{\sqrt{m^2(t) + n^2(t)}}.$$

By a direct calculation, $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed immersion and $\boldsymbol{n}_1(t) \times \boldsymbol{n}_2(t) = \boldsymbol{\mu}(t)$. We call the moving frame $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ a Frenet type frame along $\gamma(t)$. Then the Frenet–Serret type formula is given by

$$\begin{pmatrix} \dot{\boldsymbol{n}}_1(t) \\ \dot{\boldsymbol{n}}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & L(t) & M(t) \\ -L(t) & 0 & 0 \\ -M(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(t) \\ \boldsymbol{n}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t), \tag{2.5}$$

where

$$L(t) = \frac{m(t)\dot{n}(t) - \dot{m}(t)n(t) + \ell(t)(m^2(t) + n^2(t))}{m^2(t) + n^2(t)}, \ M(t) = \sqrt{m^2(t) + n^2(t)}.$$

Therefore, the curvature of the framed immersion $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is given by $(L, M, 0, \alpha)$.

Since the original frame $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$ and the Frenet type frame $\{\boldsymbol{n}_1(t), \ \boldsymbol{n}_2(t), \ \boldsymbol{\mu}(t)\}$ have the common unit vector $\boldsymbol{\mu}(t)$ and the same orientation, the Frenet type frame is one of a rotated frame along $\gamma(t)$.

Let $\gamma : I \to \mathbb{R}^3$ be nondegenerate. If we take $\nu_1(t) = \boldsymbol{n}(t)$ and $\nu_2(t) = \boldsymbol{b}(t)$, then $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve and we have $\boldsymbol{n}_1(t) = -\boldsymbol{n}(t), \boldsymbol{n}_2(t) = -\boldsymbol{b}(t), \boldsymbol{\mu}(t) = \boldsymbol{t}(t)$. This is the reason why we call $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ the Frenet type frame along $\gamma(t)$.

As a special case of a framed curve, let us consider a spherical Legendre curve (see [21] for more detail). We say that $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ is a spherical Legendre curve if $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. We call γ a frontal and ν a dual of γ.

We define $\boldsymbol{\mu}(t) = \gamma(t) \times \nu(t)$. Then $\boldsymbol{\mu}(t) \in S^2$, $\gamma(t) \cdot \boldsymbol{\mu}(t) = 0$ and $\nu(t) \cdot \boldsymbol{\mu}(t) = 0$ for all $t \in I$. It follows that $\{\gamma(t), \nu(t), \boldsymbol{\mu}(t)\}$ is a moving frame along the frontal $\gamma(t)$.

Let $(\gamma, \nu) : I \to \Delta$ be a spherical Legendre curve. Then we have

$$\begin{pmatrix} \dot{\gamma}(t) \\ \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m(t) \\ 0 & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{2.6}$$

where $m(t) = \dot{\gamma}(t) \cdot \boldsymbol{\mu}(t)$ and $n(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$.

We say that the pair of functions (m, n) is the curvature of the spherical Legendre curve $(\gamma, \nu) : I \to \Delta$.

3. Bertrand curves of framed curves

Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves with the curvature (ℓ, m, n, α) and $(\overline{\ell}, \overline{m}, \overline{n}, \overline{\alpha})$, respectively. Suppose that γ and $\overline{\gamma}$ are different curves, that is, $\gamma \not\equiv \overline{\gamma}$.

Definition 3.1 *We say that framed curves (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates (or, $(\nu_1, \overline{\nu}_1)$-mates) if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\nu_1(t)$ and $\nu_1(t) = \overline{\nu}_1(t)$ for all $t \in I$.*

We also say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a *Bertrand curve* if there exists a framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ such that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates.

Lemma 3.2 *Under the notation in Definition 3.1, if (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates, then λ is a nonzero constant.*

Proof By differentiating $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\nu_1(t)$, we have

$$\overline{\alpha}(t)\overline{\boldsymbol{\mu}}(t) = (\alpha(t) + \lambda(t)m(t))\boldsymbol{\mu}(t) + \lambda(t)\ell(t)\nu_2(t) + \dot{\lambda}(t)\nu_1(t)$$

for all $t \in I$. Since $\overline{\nu}_1(t) = \nu_1(t)$, we have $\dot{\lambda}(t) = 0$ for all $t \in I$. Therefore, λ is a constant. If $\lambda = 0$, then $\gamma(t) = \overline{\gamma}(t)$ for all $t \in I$. Hence, λ is a nonzero constant. □

We give a necessary and sufficient condition of a Bertrand curve for a framed curve.

Theorem 3.3 *Let $(\gamma,\nu_1,\nu_2): I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ,m,n,α). Then (γ,ν_1,ν_2) is a Bertrand curve if and only if there exist a nonzero constant λ and a smooth function $\theta: I \to \mathbb{R}$ such that*

$$\lambda\ell(t)\cos\theta(t) - (\alpha(t)+\lambda m(t))\sin\theta(t) = 0 \tag{3.1}$$

for all $t \in I$.

Proof Suppose that (γ,ν_1,ν_2) is a Bertrand curve. By Lemma 3.2, there exist a framed curve $(\overline{\gamma},\overline{\nu}_1,\overline{\nu}_2)$ and a nonzero constant $\lambda \in \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$ and $\nu_1(t) = \overline{\nu}_1(t)$ for all $t \in I$. By differentiating $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$, we have

$$\overline{\alpha}(t)\overline{\boldsymbol{\mu}}(t) = (\alpha(t)+\lambda m(t))\,\boldsymbol{\mu}(t) + \lambda\ell(t)\nu_2(t).$$

Since $\nu_1(t) = \overline{\nu}_1(t)$, there exists a function $\theta: I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\nu}_2(t) \\ \overline{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}. \tag{3.2}$$

Then we have

$$\overline{\alpha}(t)\sin\theta(t) = \lambda\ell(t), \quad \overline{\alpha}(t)\cos\theta(t) = \alpha(t)+\lambda m(t). \tag{3.3}$$

It follows that $\lambda\ell(t)\cos\theta(t) - (\alpha(t)+\lambda m(t))\sin\theta(t) = 0$ for all $t \in I$.

Conversely, suppose that $\lambda\ell(t)\cos\theta(t) - (\alpha(t)+\lambda m(t))\sin\theta(t) = 0$ for all $t \in I$. We define a mapping $(\overline{\gamma},\overline{\nu}_1,\overline{\nu}_2): I \to \mathbb{R}^3 \times \Delta$ by

$$\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t),\ \overline{\nu}_1(t) = \nu_1(t),\ \overline{\nu}_2(t) = \cos\theta(t)\nu_2(t) - \sin\theta(t)\,\boldsymbol{\mu}(t).$$

Then $(\overline{\gamma},\overline{\nu}_1,\overline{\nu}_2)$ is a framed curve. Therefore, (γ,ν_1,ν_2) and $(\overline{\gamma},\overline{\nu}_1,\overline{\nu}_2)$ are Bertrand mates. □

Proposition 3.4 *Suppose that (γ,ν_1,ν_2) and $(\overline{\gamma},\overline{\nu}_1,\overline{\nu}_2)$ are Bertrand mates, where $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$ and $\lambda\ell(t)\cos\theta(t) - (\alpha(t)+\lambda m(t))\sin\theta(t) = 0$ for all $t \in I$. Then the curvature $(\overline{\ell},\overline{m},\overline{n},\overline{\alpha})$ of $(\overline{\gamma},\overline{\nu}_1,\overline{\nu}_2)$ is given by*

$$\begin{aligned} \overline{\ell}(t) &= \ell(t)\cos\theta(t) - m(t)\sin\theta(t), \\ \overline{m}(t) &= \ell(t)\sin\theta(t) + m(t)\cos\theta(t), \\ \overline{n}(t) &= n(t) - \dot{\theta}(t), \\ \overline{\alpha}(t) &= \lambda\ell(t)\sin\theta(t) + (\alpha(t)+\lambda m(t))\cos\theta(t). \end{aligned} \tag{3.4}$$

Proof By the equation (3.2), we have $\overline{\nu}_2(t) = \cos\theta(t)\nu_2(t) - \sin\theta(t)\,\boldsymbol{\mu}(t)$. By differentiating, we have

$$\begin{aligned} -\overline{\ell}(t)\overline{\nu}_1(t) + \overline{n}(t)\overline{\boldsymbol{\mu}}(t) &= (-\ell(t)\cos\theta(t) + m(t)\sin\theta(t))\nu_1(t) \\ &\quad +(-\dot{\theta}(t) + n(t))\sin\theta(t)\nu_2(t) + (n(t) - \dot{\theta}(t))\cos\theta(t)\boldsymbol{\mu}(t). \end{aligned}$$

Since $\nu_1(t) = \overline{\nu}_1(t)$, we have $\overline{\ell}(t) = \ell(t)\cos\theta(t) - m(t)\sin\theta(t)$. Again by (3.2), $\overline{n}(t) = n(t) - \dot{\theta}(t)$. Moreover, by differentiating $\overline{\boldsymbol{\mu}}(t) = \sin\theta(t)\nu_2(t) + \cos\theta(t)\,\boldsymbol{\mu}(t)$, we have

$$\begin{aligned} -\overline{m}(t)\overline{\nu}_1(t) - \overline{n}(t)\overline{\nu}_2(t) &= (-\ell(t)\sin\theta(t) - m(t)\cos\theta(t))\nu_1(t) \\ &\quad +(\dot{\theta}(t) - n(t))\cos\theta(t)\nu_2(t) + (n(t) - \dot{\theta}(t))\sin\theta(t)\boldsymbol{\mu}(t). \end{aligned}$$

Since $\nu_1(t) = \overline{\nu}_1(t)$, we have $\overline{m}(t) = \ell(t)\sin\theta(t) + m(t)\cos\theta(t)$. By the equation (3.3), we also have $\overline{\alpha}(t) = \lambda\ell(t)\sin\theta(t) + (\alpha(t) + \lambda m(t))\cos\theta(t)$. □

By Theorem 3.3, we have the following result.

Corollary 3.5 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α).*

(1) If $\ell(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Bertrand curve.

(2) If there exists a nonzero constant λ such that $\alpha(t) + \lambda m(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Bertrand curve.

Proof (1) If we take $\theta(t) = 0$, then the equation (3.1) is satisfied. (2) If we take $\theta(t) = \pi/2$, then the equation (3.1) is satisfied. □

Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with curvature (ℓ, m, n, α). If we take an adapted frame $\{\widetilde{\nu}_1, \widetilde{\nu}_2, \boldsymbol{\mu}\}$, then the curvature is given by $(0, \widetilde{m}, \widetilde{n}, \alpha)$, see (2.4). By Theorem 3.3 or Corollary 3.5, we have the following.

Corollary 3.6 *For an adapted frame, $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is always a Bertrand curve.*

Remark 3.7 *By Corollary 3.6, a Bertrand curve of a framed base curve is dependent on the moving frame. Even if those are regular space curves, the same phenomenon occurs when we consider the other moving frames (for instance, Bishop frame [5]).*

Next we consider the principal normal direction like as regular cases. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). Since $\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t)$, $\boldsymbol{\mu}(t)$ is the unit tangent direction and

$$
\begin{aligned}
\ddot{\gamma}(t) &= \dot{\alpha}(t)\boldsymbol{\mu}(t) - \alpha(t)m(t)\nu_1(t) - \alpha(t)n(t)\nu_2(t), \\
\dot{\gamma}(t) \times \ddot{\gamma}(t) &= \alpha^2(t)n(t)\nu_1(t) - \alpha^2(t)m(t)\nu_2(t), \\
(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t) &= -\alpha^3(t)m(t)\nu_1(t) - \alpha^3(t)n(t)\nu_2(t),
\end{aligned}
$$

the principal normal direction is given by $\pm\boldsymbol{n}_1(t)$ away from singular points of γ (that is, $\alpha(t) \neq 0$). Hence, we consider Frenet type frame $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{\mu}\}$, see (2.5) in Section 2.4.

Corollary 3.8 *Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. Then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\overline{\gamma}, \overline{\boldsymbol{n}}_1, \overline{\boldsymbol{n}}_2)$ are Bertrand mates if and only if there exists a constant λ such that the equations (3.1) and (3.4) are satisfied, where $\theta(t)$ is a constant.*

Proof Since the curvature $n(t) = \overline{n}(t) = 0$ and Proposition 3.4, $\dot{\theta}(t) = 0$ for all $t \in I$. It follows that θ is a constant. By Theorem 3.3, we have the result. □

Remark 3.9 *When $(m(t), n(t)) = (0, 0)$ at some points, if there exist a nonnegative smooth function $r : I \to \mathbb{R}$ and a smooth function $\varphi : I \to \mathbb{R}$ such that $m(t) = r(t)\cos\varphi(t), n(t) = r(t)\sin\varphi(t)$, then we can consider the same result for Corollary 3.8.*

We give a construction of Bertrand curves of framed curves by using spherical Legendre curves, see (2.6) in Section 2.4. For regular cases see [15].

Theorem 3.10 *Let $(\gamma, \nu) : I \to \Delta$ be a spherical Legendre curve with the curvature (m, n). Suppose that λ, φ are nonzero constants with $\sin\varphi \neq 0$, $\boldsymbol{c} \in \mathbb{R}^3$ is a constant vector. Set*

$$\begin{aligned}\widetilde{\gamma}(t) &= \lambda \int m(t)\gamma(t)dt + \lambda \cot\varphi \int m(t)\nu(t)dt + \boldsymbol{c},\\ \widetilde{\nu}_1(t) &= \boldsymbol{\mu}(t) = \gamma(t) \times \nu(t),\\ \widetilde{\nu}_2(t) &= \cos\varphi\gamma(t) + \sin\varphi\nu(t).\end{aligned}$$

Then $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve with the curvature $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$,

$$\widetilde{\ell}(t) = -\cos\varphi m(t) + \sin\varphi n(t),\ \widetilde{m}(t) = -\sin\varphi m(t) - \cos\varphi n(t),\ \widetilde{n}(t) = 0,\ \widetilde{\alpha}(t) = \frac{\lambda m(t)}{\sin\varphi}.$$

Moreover, $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is a Bertrand curve.

Proof By definition, $|\widetilde{\nu}_1(t)| = |\widetilde{\nu}_2(t)| = 1$ and $\widetilde{\nu}_1(t) \cdot \widetilde{\nu}_2(t) = 0$ for all $t \in I$. By a direct calculation, we have

$$\widetilde{\boldsymbol{\mu}}(t) = \widetilde{\nu}_1(t) \times \widetilde{\nu}_2(t) = \boldsymbol{\mu}(t) \times (\cos\varphi\gamma(t) - \sin\varphi\nu(t)) = \sin\varphi\gamma(t) + \cos\varphi\nu(t).$$

Since $\dot{\widetilde{\gamma}}(t) = \lambda m(t)\gamma(t) + \lambda\cot\varphi m(t)\nu(t) = (\lambda m(t)/\sin\varphi)\widetilde{\boldsymbol{\mu}}(t)$, $\dot{\widetilde{\gamma}}(t) \cdot \widetilde{\nu}_1(t) = \dot{\widetilde{\gamma}}(t) \cdot \widetilde{\nu}_2(t) = 0$ for all $t \in I$. Therefore, $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is a framed curve. By $\dot{\widetilde{\nu}}_1(t) = -m(t)\gamma(t) - n(t)\nu(t)$ and $\dot{\widetilde{\nu}}_2(t) = (\cos\varphi m(t) - \sin\varphi n(t))\boldsymbol{\mu}(t)$, we have the curvature of $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ by

$$\begin{aligned}\widetilde{\ell}(t) &= \dot{\widetilde{\nu}}_1(t) \cdot \widetilde{\nu}_2(t) = -\cos\varphi m(t) + \sin\varphi n(t),\\ \widetilde{m}(t) &= \dot{\widetilde{\nu}}_1(t) \cdot \widetilde{\boldsymbol{\mu}}(t) = -\sin\varphi m(t) - \cos\varphi n(t),\\ \widetilde{n}(t) &= \dot{\widetilde{\nu}}_2(t) \cdot \widetilde{\boldsymbol{\mu}}(t) = 0,\\ \widetilde{\alpha}(t) &= \dot{\widetilde{\gamma}}(t) \cdot \widetilde{\boldsymbol{\mu}}(t) = \frac{\lambda m(t)}{\sin\varphi}.\end{aligned}$$

If we take $\theta(t) = -\varphi$, then we have

$$\begin{aligned}&\lambda\widetilde{\ell}(t)\cos\theta(t) - (\widetilde{\alpha}(t) + \lambda\widetilde{m}(t))\sin\theta(t)\\ &= \lambda(-\cos\varphi m(t) + \sin\varphi n(t))\cos\varphi + \left(\frac{\lambda m(t)}{\sin\varphi} + \lambda(-\sin\varphi m(t) - \cos\varphi n(t))\right)\sin\varphi\\ &= 0\end{aligned}$$

for all $t \in I$. By Theorem 3.3, $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is a Bertrand curve. □

4. Mannheim curves of framed curves

Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves with the curvatures (ℓ, m, n, α) and $(\overline{\ell}, \overline{m}, \overline{n}, \overline{\alpha})$, respectively. Suppose that γ and $\overline{\gamma}$ are different curves, that is, $\gamma \not\equiv \overline{\gamma}$.

Definition 4.1 *We say that framed curves (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates (or, $(\nu_1, \overline{\nu}_2)$-mates) if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\nu_1(t)$ and $\nu_1(t) = \overline{\nu}_2(t)$ for all $t \in I$.*

We also say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a Mannheim curve if there exists a framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ such that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates.

Lemma 4.2 *Under the notation in Definition 4.1, if* (γ, ν_1, ν_2) *and* $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ *are Mannheim mates, then* λ *is a nonzero constant.*

Proof By differentiating $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\nu_1(t)$, we have

$$\overline{\alpha}(t)\overline{\boldsymbol{\mu}}(t) = (\alpha(t) + \lambda(t)m(t))\,\boldsymbol{\mu}(t) + \lambda(t)\ell(t)\nu_2(t) + \dot{\lambda}(t)\nu_1(t)$$

for all $t \in I$. Since $\overline{\nu}_2(t) = \nu_1(t)$, we have $\dot{\lambda}(t) = 0$ for all $t \in I$. Therefore, λ is a constant. If $\lambda = 0$, then $\gamma(t) = \overline{\gamma}(t)$ for all $t \in I$. Hence, λ is a nonzero constant. □

We give a necessary and sufficient condition of a Mannheim curve for a framed curve.

Theorem 4.3 *Let* $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ *be a framed curve with the curvature* (ℓ, m, n, α). *Then* (γ, ν_1, ν_2) *is a Mannheim curve if and only if there exist a nonzero constant* λ *and a smooth function* $\varphi : I \to \mathbb{R}$ *such that*

$$\lambda\ell(t)\sin\varphi(t) + (\alpha(t) + \lambda m(t))\cos\varphi(t) = 0 \tag{4.1}$$

for all $t \in I$.

Proof Suppose that (γ, ν_1, ν_2) is a Mannheim curve. By Lemma 4.2, there exist a framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ and a nonzero constant $\lambda \in \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$ and $\nu_1(t) = \overline{\nu}_2(t)$ for all $t \in I$. Then we have $\overline{\alpha}(t)\overline{\boldsymbol{\mu}}(t) = (\alpha(t) + \lambda m(t))\,\boldsymbol{\mu}(t) + \lambda\ell(t)\nu_2(t)$. Since $\nu_1(t) = \overline{\nu}_2(t)$, there exists a function $\varphi : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\boldsymbol{\mu}}(t) \\ \overline{\nu}_1(t) \end{pmatrix} = \begin{pmatrix} \cos\varphi(t) & -\sin\varphi(t) \\ \sin\varphi(t) & \cos\varphi(t) \end{pmatrix} \begin{pmatrix} \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}. \tag{4.2}$$

Then we have

$$\overline{\alpha}(t)\cos\varphi(t) = \lambda\ell(t), \quad -\overline{\alpha}(t)\sin\varphi(t) = \alpha(t) + \lambda m(t). \tag{4.3}$$

It follows that $\lambda\ell(t)\sin\varphi(t) + (\alpha(t) + \lambda m(t))\cos\varphi(t) = 0$ for all $t \in I$.

Conversely, suppose that $\lambda\ell(t)\sin\varphi(t) + (\alpha(t) + \lambda m(t))\cos\varphi(t) = 0$ for all $t \in I$. We define a mapping $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ by

$$\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t),\ \overline{\nu}_1(t) = \sin\varphi(t)\nu_2(t) + \cos\varphi(t)\,\boldsymbol{\mu}(t),\ \overline{\nu}_2(t) = \nu_1(t).$$

Then $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ is a framed curve. Therefore, (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates. □

Proposition 4.4 *Suppose that* (γ, ν_1, ν_2) *and* $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ *are Mannheim mates, where* $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$ *and* $\lambda\ell(t)\sin\varphi(t) + (\alpha(t) + \lambda m(t))\cos\varphi(t) = 0$ *for all* $t \in I$. *Then the curvature* $(\overline{\ell}, \overline{m}, \overline{n}, \overline{\alpha})$ *of* $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ *is given by*

$$\begin{aligned} \overline{\ell}(t) &= -\ell(t)\sin\varphi(t) - m(t)\cos\varphi(t), \\ \overline{m}(t) &= \dot{\varphi}(t) - n(t), \\ \overline{n}(t) &= \ell(t)\cos\varphi(t) - m(t)\sin\varphi(t), \\ \overline{\alpha}(t) &= \lambda\ell(t)\cos\varphi(t) - (\alpha(t) + \lambda m(t))\sin\varphi(t). \end{aligned} \tag{4.4}$$

Proof By the equation (4.2), we have $\overline{\boldsymbol{\mu}}(t) = \cos\theta(t)\nu_2(t) - \sin\theta(t)\boldsymbol{\mu}(t)$. By differentiating, we have

$$\begin{aligned} -\overline{m}(t)\overline{\nu}_1(t) + \overline{n}(t)\overline{\nu}_2(t) &= (-\ell(t)\cos\varphi(t) + m(t)\sin\varphi(t))\nu_1(t) \\ &\quad +(-\dot{\varphi}(t) + n(t))\sin\varphi(t)\nu_2(t) + (n(t) - \dot{\varphi}(t))\cos\varphi(t)\boldsymbol{\mu}(t). \end{aligned}$$

Since $\nu_1(t) = \overline{\nu}_2(t)$, we have $\overline{n}(t) = \ell(t)\cos\varphi(t) - m(t)\sin\varphi(t)$. Again by (4.2), $\overline{m}(t) = \dot{\varphi}(t) - n(t)$. Moreover, by differentiating $\overline{\nu}_1(t) = \sin\varphi(t)\nu_2(t) + \cos\varphi(t)\boldsymbol{\mu}(t)$, we have

$$\begin{aligned} \overline{\ell}(t)\overline{\nu}_2(t) + \overline{m}(t)\overline{\boldsymbol{\mu}}(t) &= (-\ell(t)\sin\varphi(t) - m(t)\cos\varphi(t))\nu_1(t) \\ &\quad +(\dot{\varphi}(t) - n(t))\cos\varphi(t)\nu_2(t) + (n(t) - \dot{\varphi}(t))\sin\varphi(t)\boldsymbol{\mu}(t). \end{aligned}$$

Since $\nu_1(t) = \overline{\nu}_2(t)$, we have $\overline{\ell}(t) = -\ell(t)\sin\varphi(t) - m(t)\cos\varphi(t)$. By the equation (4.3), we have $\overline{\alpha}(t) = \lambda\ell(t)\cos\varphi(t) + (\alpha(t) + \lambda m(t))\sin\varphi(t)$. □

As a difference between nondegenerate regular space curves and framed curves, we have a relation between Bertrand and Mannheim curves of framed curves.

Theorem 4.5 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). Then (γ, ν_1, ν_2) is a Bertrand curve if and only if (γ, ν_1, ν_2) is a Mannheim curve.*

Proof Suppose that (γ, ν_1, ν_2) is a Bertrand curve. By Theorem 3.3, there exist a nonzero constant λ and a smooth function $\theta : I \to \mathbb{R}$ such that $\lambda\ell(t)\cos\theta(t) - (\alpha(t) + \lambda m(t))\sin\theta(t) = 0$ for all $t \in I$. If $\varphi(t) = \theta(t) + \pi/2$, then we have $\lambda\ell(t)\sin\varphi(t) + (\alpha(t) + \lambda m(t))\cos\varphi(t) = 0$ for all $t \in I$. By Theorem 4.3, (γ, ν_1, ν_2) is a Mannheim curve.

Conversely, suppose that (γ, ν_1, ν_2) is a Mannheim curve. By Theorem 4.3, there exist a nonzero constant λ and a smooth function $\varphi : I \to \mathbb{R}$ such that $\lambda\ell(t)\sin\varphi(t) + (\alpha(t) + \lambda m(t))\cos\varphi(t) = 0$ for all $t \in I$. If $\theta(t) = \varphi(t) - \pi/2$, then we have $\lambda\ell(t)\cos\theta(t) - (\alpha(t) + \lambda m(t))\sin\theta(t) = 0$ for all $t \in I$. By Theorem 3.3, (γ, ν_1, ν_2) is a Bertrand curve. □

If (γ, ν_1, ν_2) is a framed curve, then (γ, ν_2, ν_1) is also a framed curve. Since we can choose a moving frame, the above result can be proved by the definitions of Bertrand and Mannheim curves of framed curves. On the other hand, for regular cases, the moving frame is fixed. Therefore, the above result does not hold (cf. Remark 2.9).

By Theorem 4.3, we have the following result.

Corollary 4.6 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α).*

(1) *If $\ell(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Mannheim curve.*

(2) *If there exists a nonzero constant λ such that $\alpha(t) + \lambda m(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Mannheim curve.*

Proof (1) If we take $\varphi(t) = \pi/2$, then the equation (4.1) is satisfied. (2) If we take $\theta(t) = 0$, then the equation (4.1) is satisfied. □

If we take an adapted frame $\{\widetilde{\nu}_1, \widetilde{\nu}_2, \boldsymbol{\mu}\}$, then the curvature is given by $(0, \widetilde{m}, \widetilde{n}, \alpha)$, see (2.4). By Theorem 4.3 or Corollary 4.6, we have the following.

Corollary 4.7 *For an adapted frame, $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is always a Mannheim curve.*

Remark 4.8 *By Corollary 4.7, a Mannheim curve of a framed base curve is dependent on the moving frame. By Corollaries 3.6 and 4.7,* $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ *is not only a Bertrand curve but also a Mannheim curve (cf. Remark 2.9).*

Next we consider the principal normal direction like as regular cases, see (2.5) and Corollary 3.8.

Corollary 4.9 *Let* (γ, ν_1, ν_2) *and* $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ *be framed curves. Then* $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ *and* $(\overline{\gamma}, \overline{\boldsymbol{n}}_1, \overline{\boldsymbol{n}}_2)$ *are Mannheim mates if and only if there exist a constant* λ *and a smooth function* $\varphi : I \to \mathbb{R}$ *such that*

$$\begin{aligned}
&\lambda L(t) \sin\varphi(t) + (\alpha(t) + \lambda M(t)) \cos\varphi(t) = 0, \\
&L(t) \cos\varphi(t) - M(t) \sin\varphi(t) = 0, \\
&\overline{L}(t) = -L(t) \sin\varphi(t) - M(t) \cos\varphi(t), \\
&\overline{M}(t) = \dot{\varphi}(t), \\
&\overline{\alpha}(t) = \lambda L(t) \cos\varphi(t) - (\alpha(t) + \lambda M(t)) \sin\varphi(t)
\end{aligned}$$

for all $t \in I$.

Proof Since the curvature $n(t) = \overline{n}(t) = 0$ and Proposition 4.4, $\overline{M}(t) = \dot{\varphi}(t)$ and $L(t)\cos\varphi(t) - M(t)\sin\varphi(t) = 0$ for all $t \in I$. By Theorem 4.3, we have the result. □

However, if we consider $(\overline{\gamma}, \overline{\boldsymbol{n}}_2, \overline{\boldsymbol{n}}_1)$, we have the following result.

Corollary 4.10 *Let* (γ, ν_1, ν_2) *and* $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ *be framed curves. Then* $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ *and* $(\overline{\gamma}, \overline{\boldsymbol{n}}_2, \overline{\boldsymbol{n}}_1)$ *are Mannheim mates if and only if there exists a constant* λ *such that the equations (4.1) and (4.4) are satisfied, where* $\varphi(t)$ *is a constant.*

Proof Since the curvature $n(t) = \overline{m}(t) = 0$ and Proposition 4.4, $\dot{\varphi}(t) = 0$. It follows that φ is a constant. By Theorem 4.3, we have the result. □

By Theorem 3.10 and Corollary 4.10, we can construct Mannheim curves of framed curves by using spherical Legendre curves.

As related topics, we investigate singularities of parallel curves (circler evolutes) in [Honda S, Takahashi M. Evolutes and involutes of Bishop directions of framed curves in the Euclidean space. In preparation]. Also, the definition and properties of evolutes (spherical evolutes) of framed immersions are given in [13]. Furthermore, we consider higher dimensional cases of Bertrand and Mannheim curves of framed curves in [Honda S, Takahashi M. Bertrand and Mannheim curves of framed curves in higher dimensional Euclidean space. In preparation].

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