Inverse problems for differential operators with two delays larger than half the length of the interval and Dirichlet conditions

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Abstract: This paper deals with nonself-adjoint second-order differential operators with two constant delays from $\left[\frac{\pi}{2}, \pi\right)$ and two real-valued potentials from $L_2[0, \pi]$. An inverse spectral problem of recovering operators from the spectra of four boundary value problems is studied.

Key words: Operators with delays, inverse problems, Fourier coefficients

1. Introduction

Spectral problems can be regarded as splitting into two branches, direct and inverse. The direct one includes investigating the properties of eigenvalues and eigenfunctions. The inverse problem, on the other hand, includes recovering the differential equation and the boundary conditions of the problem from the knowledge of some suitable data.

The theory of differential equations with delays is a rapidly developing branch of the theory of ordinary differential equations and it is supported by solid mathematical literature (see [4] and the references therein).

For a number of results revealing inverse spectral problems for classical Sturm-Liouville operators we refer the reader to [2], while some aspects of the direct and inverse problems for operators with *one delay* can be found in [3, 6, 7, 9]. In [3] nonself-adjoint second-order differential operator with a constant delay is studied. The properties of the spectral characteristics are established and the inverse problem of recovering operators from their spectra is investigated. The uniqueness theorem for this inverse problem is proved. In [6] the Borg-type inverse problem for the Sturm-Liouville equation with one delay is considered. It is shown that the asymptotic behavior of the eigenvalues determines the delay in the differential equation. The asymptotic behavior of the characteristic function in the differential equation and the coefficients in the boundary conditions from two sets of eigenvalues is explained. Necessary conditions for uniqueness of the solution of this reconstruction are given. In [7] the inverse problem of the boundary spectral assignment of the Sturm-Liouville operator with one delay is studied. It is argued that if the sequence of the eigenvalues is given, then the delay factor and the potential function in the differential equation are unambiguous. The potential function is composed by means of the trigonometric Fourier coefficients. In [9] an inverse problem for the nonself-adjoint second-order differential

operator with constant delay is studied. The delay and the potential function are uniquely determined from two spectra and reconstruction procedures of determining the delay and the potential function are given.

Although, there are a number of results about both direct and inverse problems for operators with *one delay*, there are just a few results related to the operators with *two or more delays* [5, 8, 10]. In [5] a boundary value problem consisting of a differential equation with two delays is observed. The characteristic function and the asymptotics of the eigenvalues for this boundary value problem are constructed. The first regularized trace is calculated. In [8] nonself-adjoint second-order differential operators with two constant delays are studied. The properties of the spectral characteristics and the inverse problem of recovering operators from their spectra are investigated. An inverse spectral problem of recovering the potential from the spectra of two boundary value problems with one common boundary condition is examined and the uniqueness theorem for this inverse problem is proved. In [10] the boundary value problem for the operator of Sturm-Liouville type with N constant delays is studied. The solution of the boundary value problem is constructed by the method of successive approximation. The characteristic function of this boundary value problem is determined and asymptotic behavior of the eigenvalues is investigated. In what follows, we always take $k = 0, 1$ and $i = 1, 2$.

We consider the boundary value problems $D_{1,k}$

$$-y''(x) + q_1(x) y(x - \tau_1) + q_2(x) y(x - \tau_2) = \lambda y(x), \ x \in [0, \pi], \tag{1.1}$$

$$y(0) = y^{(k)}(\pi) = 0,$$

and the boundary value problems $D_{2,k}$ with a minus sign in front of q_2 in (1.1), under the same boundary conditions. We assume that λ is a spectral parameter, $\frac{\pi}{2} \leq \tau_2 < \tau_1 < \pi$ and $q_i(x)$ are real-valued functions such that $q_i \in L_2[\tau_i, \pi]$ and $q_i(x) = 0$ as $x \in [0, \tau_i)$. It is known that the spectrum of $D_{i,k}$ is countable. We prove that delays τ_i and potential functions q_i are uniquely determined from the spectra of $D_{i,k}$.

To emphasize the distinctness of our paper from the abovementioned papers [5, 8–10], it is worth noting that although the problem that [5] investigates has two and the problem that [10] investigates has N delays in the differential equation, they both study the direct spectral problem for the operators.

If we compare our paper with [9], we can easily notice that the results in our paper are the generalization of the results in [9]. Indeed, if we take $q_2(x) = 0$ in our paper, we get the results of [9]. In the same way, a generalization of the results for the operators with one delay less than $\frac{\pi}{2}$ to the operators with two or more delays, can be obtained. Accordingly, there is a number of opportunities for the generalization of the existing results for the operators with one delay to the operators with two and more delays which can be regarded as open problems in this area. Furthermore, in [8] the author studies an inverse problem of recovering the potential functions from the spectra of two boundary value problems with one common boundary condition: $y(0) = y^{(k)}(\pi) = 0, \ k = 0, 1$. The uniqueness theorem for this inverse problem is proved: if the eigenvalues of these boundary value problems are the same with the eigenvalues of the boundary value problems with zero potential functions, then $q_1(x) = q_2(x) = 0$.

In our paper, we investigate an inverse spectral problem of recovering operators from the spectra of four boundary value problems. We show that the potential functions q_1 and q_2 are uniquely determined by the eigenvalues and then we construct them, while the uniqueness theorem in [8] was proved only according to the zero potential functions. Also, the method which we use to solve the inverse problem, completely differs from the one that has been applied to solve the inverse problem in [8]. In order to solve the inverse problem, we

transform the characteristic functions and for this purpose we define transitional functions. Then we show that the Fourier coefficients of transitional functions are uniquely determined by the eigenvalues which allows us to construct the potential functions.

The structure of the paper is as follows. In Section 2, we investigate the spectral properties of the boundary value problems $D_{i,k}$. In Section 3, by the method of Fourier coefficients (see [6]), we prove that delays and potential functions are uniquely determined from the spectra and we construct the potential functions.

2. Spectral properties

Let $S(x, \lambda)$ be the solution of the equation (1.1) with the initial conditions $S(0, \lambda) = 0$, $S'(0, \lambda) = 1$. It can be easily shown that the function $S(x, \lambda)$ is the unique solution of the integral equation

$$S(x, z) = \frac{\sin xz}{z} + \sum_{m=1}^{2} \int_{\tau_m}^{x} q_m(t) \frac{\sin z(x - t)}{z} S(t - \tau_m, z)dt \tag{2.1}$$

where $\lambda := z^2$. Assuming that $q_i(x) = 0$ as $x \in [0, \tau_i)$, by the method of steps, it can be easily verified that the integral equation (2.1) has the solution

$$S(x, z) = \frac{\sin xz}{z} + \frac{1}{z^2} \sum_{m=1}^{2} \int_{\tau_m}^{x} q_m(t) \sin z(x - t) \sin z(x - \tau_m)dt \tag{2.2}$$

on $(\tau_1, \pi]$. Now, by denoting $\Delta_{1,k}(\lambda) = F_k(z) = S^{(k)}(\pi, z)$ and using (2.2) we obtain

$$\Delta_{1,0}(\lambda) = F_0(z) = \frac{\sin \pi z}{z} + \frac{1}{z^2} \left(b_{s^2}^{(1)}(z) + b_{s^2}^{(2)}(z) \right), \tag{2.3}$$

$$\Delta_{1,1}(\lambda) = F_1(z) = \frac{\cos \pi z}{z} + \frac{1}{z} \left(b_{cs^2}^{(1)}(z) + b_{cs^2}^{(2)}(z) \right), \tag{2.4}$$

where

$$b_{s^2}^{(i)}(z) = \int_{\tau_i}^{\pi} q_i(t) \sin z(\pi - t) \sin z(t - \tau_i)dt,$$

$$b_{cs}^{(i)}(z) = \int_{\tau_i}^{\pi} q_i(t) \cos z(\pi - t) \sin z(t - \tau_i)dt.$$

The functions $\Delta_{1,k}(\lambda)$ are entire in λ of order $\frac{1}{2}$. The zeros of $\Delta_{1,k}(\lambda)$ coincide with the eigenvalues of $D_{1,k}$. Therefore, the functions $\Delta_{1,k}(\lambda)$ are the characteristic functions of the boundary value problems $D_{1,k}$.

In order to solve the inverse problem, we should transform the characteristic functions and for this purpose we define the transitional function $\tilde{q}$ as follows

$$\tilde{q}(t) = \begin{cases} q_1 \left(t + \frac{\tau_1}{2} \right) + q_2 \left(t + \frac{\tau_2}{2} \right), & t \in \left[\frac{\tau_1}{2}, \pi - \frac{\tau_1}{2} \right], \\ q_1 \left(t + \frac{\tau_1}{2} \right), & t \in \left[\frac{\tau_2}{2}, \frac{\tau_1}{2} \right) \cup \left(\pi - \frac{\tau_1}{2}, \pi - \frac{\tau_2}{2} \right], \\ 0, & t \in \left[0, \frac{\tau_2}{2} \right) \cup \left(\pi - \frac{\tau_2}{2}, \pi \right]. \end{cases} \tag{2.5}$$

Denote $J_1^{(i)} := \int_{\tau_i}^{\pi} q_i(t)dt$ and

$$\tilde{a}_c(z) = \int_0^{\pi} \tilde{q}(t) \cos z(\pi - 2t)dt, \quad \tilde{a}_s(z) = \int_0^{\pi} \tilde{q}(t) \sin z(\pi - 2t)dt.$$

Then the following relations can be easily obtained

$$b_{s^2}^{(1)}(z) + b_{s^2}^{(2)}(z) = \frac{1}{2}\left(\tilde{a}_c(z) - J_1^{(1)}\cos z(\pi - \tau_1) - J_1^{(2)}\cos z(\pi - \tau_2)\right), \tag{2.6}$$

$$b_{cs}^{(1)}(z) + b_{cs}^{(2)}(z) = \frac{1}{2}\left(-\tilde{a}_c(z) - J_1^{(1)}\sin z(\pi - \tau_1) - J_1^{(2)}\sin z(\pi - \tau_2)\right). \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.3) and (2.4) respectively, we can rewrite the characteristic functions as follows:

$$\Delta_{1,0}(\lambda) = F_0(z) = \frac{\sin z}{z} + \frac{1}{2z^2}\left(\tilde{a}_c(z) - J_1^{(1)}\cos z(\pi - \tau_1) - J_1^{(2)}\cos(\pi - \tau_2)\right), \tag{2.8}$$

$$\Delta_{1,1}(\lambda) = F_1(z) = \cos z + \frac{1}{2z}\left(-\tilde{a}_s(z) + J_1^{(1)}\sin z(\pi - \tau_1) + J_1^{(2)}\sin(\pi - \tau_2)\right). \tag{2.9}$$

Using (2.8) and (2.9) in the well-known method (see [2]), we obtain the asymptotic formulas for the eigenvalues $\lambda_{n,k} := z_{n,k}^2$ of $D_{1,k}$ as:

$$\begin{cases} \lambda_{n,0} = n^2 + \frac{J_1^{(1)}}{\pi}\cos n\tau_1 + \frac{J_1^{(2)}}{\pi}\cos n\tau_2 + o(1), \\ \lambda_{n,1} = \left(n - \frac{1}{2}\right)^2 + \frac{J_1^{(1)}}{\pi}\cos\left(n - \frac{1}{2}\right)\tau_1 + \frac{J_1^{(2)}}{\pi}\cos\left(n - \frac{1}{2}\right)\tau_2 + o(1). \end{cases} \tag{2.10}$$

Now, let us follow the same procedure for the boundary value problems $D_{2,k}$ and define the transitional function $\tilde{Q}(t)$ which differs from the transitional function $\tilde{q}(t)$ in (2.5) only with the minus sign in front of q_2.

Introducing the following notations

$$\tilde{A}_c(z) = \int_0^\pi \tilde{Q}(t)\cos z(\pi - 2t)dt, \ \ \tilde{A}_s(z) = \int_0^\pi \tilde{Q}(t)\sin z(\pi - 2t)dt$$

allows us to show that the characteristic functions for the boundary value problems $D_{2,k}$ are in the form of

$$\Delta_{2,0}(\lambda) = G_0(z) = \frac{\sin \pi z}{z} + \frac{1}{2z^2}\left(\tilde{A}_c(z) - J_1^{(1)}\cos z(\pi - \tau_1) + J_1^{(2)}\cos z(\pi - \tau_2)\right),$$

$$\Delta_{2,1}(\lambda) = G_1(z) = \cos z + \frac{1}{2z}\left(-\tilde{A}_s(z) + J_1^{(1)}\sin z(\pi - \tau_1) - J_1^{(2)}\sin z(\pi - \tau_2)\right).$$

Thus, the asymptotic formulas for the eigenvalues $\{\mu_{n,k}\}_{n=1}^\infty$ of $D_{2,k}$ can be given as

$$\begin{cases} \mu_{n,0} = n^2 + \frac{J_1^{(1)}}{\pi}\cos n\tau_1 - \frac{J_1^{(2)}}{\pi}\cos n\tau_2 + o(1), \\ \mu_{n,1} = \left(n - \frac{1}{2}\right)^2 + \frac{J_1^{(1)}}{\pi}\cos\left(n - \frac{1}{2}\right)\tau_1 - \frac{J_1^{(2)}}{\pi}\cos\left(n - \frac{1}{2}\right)\tau_2 + o(1). \end{cases} \tag{2.11}$$

Now, by Hadamard's factorization theorem (see [1]) we can construct the characteristic functions $\Delta_{i,k}(\lambda)$ from the spectra of $D_{i,k}$. Thus, the following lemma is proved.

Lemma 2.1 *The specification of the spectra $\{\{\lambda_{n,k}\}, \{\mu_{n,k}\}\}_{n=1}^\infty$ uniquely determines the characteristic functions $\Delta_{i,k}(\lambda)$ by the formulas*

$$\begin{cases} \Delta_{1,0}(\lambda) = \pi\prod_{n=1}^\infty \frac{\lambda_{n,0}-\lambda}{n^2}, \quad \Delta_{1,1}(\lambda) = \prod_{n=1}^\infty \frac{\lambda_{n,1}-\lambda}{\left(n-\frac{1}{2}\right)^2}, \\ \Delta_{2,0}(\lambda) = \pi\prod_{n=1}^\infty \frac{\mu_{n,0}-\lambda}{n^2}, \quad \Delta_{2,1}(\lambda) = \prod_{n=1}^\infty \frac{\mu_{n,1}-\lambda}{\left(n-\frac{1}{2}\right)^2} \end{cases} \tag{2.12}$$

respectively.

3. Main results

This section is devoted to show that the delays are uniquely determined from the spectra of the boundary value problems $D_{i,0}$.

Lemma 3.1 *Delays τ_i and integrals $J_1^{(i)}$ are uniquely determined by the eigenvalues $\{\lambda_{n,0}\}_{n=1}^{\infty}$ and $\{\mu_{n,0}\}_{n=1}^{\infty}$ of the boundary value problems $D_{i,0}$.*

Proof Let us consider the sequences $\{\rho_n\}_{n=1}^{\infty} = \frac{1}{2}\{\lambda_{n,0} + \mu_{n,0}\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty} = \frac{1}{2}\{\lambda_{n,0} - \mu_{n,0}\}_{n=1}^{\infty}$. From (2.10) and (2.11), we obtain the asymptotic formulas for these sequences as

$$\rho_n = n^2 + \frac{J_1^{(1)}}{\pi}\cos n\tau_1 + o(1),$$

$$\sigma_n = \frac{J_1^{(2)}}{\pi}\cos n\tau_2 + o(1).$$

Since, each of the sequences $\{\rho_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty}$ contains only one delay, we determine τ_2 and $J_1^{(2)}$ from $\{\sigma_n\}_{n=1}^{\infty}$, and τ_1 and $J_1^{(1)}$ from $\{\rho_n\}_{n=1}^{\infty}$ in the same way as in the operators with one delay (see [9]). □

Denote $\tilde{a}_{2n} = \int_0^{\pi} \tilde{q}(t)\cos 2nt\, dt$, $\tilde{b}_{2n} = \int_0^{\pi} \tilde{q}(t)\sin 2nt\, dt$.

Theorem 3.2 *The coefficients $\tilde{a}_{2n}$ and $\tilde{b}_{2n}$, $n \in \mathbf{N}$ are uniquely determined by the eigenvalues $\{\lambda_{n,k}\}_{n=1}^{\infty}$.*

Proof Substituting $\lambda = n^2$, $n \in \mathbf{N}$ into (2.12) and taking (2.8) and (2.9) into account, we obtain

$$F_0(n) = \frac{(-1)^n}{2n^2}\left(\tilde{a}_{2n} - J_1^{(1)}\cos n\tau_1 - J_1^{(2)}\cos n\tau_2\right), \tag{3.1}$$

$$F_1(n) = \frac{(-1)^n}{2n}\left(2n + \tilde{b}_{2n} - J_1^{(1)}\sin n\tau_1 - J_1^{(2)}\sin n\tau_2\right). \tag{3.2}$$

From (3.1) and (3.2) we get, $\tilde{a}_{2n} = 2n^2(-1)^n F_0(n) + J_1^{(1)}\cos n\tau_1 + J_1^{(2)}\cos n\tau_2$, $\tilde{b}_{2n} = 2n^2(-1)^n F_1(n) + J_1^{(1)}\sin n\tau_1 + J_1^{(2)}\sin n\tau_2$ and this proves the theorem. □

Theorem 3.3 *The potential functions q_i are uniquely determined by the eigenvalues $\{\lambda_{n,k}\}_{n=1}^{\infty}$ and $\{\mu_{n,k}\}_{n=1}^{\infty}$.*

Proof By virtue of Theorem 3.2, the coefficients $\tilde{a}_{2n}$ and $\tilde{b}_{2n}$ $(n \in \mathbf{N})$ are uniquely determined by $\{\lambda_{n,k}\}_{n=1}^{\infty}$. Since, $\tilde{a}_0 = J_1^{(1)} + J_1^{(2)}$, with the help of Lemma 3.1, the coefficient $\tilde{a}_0$ is also uniquely determined from the spectra. So the uniqueness of the transitional function is proved. Thus, we construct

$$\tilde{q}(t) = \frac{\tilde{a}_0}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\tilde{a}_{2n}\cos 2nt + \tilde{b}_{2n}\sin 2nt, \quad t \in \left[\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}\right]. \tag{3.3}$$

In the same way, we show that coefficients $\tilde{A}_{2n} = \int_0^{\pi}\tilde{Q}(t)\cos 2nt\, dt$, $\tilde{B}_{2n} = \int_0^{\pi}\tilde{Q}(t)\sin 2nt\, dt$ $(n \in \mathbf{N})$ and $\tilde{A}_0 = J_1^{(1)} - J_1^{(2)}$ are uniquely determined by $\{\mu_{n,k}\}_{n=1}^{\infty}$. Then, we obtain

$$\tilde{Q}(t) = \frac{\tilde{A}_0}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\tilde{A}_{2n}\cos 2nt + \tilde{B}_{2n}\sin 2nt, \quad t \in \left[\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}\right]. \tag{3.4}$$

Now, using the definition of the transitional functions, from (3.3) and (3.4), we obtain that the potentials q_1 and q_2 are uniquely determined by

$$q_1(x) = \frac{1}{2}\left(\tilde{q}\left(x - \frac{\tau_1}{2}\right) + \tilde{Q}\left(x - \frac{\tau_1}{2}\right)\right),\ x \in [\tau_1, \pi],$$

$$q_2(x) = \frac{1}{2}\left(\tilde{q}\left(x - \frac{\tau_2}{2}\right) - \tilde{Q}\left(x - \frac{\tau_2}{2}\right)\right),\ x \in [\tau_2, \pi].$$

Thus, proving the theorem. □

4. Conclusion

In this paper, we investigate nonself-adjoint second-order differential operators with two constant delays from $\left[\frac{\pi}{2}, \pi\right)$ and two real-valued potentials from $L_2[0, \pi]$. We investigate the spectral properties of four boundary value problems. We prove that delays and potential functions are uniquely determined from the spectra and we construct the potential functions.

As we mentioned before, presented results of this paper are a generalization of the results to the operators with one delay. Using the method presented in this paper, a generalization of the results for the operators with one delay less than $\frac{\pi}{2}$ to the operators with two or more delays, can be obtained. Accordingly, there is a number of opportunities for the generalization of the existing results for the operators with one delay to the operators with two and more delays. which can be regarded as open problems in this area. This paper not only proves the uniqueness of the potential functions, but also provides an algorithm for their construction, which is a completely new result for the operators with two delays.

References

[1] Conway JB. Function of One Complex Variable, Second Edition. New York, NY, USA: Springer-Verlag, 1995.

[2] Freiling G, Yurko VA. Inverse Sturm-Liouville Problems and Their Applications. New York, NY, USA: NOVA Science Publishers, 2001.

[3] Freiling G, Yurko VA. Inverse Sturm-Liouville differential operators with a constant delay. Applied Mathematics Letters 2012; 25 (11): 1999-2004. doi:10.16/j.aml.2012.03.026

[4] Norkin SB. Second Order Differential Equations with a Delay Argument. Moscow, Russia: Nauka, 1965.

[5] Pavlovic N, Pikula M, Vojvodic B. First regularized trace formula of the limit assignment of Sturm-Liouville type with two constant delays. Filomat 2015; 29 (1): 51-62. doi:10.2298/FIL1501063S

[6] Pikula M. Determination of a Sturm-Liouville type differential operator with delay argument from two spectra. Matematicki Vesnik 1991; 43 (3-4): 159-171.

[7] Pikula M, Vladicic V, Markovic O. A solution to the inverse problem for Sturm-Liouville type equation with a delay. Filomat 2013; 27 (7): 237-1245. doi:10.2298/FIL130727P

[8] Shahriari M. Inverse problem for Sturm-Liouville differential operators with two constant delays. Turkish Journal of Mathematics 2019; 43 (2): 965-976. doi:10.3906/mat-1811-113

[9] Vladicic V, Pikula M. An inverse problem for Sturm-Liouville type differential equation with a constant delay. Sarajevo Journal of Mathematics 2016; 12 (24-1): 83-88. doi:10.5644/SJM.12.1.06

[10] Vojvodic B, Pikula M. Boundary value problem of differential operator of the Sturm-Liouville with n constant delays and asymptotic of the eigenvalues. Mathematica Montisnigri 2016; XXXV: 5-21.