Solution of nonlinear ordinary differential equations with quadratic and cubic terms by Morgan-Voyce matrix-collocation method

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Abstract: Nonlinear differential equations have many applications in different science and engineering disciplines. However, a nonlinear differential equation cannot be solved analytically and so must be solved numerically. Thus, we aim to develop a novel numerical algorithm based on Morgan-Voyce polynomials with collocation points and operational matrix method to solve nonlinear differential equations. In the our proposed method, the nonlinear differential equations including quadratic and cubic terms having the initial conditions are converted to a matrix equation. In order to obtain the matrix equations and solutions for the selected problems, code was developed in MATLAB. The solution of this method for the convergence and efficiency was compared with the equations such as Van der Pol differential equation calculated by different methods.

Key words: Nonlinear ordinary differential equations, Morgan-Voyce polynomials, matrix-collocation method, residual error analysis

1. Introduction

Numerous phenomena in various areas of physical and engineering such as solid state, electrical engineering, mechanical engineering, economics, chemical reactions, spring-mass systems, bending of beams, fluid mechanics, epidemic model in biology and nonlinear optics can be modeled by a nonlinear class of ordinary differential equations [1–5, 8, 9, 11, 16, 17, 19]. Therefore, analytical and numerical solutions of the equations play an important role in the fields of applied mathematics and engineering.

In this study, the high-order nonlinear differential equation involving quadratic and cubic nonlinear terms

$$\sum_{k=0}^{m} P_k(t) y^{(k)}(t) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(t) y^{(p)}(t) y^{(q)}(t) + \sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{pqr}(t) y^{(p)}(t) y^{(q)}(t) y^{(r)}(t) = g(t) \qquad (1.1)$$

subject to the initial and boundary(mixed) conditions

$$\sum_{k=0}^{m-1} (a_{kj} y^{(k)}(a) + b_{kj} y^{(k)}(b)) = \lambda_j, \; j = 0, 1, \ldots, m-1 \qquad (1.2)$$

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is considered, where $P_k(t), Q_{pq}(t), Q_{pqr}(t)$ and $g(t)$ are the given analytic functions defined on the interval $a \leq t \leq b$; λ_j, a_{kj} and b_{kj} are the known rael coefficients. In order to solve the nonlinear problem (1.1)-(1.2), we utilize the matrix-collocation method, which have been developed by Sezer and Coworkers[7, 10, 12–14], and research the numerical solution in the truncated Morgan-Voyce series form

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} a_n B_n(t), \ \ a \leq t \leq b \tag{1.3}$$

where a_n, $n = 0, 1, \ldots, N$ are the unknown coefficients to be determined and $B_n(t)$, $n = 0, 1, \ldots, N$, $N \geq m$, are the Morgan-Voyce polynomials [13–15, 17] defined by, recursively [6, 13–15, 17, 18],

$$B_n(t) = (t+2)B_{n-1}(t) - B_{n-2}(t), \ n \geq 2$$

with $B_o(t) = 1$ and $B_1(t) = t + 2$ or explicitly, for $n \geq 1$,

$$B_n(t) = \sum_{j=0}^{n} \binom{n+j+1}{n-j} t^j. \tag{1.4}$$

Also, these polynomials, for $n = 0, 1, \ldots$ are solutions of the differential equation

$$t(t+4)B_n''(t) + 3(t+2)B_n'(t) - n(n+2)B_n(t) = 0.$$

2. Fundamental matrix relations

In this section, we consider (1.1) and create the matrix forms of each term in the equation. For our purpose, firstly we transform the truncated Morgan-Voyce series defined by (1.3) into the matrix form

$$y(t) \cong y_N(t) = \boldsymbol{B}(t)A, \tag{2.1}$$

where

$$\boldsymbol{B}(t) = [B_o(t), \ B_1(t), \ \ldots, \ B_N(t)],$$

$$\boldsymbol{A} = [a_o, a_1, \ldots, a_N]^T.$$

Also, the matrix $\boldsymbol{B}(t)$ and its derivative $\boldsymbol{B}'(t)$ can be written in the matrix forms

$$\boldsymbol{B}(t) = \boldsymbol{T}(t)\boldsymbol{R} \ \ and \ \ \boldsymbol{B}'(t) = \boldsymbol{T}(t)\boldsymbol{M}\boldsymbol{R}, \tag{2.2}$$

where

$$\boldsymbol{T}(t) = \left[1, \ t, \ t^2, \ldots, t^N\right].$$

$$\boldsymbol{R} = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 3 \\ 2 \end{pmatrix} & \cdots & \begin{pmatrix} N+1 \\ N \end{pmatrix} \\ 0 & \begin{pmatrix} 3 \\ 0 \end{pmatrix} & \begin{pmatrix} 4 \\ 1 \end{pmatrix} & \cdots & \begin{pmatrix} N+2 \\ N-1 \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} 5 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} N+3 \\ N-2 \end{pmatrix} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \begin{pmatrix} 2N+1 \\ 0 \end{pmatrix} \end{bmatrix},$$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By using the relations (2.2), we obtain the following matrix relations:

$$\left. \begin{aligned} \boldsymbol{T}(t) &= \boldsymbol{B}(t)\boldsymbol{R}^{-1} \\ \boldsymbol{B}'(t) &= \boldsymbol{T}(t)\boldsymbol{M}\boldsymbol{R} \end{aligned} \right\} \Rightarrow \boldsymbol{B}'(t) = \boldsymbol{B}(t)\boldsymbol{C}$$

and, by extansion, for λ_k, $k = 0, 1, \ldots$

$$\boldsymbol{B}^{(k)}(t) = \boldsymbol{B}(t)\boldsymbol{C}^k, \tag{2.3}$$

where

$$\boldsymbol{C} = \boldsymbol{R}^{-1}\boldsymbol{M}\boldsymbol{R}, \quad \boldsymbol{C}^o \text{unit matrix.}$$

Moreover, by means of the relations (2.1) and (2.3), we have the expression

$$y^{(k)}(t) = \boldsymbol{B}^{(k)}(t)\boldsymbol{A} = \boldsymbol{B}(t)\boldsymbol{C}^{(k)}\boldsymbol{A}. \tag{2.4}$$

In addition, we can obtain the general matrix forms of the nonlinear quadratic and cubic parts by similar operations as (2.1)-(2.4) [13, 14], for $p, q, r = 0, 1, 2$, as follows:

$$y^{(p)}(t)y^{(q)}(t) = \boldsymbol{B}(t)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t)\overline{\boldsymbol{C}}^q\overline{\boldsymbol{A}} \tag{2.5}$$

and

$$y^{(p)}(t)y^{(q)}(t)y^{(r)}(t) = \boldsymbol{B}(t)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t)\overline{\boldsymbol{C}}^q\overline{\overline{\boldsymbol{B}}}(t)\overline{\overline{\boldsymbol{C}}}^r\overline{\overline{\boldsymbol{A}}}, \tag{2.6}$$

where

$$\overline{\boldsymbol{A}} = \left[a_o\boldsymbol{A}, \ a_1\boldsymbol{A}, \ \ldots, a_N\boldsymbol{A}\right]^T, \qquad \overline{\overline{\boldsymbol{A}}} = \left[a_o\overline{\boldsymbol{A}}, \ a_1\overline{\boldsymbol{A}}, \ \ldots, a_N\overline{\boldsymbol{A}}\right]^T,$$

$$\overline{\boldsymbol{B}}(t) = diag\left[\boldsymbol{B}(t), \ \boldsymbol{B}(t), \ \ldots, \boldsymbol{B}(t)\right], \qquad \overline{\overline{\boldsymbol{B}}}(t) = diag\left[\overline{\boldsymbol{B}}(t), \ \overline{\boldsymbol{B}}(t), \ \ldots, \overline{\boldsymbol{B}}(t)\right],$$

$$\overline{\boldsymbol{C}}^q = diag\left[\boldsymbol{C}^q, \ \boldsymbol{C}^q, \ \ldots, \boldsymbol{C}^q\right], \qquad \overline{\overline{\boldsymbol{C}}}^r = diag\left[\overline{\boldsymbol{C}}^r, \ \overline{\boldsymbol{C}}^r, \ \ldots, \overline{\boldsymbol{C}}^r\right].$$

3. Morgan-Voyce matrix-collocation method

To construct the matrix-collocation based on Morgan-Voyce polynomials [13, 14], we first constitute the matrix equation of (1.1), by putting the matrix forms (2.4), (2.5), and (2.6) in (1.1),

$$\sum_{k=0}^{m} P_k(t)\boldsymbol{B}(t)\boldsymbol{C}^k\boldsymbol{A} + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(t)\boldsymbol{B}(t)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t)\overline{\boldsymbol{C}}^q\overline{\boldsymbol{A}} + \sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{pqr}(t)\boldsymbol{B}(t)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t)\overline{\boldsymbol{C}}^q\overline{\overline{\boldsymbol{B}}}(t)\overline{\overline{\boldsymbol{C}}}^r\overline{\overline{\boldsymbol{A}}} = g(t) \quad (3.1)$$

Then by using the collocation points defined by

$$t_i = a + \frac{b-a}{N}i, \quad i = 0, 1, \ldots, N$$

in (3.1) and simplifying, we achieve the fundamental matrix equation

$$\sum_{k=0}^{m} \boldsymbol{P}_k\boldsymbol{B}\boldsymbol{C}^k\boldsymbol{A} + \sum_{p=0}^{2}\sum_{q=0}^{p} \boldsymbol{Q}_{pq}\boldsymbol{B}\boldsymbol{C}^p\overline{\boldsymbol{B}}\,\overline{\boldsymbol{C}}^q\,\overline{\boldsymbol{A}} + \sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} \boldsymbol{Q}_{pqr}\boldsymbol{B}\boldsymbol{C}^p\overline{\boldsymbol{B}}\,\overline{\boldsymbol{C}}^q\overline{\overline{\boldsymbol{B}}}\,\overline{\overline{\boldsymbol{C}}}^r\,\overline{\overline{\boldsymbol{A}}} = \boldsymbol{G}$$

or briefly

$$\sum_{k=0}^{m} \boldsymbol{P}_k\boldsymbol{B}\boldsymbol{C}^k\boldsymbol{A} + \sum_{p=0}^{2}\sum_{q=0}^{p} \boldsymbol{Q}_{pq}\boldsymbol{B}^*_{pq}\overline{\boldsymbol{A}} + \sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} \boldsymbol{Q}_{pqr}\boldsymbol{B}^*_{pqr}\overline{\overline{\boldsymbol{A}}} = \boldsymbol{G} \tag{3.2}$$

where

$$\boldsymbol{P}_k = diag\left[P_k(t_o),\ P_k(t_1),\ \ldots, P_k(t_N)\right],$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}(t_o) \\ \boldsymbol{B}(t_1) \\ \vdots \\ \boldsymbol{B}(t_N) \end{bmatrix} = \begin{bmatrix} B_o(t_o) & B_1(t_o) & \cdots & B_N(t_o) \\ B_o(t_1) & B_1(t_1) & \cdots & B_N(t_1) \\ \vdots & \vdots & \cdots & \vdots \\ B_o(t_N) & B_1(t_N) & \cdots & B_N(t_N) \end{bmatrix},$$

$$\boldsymbol{Q}_{pq} = diag\left[Q_{pq}(t_o),\ Q_{pq}(t_1),\ \ldots, Q_{pq}(t_N)\right], \quad \boldsymbol{Q}_{pqr} = diag\left[Q_{pqr}(t_o),\ Q_{pqr}(t_1),\ \ldots, Q_{pqr}(t_N)\right],$$

$$\boldsymbol{G} = \left[g(t_o),\ g(t_1),\ \ldots, g(t_N)\right],$$

$$\boldsymbol{B}^*_{pq} = \begin{bmatrix} \boldsymbol{B}(t_o)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t_o)\overline{\boldsymbol{C}}^q \\ \boldsymbol{B}(t_1)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t_1)\overline{\boldsymbol{C}}^q \\ \vdots \\ \boldsymbol{B}(t_N)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t_N)\overline{\boldsymbol{C}}^q \end{bmatrix}, \quad \boldsymbol{B}^*_{pqr} = \begin{bmatrix} \boldsymbol{B}(t_o)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t_o)\overline{\boldsymbol{C}}^q\overline{\overline{\boldsymbol{B}}}(t_o)\overline{\overline{\boldsymbol{C}}}^r \\ \boldsymbol{B}(t_1)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t_1)\overline{\boldsymbol{C}}^q\overline{\overline{\boldsymbol{B}}}(t_1)\overline{\overline{\boldsymbol{C}}}^r \\ \vdots \\ \boldsymbol{B}(t_N)\boldsymbol{C}^p\overline{\boldsymbol{B}}(t_N)\overline{\boldsymbol{C}}^q\overline{\overline{\boldsymbol{B}}}(t_N)\overline{\overline{\boldsymbol{C}}}^r \end{bmatrix}.$$

Next, the fundamental matrix equation (3.2) of (1.1) can be expressed in the form

$$\boldsymbol{W}\boldsymbol{A} + \boldsymbol{V}\overline{\boldsymbol{A}} + \boldsymbol{Z}\overline{\overline{\boldsymbol{A}}} = \boldsymbol{G} \Longleftrightarrow [\boldsymbol{W}; \boldsymbol{V}; \boldsymbol{Z} : \boldsymbol{G}], \tag{3.3}$$

where

$$\boldsymbol{W} = [w_{ij}] = \sum_{k=0}^{m} \boldsymbol{P}_k \boldsymbol{B} \boldsymbol{C}^k, \ \ i, j = 0, 1, \ldots, N$$

$$\boldsymbol{V} = [v_{ln}] = \sum_{p=0}^{2} \sum_{q=0}^{p} \boldsymbol{Q}_{pq} \boldsymbol{B}_{pq}^*, \ \ l = 0, 1, \ldots, N, n = 0, 1, \ldots, (N+1)^2 - 1$$

$$\boldsymbol{Z} = [z_{sm}] = \sum_{p=0}^{2} \sum_{q=0}^{p} \sum_{r=0}^{q} \boldsymbol{Q}_{pqr} \boldsymbol{B}_{pqr}^*, \ \ s = 0, 1, \ldots, N, m = 0, 1, \ldots, (N+1)^3 - 1.$$

Furthermore, by means of the relation (2.4), the matrix form of the conditions (1.2) is obtained as

$$\sum_{k=0}^{m-1} \left(a_{jk}\boldsymbol{B}\left(a\right) + b_{jk}\boldsymbol{B}\left(b\right)\right)\boldsymbol{C}^k \boldsymbol{A} = \lambda_j, \ j = 0, 1, \ldots, m-1$$

or briefly

$$\boldsymbol{U}\boldsymbol{A} + \boldsymbol{O}^*\overline{\boldsymbol{A}} + \boldsymbol{O}^{**}\overline{\overline{\boldsymbol{A}}} = \lambda \Longleftrightarrow [\boldsymbol{U}; \boldsymbol{O}^*; \boldsymbol{O}^{**} : \lambda], \tag{3.4}$$

where for

$$j = 0, 1, \ldots, m-1$$

$$\boldsymbol{U} = [u_{jo}, u_{j1}, \ldots, u_{jN}]; \ \lambda = [\lambda_o, \lambda_1, \ldots, \lambda_{m-1}]^T$$

$$\boldsymbol{O}^* = [0, 0, \ldots, 0]_{m \times (N+1)^2}, \quad and \quad \boldsymbol{O}^{**} = [0, 0, \ldots, 0]_{m \times (N+1)^3}$$

are zero matrices.

To obtain the solution of (1.1) with the mixed conditions (1.2), we replace the any m rows of (3.3) by the m row matrices (3.4) and thus, the desired augmented matrix equation is obtained as

$$[\boldsymbol{W}; \boldsymbol{V}; \boldsymbol{Z} : \boldsymbol{G}] \Longleftrightarrow \boldsymbol{W}\boldsymbol{A} + \boldsymbol{V}\overline{\boldsymbol{A}} + \boldsymbol{Z}\overline{\overline{\boldsymbol{A}}} = \boldsymbol{G}$$

which corresponds to the system of the nonlinear algebraic equations with the Morgan-Voyce coefficients $a_n, n = 0, 1, \ldots, N$.

4. Residual error estimation and convergency test

In this section, we will give an error analysis based on the residual function [7, 10, 12–14] for the present method. Furthermore, we will improve the Morgan-Voyce polynomial solutions by means of the residual error function. For our purpose, we define the residual function using both the linear and nonlinear parts of (1.1) for the present method as

$$R_N(t) = L[y_N(t)] + N[y_N(t)] - g(t), \tag{4.1}$$

where the linear part is

$$L[y_N(t)] = \sum_{k=0}^{m} P_k(t) y^{(k)}(t)$$

and the nonlinear part is

$$N[y_N(t)] = \sum_{p=0}^{2} \sum_{q=0}^{p} Q_{pq}(t) y^{(p)}(t) y^{(q)}(t) + \sum_{p=0}^{2} \sum_{q=0}^{p} \sum_{r=0}^{q} Q_{pqr}(t) y^{(p)}(t) y^{(q)}(t) y^{(r)}(t).$$

The $y_N(t)$ represents the Morgan-Voyce polynomial solutions given by (1.3) of the problem (1.1). Thus, the $y_N(t)$ satisfies the problem (1.1)-(1.2) under the conditions (1.2). Also, the error function $e_N(t)$ can be defined as

$$e_N(t) = y(t) - y_N(t), \tag{4.2}$$

where $y(t)$ is the exact solution of the problem (1.1)-(1.2). From (1.1), (1.2), (4.1), and (4.2), we obtain the error equation

$$L[e_N(t)] + N[y_N(t) + e_N(t)] - N[y_N(t)] = -R_N(t)$$

with homogeneous conditions

$$\sum_{k=0}^{m-1} (a_{kj} e_N^{(k)}(a) + b_{kj} e_N^{(k)}(b)) = 0$$

or briefly, the error problem remarked by

$$\left. \begin{array}{c} L[e_N(t)] + N[e_N(t)] = -R_N(t) \\ \sum_{k=0}^{m-1} (a_{kj}(t) e_N^{(k)}(a) + b_{kj}(t) e_N^{(k)}(b)) = 0 \end{array} \right\} \tag{4.3}$$

Solving the problem (4.3) in a similar manner as in Section 3, we obtain the approximation $e_{N,M}(t)$ to $e_N(t)$, $(M \geq N)$.

As a result of this, the corrected Morgan-Voyce polynomial solution $y_{N,M}(t) = y_N(t) + e_{N,M}(t)$ is obtained by the polynomials $y_N(t)$ and $e_{N,M}(t)$. So, we establish the error function $e_N(t) = y(t) - y_N(t)$, the estimated error function $e_{N,M}(t)$ and the corrected error function $E_{N,M}(t) = e_N(t) - e_{N,M}(t) = y(t) - y_{N,M}(t)$.

A study on the convergence of Homotopy perturbation method has been presented for nonlinear differential equations in the investigation the rate of convergence in Banach space by Ayati and Biazar[3] . In addition to this, the convergence of Dickson polynomial solution of the nonlinear model problem has been developed using the residual function in Banach space by Kürkçü and Coworkers[10].

In this study, taking account of these two studies, we reveal the following convergence criterions for Morgan-Voyce polynomial solutions. For this purpose, the residual function $R_N(t)$ given by (4.1) can be defined on the interval $[a, b]$ or (a, b) as $R_N(t) : [a, b] \longrightarrow \mathbb{R}$ or $R_N(t) : [a + \varepsilon, b - \varepsilon] \longrightarrow \mathbb{R}$ (ε is a sufficiently small value) and $R_N(t)$ can be written in the Taylor series form

$$R_N(t) = r_0 + r_1 t + r_2 t^2 + ... + r_N t^N = \sum_{n=0}^{N} r_n t^n$$

where $\mathbb{R}$ is the set of real numbers. Now, we can use the following theorem for our investigation.

Theorem 4.1 *[10] Let B be a Banach space. The residual function sequence $\{R_N(t)\}_{N=2}^{\infty}$ is convergent in B and the following inequality is satisfied so that $0 < \mu_N < 1$. Here μ_N is constant in B:*

$$||R_{N+1}(t)|| < \mu_N ||R_N(t)|.$$ (4.4)

5. Numerical examples

In order to show the advantage and the accuracy of our method, the several numerical examples are given in the following.

Example 1. Let us first consider the second-order differential equation

$$y''(t) + y(t) = g(t)$$ (5.1)

with the condition $y(0) = 1, y'(0) = 1$, $0 \le t \le 1$ and the exact solution $y(t) = e^t$. Here, $g(t) = 2e^t$, $m = 2, P_o(t) = 1, P_1(t) = 0, P_2(t) = 1$. For $N = 2$, the collocation points are computed as $t_o = 0, t_1 = \frac{1}{2}, t_2 = 1$. By using (3.2), the fundamental matrix equation of the problem is written as

$$\sum_{k=0}^{2} P_k B C^k A = G.$$

Here,

$$C^o = I(\text{identity matrix}), \quad C^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad C^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$P_o = I \text{ (identity matrix)}, \ P_1 = O \text{ (zero matrix)}, \ P_2 = I \text{ (identity matrix)}.$$

The augmented matrix of the fundamental matrix equation and their conditions are written as

$$[W; G] = \begin{bmatrix} 1.0000 & 2.0000 & 5.0000 & 2.0000 \\ 1.0000 & 2.0000 & 3.0000 & 1.0000 \\ 0.0000 & 1.0000 & 4.0000 & 1.0000 \end{bmatrix}$$

and $[U_o; \lambda_o] = \begin{bmatrix} 1 & 2 & 3 & | & 1 \end{bmatrix}$ and $[U_1; \lambda_1] = \begin{bmatrix} 0 & 1 & 4 & | & 1 \end{bmatrix}$, respectively.

So, we have the solution for N=2,

$$y_2(t) = \sum_{n=0} a_n B_n(t) = 1 + t + 0.5000 \, t^2.$$

The approximate, corrected solutions, and errors are given in Table 1 and Figure 1.

By using the Theorem 4.1, the residual functions sequence can be calculated as

$$\{|R_N(1)|\}_{N=2}^{\infty} = \left\{|R_2(1)|, |R_3(1)|, |R_4(1)|, |R_5(1)|, |R_6(1)|, \dots\right\} = \{0.0208, 0.0020, 0.000129, 0.000008, 0.00000040, \dots\}$$

Table 1. Numerical results of Example 1 for some N values.

		$N = 4, M = 5$		$N = 7, M = 8$	
		Appr.	Corrected	Appr.	Corrected
t	Exact	Solutions			
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.2214	1.2214	1.2214	1.2214	1.2214
0.4	1.4918	1.4918	1.4918	1.4918	1.4918
0.6	1.8221	1.8221	1.8221	1.8221	1.8221
0.8	2.2255	2.2252	2.2255	2.2255	2.2255
1.0	2.7183	2.7163	2.7182	2.7183	2.7183
t	Exact	Errors			
0.0	1.0000	1.11E-16	4.44E-16	2.22E-16	4.44E-16
0.2	1.2214	1.69E-05	1.20E-06	3.24E-09	1.34E-10
0.4	1.4918	4.93E-05	2.68E-06	6.72E-09	2.83E-10
0.6	1.8221	5.37E-05	4.45E-06	9.89E-09	4.20E-10
0.8	2.2255	3.24E-04	1.48E-06	1.26E-08	5.01E-10
1.0	2.7183	2.02E-03	1.29E-04	3.96E-07	1.92E-08
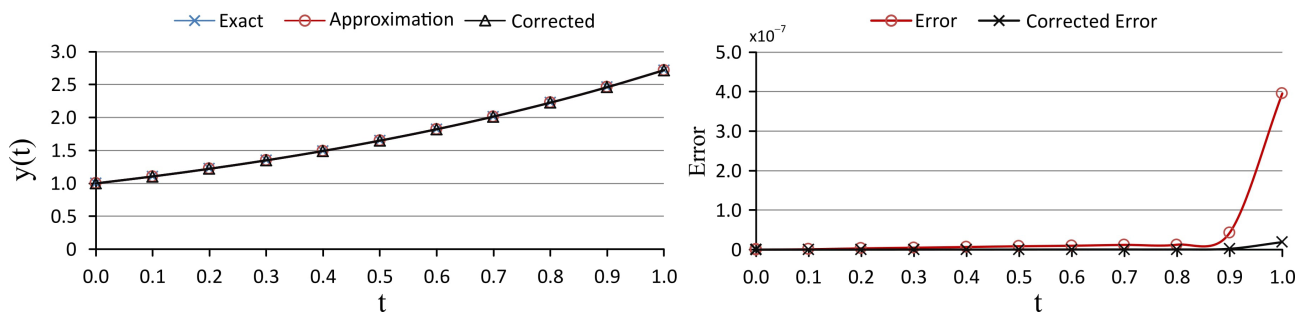
Figure 1. Solutions and errors of Example 1 for $N = 7$.

$$\mu_N = \left\{ \frac{|R_3(1)|}{|R_2(1)|}, \frac{|R_4(1)|}{|R_3(1)|}, \frac{|R_5(1)|}{|R_4(1)|}, \frac{|R_6(1)|}{|R_5(1)|}, \dots \right\} = \{0.097, 0.064, 0, 065, 0.047, \dots\}$$

so,

$$\frac{|R_{N+1}(1)|}{|R_N(1)|} < 1.$$

This shows us that the ratio is approaching zero as N increases. Thus, the residual function sequence $\{R_N(1)\}_{N=2}^{\infty}$ is convergent in B Banach space.

Example 2. Let us now consider the nonlinear second order differential equation

$$y''(t) + y'(t)y(t) = e^t(1 + e^t) \tag{5.2}$$

with the condition $y(0) = 1, y'(0) = 1$, $0 \le t \le 1$ and the exact solution $y(t) = e^t$ and having the exact solution $y(t) = e^t$. For $N = 4$ and $N = 7$, the comparative solutions are given in Table 2 and Figure 2.

Table 2. Numerical results of Example 2 for some N values.

		$N=4, M=5$		$N=7, M=8$	
		Appr.	Corrected	Appr.	Corrected
t	Exact	Solutions			
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.2214	1.2214	1.2214	1.2214	1.2214
0.4	1.4918	1.4918	1.4918	1.4918	1.4918
0.6	1.8221	1.8221	1.8221	1.8221	1.8221
0.8	2.2255	2.2252	2.2256	2.2255	2.2255
1.0	2.7183	2.7162	2.7183	2.7183	2.7183
t	Exact	Errors			
0.0	1.0000	0.00E+00	0.00E+00	4.44E-16	6.66E-16
0.2	1.2214	1.49E-05	3.87E-06	2.81E-09	5.43E-10
0.4	1.4918	3.80E-05	1.53E-05	5.11E-09	1.97E-09
0.6	1.8221	3.14E-05	3.11E-05	6.46E-09	4.30E-09
0.8	2.2255	3.10E-04	5.62E-05	6.54E-09	7.54E-09
1.0	2.7183	2.07E-03	3.59E-05	4.21E-07	2.71E-08
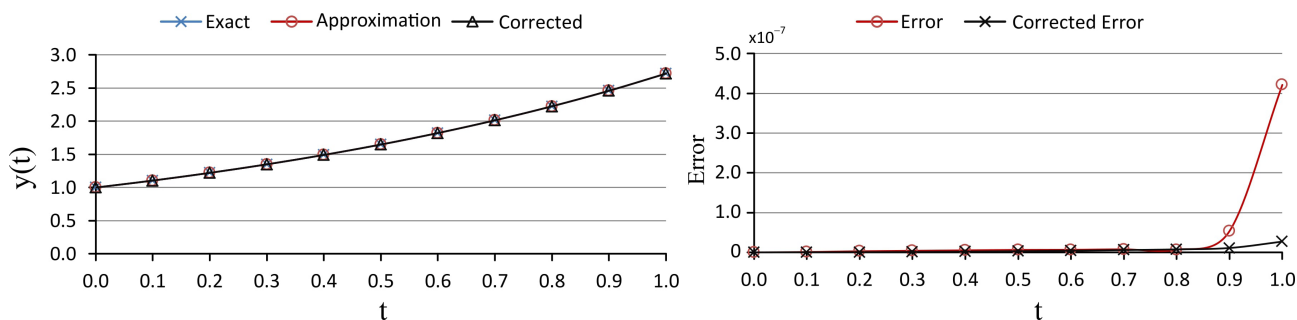
Figure 2. Solutions and errors of Example 1 for $N=7$.

Example 3. Let us now consider the differential equation

$$y''(t) - \mu(1 - y^2(t))y'(t) + y(t) = g(t) \tag{5.3}$$

with the condition $y(0) = 1, y'(0) = 0$, $0 \le t \le 1$, $g(t) = 2sin^3 t$, the constant $\mu = 2$ and having the exact solution $y(t) = cost$. The numerical results for this problem are illustrated in Table 3 and Figure 3 for $N = 4$ and $N = 7$.

Considering the results of the above three examples, it is seen that the difference between the exact and approximate solution is lower than 10^{-7} for $N > 7$.

Example 4. Let us now consider the differential equation

$$y''(t) - \mu(1 - y^2(t))y'(t) + y(t) = Asin(\Omega t), \tag{5.4}$$

where $\mu = 0.04, A = 0.04$ and $\Omega = 1.4$ with the condition $y(0) = 1, y'(0) = 0$, $0 \le t \le 1$.

Table 3. Numerical results of Example 3 for some N values.

		$N = 4, M = 5$		$N = 7, M = 8$	
		Appr.	Corrected	Appr.	Corrected
t	Exact	Solutions			
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	0.9801	0.9801	0.9801	0.9801	0.9801
0.4	0.9211	0.9211	0.9211	0.9211	0.9211
0.6	0.8253	0.8253	0.8253	0.8253	0.8253
0.8	0.6967	0.6968	0.6967	0.6967	0.6967
1.0	0.5403	0.5408	0.5403	0.5403	0.5403
t	Exact	Errors			
0.0	1.0000	4.44E-16	6.66E-16	4.44E-16	4.44E-16
0.2	0.9801	2.87E-06	2.04E-06	2.26E-09	5.43E-10
0.4	0.9211	8.58E-06	8.01E-06	4.82E-09	2.52E-09
0.6	0.8253	8.99E-06	1.73E-05	7.65E-09	6.17E-09
0.8	0.6967	7.41E-05	2.88E-05	1.12E-08	1.17E-08
1.0	0.5403	5.04E-04	7.57E-06	2.19E-07	1.11E-08
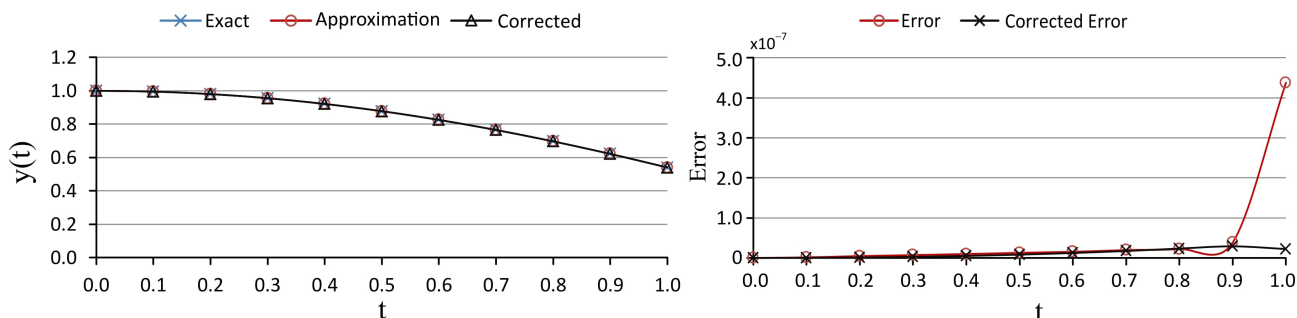
Figure 3. Solutions and errors of Example 3 for $N = 7$.

Electrical circuit involving a semiconductor results a forced Van der Pol oscillator (5.4), see Figure 4. The circuit contains a semiconductor (nonlinear term), a inductor L, a capacitor C and external voltage $E(t)$. The nonlinear equation obtained by the analysis of the electric circuit can be reached to (5.4) by making it dimensionless by means of the transformations given in [1, 8, 19]. The solution of this problem is compared with the results of the modified differential transform method (MDTM) developed by Abdelhafez [1] in Table 4 and Figure 5.

6. Conclusion

We have developed a matrix collocation method using Morgan-Voyce polynomials for the solution of nonlinear differential equations containing quadratic and cubic nonlinear terms that have not an exact solution analytically and be required numerical solutions. This approach allowed us to obtain the most appropriate numerical solutions by converting differential equations containing quadratic and cubic terms into nonlinear matrix equations. When the results of the above examples having exact solution are examined, it is seen that the
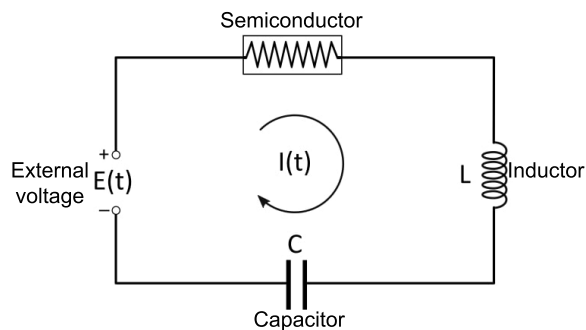
Figure 4. Electric circuit resulting in a forced Van der Pol oscillator.

Table 4. Numerical results of Example 4 for some N values.

| | | $N=4, M=5$ | | $N=7, M=8$ | |
		Appr.	Corrected	Appr.	Corrected
t	MDTM	Solutions			
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	0.9801	0.9801	0.9801	0.9801	0.9801
0.4	0.9216	0.9216	0.9216	0.9216	0.9216
0.6	0.8271	0.8271	0.8271	0.8271	0.8271
0.8	0.7005	0.7006	0.7005	0.7005	0.7005
1.0	0.5469	0.5476	0.5468	0.5468	0.5468
t	MDTM	Errors			
0.0	1.0000	2.22E-16	2.22E-16	3.33E-16	7.77E-16
0.2	0.9801	7.28E-06	2.97E-07	1.14E-08	2.83E-09
0.4	0.9216	2.13E-05	8.01E-07	3.40E-08	4.42E-09
0.6	0.8271	2.30E-05	1.90E-06	6.49E-07	6.05E-07
0.8	0.7005	1.22E-04	9.55E-06	8.01E-06	7.96E-06
1.0	0.5469	7.12E-04	5.67E-05	5.50E-05	5.65E-05
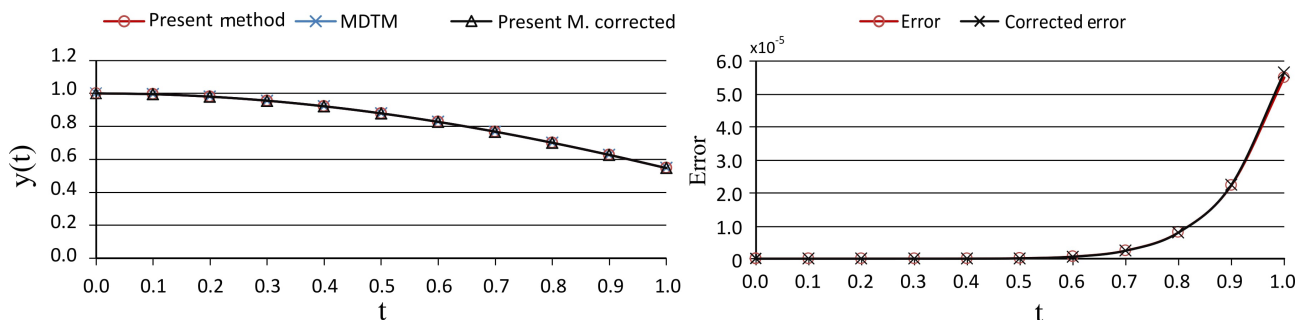
Figure 5. Comparative solutions and errors of Example 4 for $N=7$.

approximate and the corrected solutions are obtained with very high accuracy even at $N=8$ collocation points. Root mean square error(RMSE) for the approximate and corrected solutions are at 10^{-8} levels in
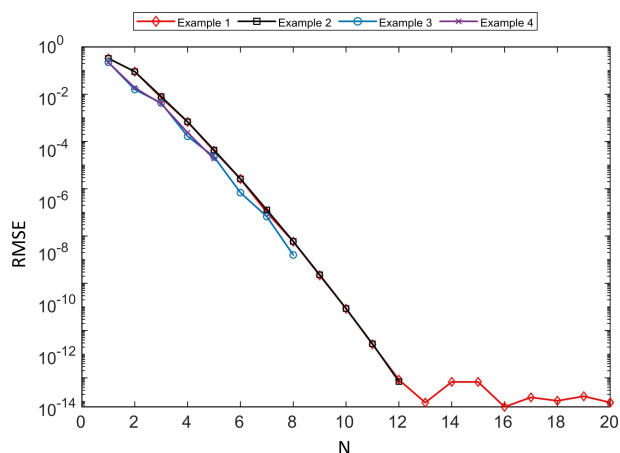
Figure 6. Comparative RMSE of all examples.

Figure 6. For $N > 8$, the reason for the fluctuation seen in calculations is due to the truncation errors. The reason for using RMSE instead of $|R_N(t)|$ to show the convergency for our method is that the convergence can be shown more simply in a graph for all examples .

In addition, the final example (Van der Pol differential equation) was compared with the results solved by the MDTM method. It is seen that the present solution is too close with the MDTM solution and to obtain more sensitive solutions, the N value and the real number format defined in the computer program (MATLAB, C++) should be increased by the significant number of digits.

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