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The statistically unbounded τ-convergence on locally solid Riesz spaces

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Abstract: A sequence (x_n) in a locally solid Riesz space (E,τ) is said to be statistically unbounded τ-convergent to $x \in E$ if, for every zero neighborhood U, $\frac{1}{n}\big|\{k \leq n : |x_k - x| \wedge u \notin U\}\big| \to 0$ as $n \to \infty$. In this paper, we introduce the concept of the st-u_τ-convergence and give the notions of st-u_τ-closed subset, st-u_τ-Cauchy sequence, st-u_τ-continuous and st-u_τ-complete locally solid vector lattice. Also, we give some relations between the order convergence and the st-u_τ-convergence.

Key words: Statistically u_τ-convergence, statistically u_τ-cauchy, locally solid Riesz space, order convergence, Riesz space

1. Preliminary and introductory facts

Vector lattices and the statistical convergence are natural and efficient tools in the theory of functional analysis. A vector lattice is an ordered vector space that has many applications in measure theory, operator theory and applications in economics (see e.g. [1, 2, 12]), it was introduced by F. Riesz in [11]. On the other hand, the statistical convergence is a generalization of the ordinary convergence of a real sequence (see e.g. [9]). Studies related to this paper are done by Maddox in [8], where the statistical convergence was introduced in a topological vector space and by Albayrak and Pehlivan in [3], where the statistical convergence was introduced on locally solid vector lattice. Otherwise, the unbounded order convergence was defined for order complete Riesz spaces by H. Nakano in [10]. Recently, some papers have been studied on the unbounded convergence on vector lattices (see. e.g. [4–7]). However, as far as we know, the concept of unbounded order convergence related to the statistical convergence has not been done before. In this paper, we aim to introduce this concept on locally solid vector lattice.

Now, let give some basic notations and terminologies that will be used in this paper. Every linear topology τ on a vector space E has a base $\mathcal{N}$ for the zero neighborhoods satisfying the following four properties; for each $V \in \mathcal{N}$, we have $\lambda V \subseteq V$ for all scalar $|\lambda| \leq 1$; for any $V_1, V_2 \in \mathcal{N}$ there is another $V \in \mathcal{N}$ such that $V \subseteq V_1 \cap V_2$; for each $V \in \mathcal{N}$ there exists another $U \in \mathcal{N}$ with $U + U \subseteq V$; for any scalar λ and each $V \in \mathcal{N}$, the set λV is also in $\mathcal{N}$. We refer the reader for an exposition on the linear topology [1, 2]. In this article, unless otherwise, when we mention a zero neighborhood, it means that it always belongs to a base that holds the above properties. Let E be a real-valued vector space. If there is an order relation "$\leq$" on E, i.e., it is antisymmetric, reflexive and transitive, then E is called ordered vector space whenever the following conditions

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hold: for every $x, y \in E$ such that $x \leq y$, we have $x+z \leq y+z$ and $\alpha x \leq \alpha y$ for all $z \in E$ and $\alpha \in \mathbb{R}_+$. An ordered vector space E is called Riesz space or vector lattice if, for any two vectors $x, y \in E$, the infimum $x \wedge y = \inf\{x, y\}$ and the supremum $x \vee y = \sup\{x, y\}$ exist in E. Let E be a vector lattice. Then, for any $x \in E$, the positive part of x is $x^+ := x \vee 0$, the negative part of x is $x^- := (-x) \vee 0$ and absolute value of x is $|x| := x \vee (-x)$. Moreover, any two elements x, y in a vector lattice is called disjoint whenever $|x| \wedge |y| = 0$. An operator T between two vector lattice E and F is said to be a lattice homomorphism or Riesz homomorphism whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in E$. A vector lattice is called order complete if every nonempty bounded above subset has a supremum (or, equivalently, whenever every nonempty bounded below subset has an infimum). A vector lattice is order complete iff $0 \leq x_n \uparrow \leq x$ implies the existence of $\sup x_n$. We refer the reader for an exposition on vector lattices [1, 2, 11, 12].

Recall that a subset A of a vector lattice E is called solid if, for each $x \in A$ and $y \in E$, $|y| \leq |x|$ implies $y \in A$. A solid vector subspace of a vector lattice is referred to as an ideal. An order closed ideal is called a band. Let E be vector lattice E and τ be a linear topology on it. Then (E, τ) is said a *locally solid vector lattice* (or, *locally solid Riesz space*) if τ has a base which consists of solid sets; for more details on these notions (see e.g. [1, 2, 6, 12]). Recall that a net $(x_\alpha)_{\alpha \in A}$ in a vector lattice X is *order convergent* to $x \in X$, if there exists another net $(y_\beta)_{\beta \in B}$ satisfying $y_\beta \downarrow 0$, and for any $\beta \in B$, there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. In this case, we write $x_\alpha \xrightarrow{0} x$. In a vector lattice X, a net (x_α) is unbounded order convergent to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in X_+$ (see e.g. [4, 10]). Also, these notions can be written for sequence. A vector lattice E is called *Archimedean* whenever $\frac{1}{n}x \downarrow 0$ holds in E for each $x \in E_+$. In this article, unless otherwise, all vector lattices are assumed to be real and Archimedean.

Consider a subset K of the set $\mathbb{N}$ of all natural numbers. Let's define a new set $K_n = \{k \in K : k \leq n\}$. Then we denote $|K_n|$ for the cardinality of that the set K_n. If the limit of $\mu(K) = \lim\limits_{n \to \infty} |K_n|/n$ exists then $\mu(K)$ is called the asymptotic density of the set K. Let X be a topological space and (x_n) be a sequence in X. Then (x_n) is said to be statistically convergent to $x \in X$ whenever, for each neighborhood U of x, we have $\mu\big(\{n \in \mathbb{N} : x_n \notin U\}\big) = 0$ (see e.g. [3, 8, 9]). Similarly, a sequence (x_n) in a locally solid Riesz space (E, τ) is said to be statistically τ-convergent to $x \in E$ if it is provided that, for every τ-neighborhood U of zero, $\lim\limits_{n \to \infty} \frac{1}{n}\big|\{k \leq n : (x_k - x) \notin U\}\big| = 0$ holds (see [3]). Motivated by the above definitions, we give the following notion.

Definition 1.1 *Let (E, τ) be a locally solid vector lattice and (x_n) be a sequence in E. Then (x_n) is said to be statistically unbounded τ-convergent to $x \in E$ if, for every zero neighborhood U, it satisfies the following limit*

$$\lim_{n \to \infty} \frac{1}{n}\big|\{k \leq n : |x_k - x| \wedge u \notin U\}\big| = 0 \tag{1}$$

for all $u \in E_+$. We abbreviate this convergence as $x_n \xrightarrow{st-u_\tau} x$, or shortly, we say that (x_n) st-u_τ-converges to x.

Let's take $K_n = \{k \leq n : |x_k - x| \wedge u \notin U\}$ for arbitrary zero neighborhood U in any locally solid vector lattice E. Then (x_n) is statistically unbounded τ-convergent to $x \in E$ whenever $\mu(K_n) = 0$ in (1) for every $u \in E_+$. In this paper, we use the following fact without giving the reference, and so we shall keep in mind it.

Remark 1.2 *Let (E,τ) be a locally solid Riesz space and U be a zero neighborhood. If $y \leq x$ for any $x, y \in E$ then we have that $y \notin U$ implies $x \notin U$. Indeed, assume $y \notin U$ and $x \in U$. Then we have $y \in U$ because U is solid and $y \leq x$. So, we have a contradict, and so we get $x \notin U$.*

Example 1.3 *Let's consider the locally solid Riesz space $(c_0, \|\cdot\|)$ the set of null sequences with the supremum norm $\|\cdot\|$ on c_0. Consider the sequence (e_n) of the standard unit vectors in c_0. Then the base $\mathcal{N}$ consists of the zero neighborhoods $U_r = \{x \in c_0 : \|x\| < r\}$, where r is a positive real number. Thus, we get that $e_n \xrightarrow{st-u_\tau} 0$ in c_0. Indeed, for arbitrary zero neighborhood U, there exist some U_r in $\mathcal{N}$ such that $U_r \subseteq U$, and also, $K_n = \{n \in \mathbb{N} : |x_n - 0| \wedge u \in U_r\} = \{n \in \mathbb{N} : \|e_n \wedge u\| < r\}$ for all $u \in E_+$. It can be seen that $\mu(K_n) = 1$. Thus, we have $e_n \xrightarrow{st-u_\tau} 0$.*

Remark 1.4 *The statistically τ-convergence implies statistically unbounded τ-convergence. Indeed, for arbitrary zero neighborhood U and fixed $u \in E_+$, by applying the formula $|x_k - x| \wedge u \leq |x_k - x|$, we can see that $|x_k - x| \wedge u \notin U$ implies $|x_k - x| \notin U$. Otherwise, it contradicts with being solid of U. Hence, $\{k \leq n : (|x_k - x| \wedge u) \notin U\} \subseteq \{k \leq n : (x_k - x) \notin U\}$ holds for all $u \in E_+$. It means that st-τ-convergence implies st-u_τ-convergence*

For the converse of Remark 1.4, we give Theorem 2.7 in the next section.

2. Main results

It follows from Theorem 2.17. in [1] that the lattice operations are uniformly continuous in locally solid vector lattice. Similarly, the lattice operations are continuous in the following sense.

Theorem 2.1 *Let (x_n) and (y_n) be two sequences in a locally solid vector lattice (E,τ). If $x_n \xrightarrow{st-u_\tau} x$ and $y_n \xrightarrow{st-u_\tau} y$ then $(x_n \vee y_n) \xrightarrow{st-u_\tau} x \vee y$.*

Proof Let U be an arbitrary zero neighborhood in E. Thus, there is another zero neighborhood V in $\mathcal{N}$ such that $V + V \subseteq U$. Consider the set $K_{yn} = \{n \in \mathbb{N} : |y_n - y| \wedge u \in V\}$ and $K_{xn} = \{n \in \mathbb{N} : |x_n - x| \wedge u \in V\}$ for every $u \in E_+$. Then $\mu(K_{xn}) = \mu(K_{yn}) = 1$ because of $x_n \xrightarrow{st-u_\tau} x$ and $y_n \xrightarrow{st-u_\tau} y$.

On the other hand, by applying Theorem 1.9(2) in [2], i.e., $|a \vee b - b \vee c| \leq |a - c|$ for any vectors in a vector lattice, we can get

$$\begin{aligned} |x_n \vee y_n - x \vee y| \wedge u &= |x_n \vee y_n - y_n \vee x + y_n \vee x - x \vee y| \wedge u \\ &\leq |x_n \vee y_n - y_n \vee x| \wedge u + |y_n \vee x - x \vee y| \wedge u \\ &\leq |x_n - x| \wedge u + |y_n - y| \wedge u \end{aligned}$$

for all $u \in E_+$. So, we have $|x_n \vee y_n - x \vee y| \wedge u \in U$ for all $u \in E_+$ and for each $n \in \mathbb{N}$ because of $|x_n - x| \wedge u + |y_n - y| \wedge u \in V + V \subseteq U$. Thus, we have $\mu\big(\{n \in \mathbb{N} : |x_n \vee y_n - x \vee y| \wedge u \in U\}\big) = 1$ for all $u \in E_+$. That is, we get $(x_n \vee y_n) \xrightarrow{st-u_\tau} x \vee y$. □

We continue with several basic results that are motivated by their analogies from vector lattice theory.

Theorem 2.2 *Let* $x_n \xrightarrow{st-u_\tau} x$ *and* $y_n \xrightarrow{st-u_\tau} y$ *in a locally solid Riesz space* (E,τ)*. Then we have the following facts:*

(i) $x_n \xrightarrow{st-u_\tau} x$ *iff* $(x_n - x) \xrightarrow{st-u_\tau} 0$ *iff* $|x_n - x| \xrightarrow{st-u_\tau} 0$*;*

(ii) $rx_n + sy_n \xrightarrow{st-u_\tau} rx + sy$ *for any* $r, s \in \mathbb{R}$*;*

(iii) $x_{n_k} \xrightarrow{st-u_\tau} x$ *for any subsequence* (x_{n_k}) *of* x_n*;*

(iv) $|x_n| \xrightarrow{st-u_\tau} |x|$*;*

(v) if τ *Hausdroff topology,* $x_n \xrightarrow{st-u_\tau} x$ *and* $x_n \xrightarrow{st-u_\tau} y$ *then* $x = y$*.*

Proof The (i) comes directly from the definition of the st-u_τ-convergence.

(ii) Firstly, we show that $rx_n \xrightarrow{st-u_\tau} rx$ for every $r \in \mathbb{R}$. Let U be an arbitrary zero neighborhood and r be a real number. Thus, we have $\mu\big(\{n \in \mathbb{N} : |x_n - x| \wedge u \in U\}\big) = 1$ for all $u \in E_+$. If $|r| \leq 1$ then, by applying the properties of $\mathcal{N}$, we can get $|r|(|y_n - y| \wedge u) = |rx_n - rx| \wedge (|r|u) \leq |x_n - x| \wedge u \in U$. In special, if we take u as $|r|u$ then $|rx_n - rx| \wedge u \in U$. So, we get $\mu\big(\{n \in \mathbb{N} : |rx_n - rx| \wedge u \in U\}\big) = 1$ for all $|r| \leq 1$ and every $u \in E_+$ because of $\{n \in \mathbb{N} : |x_n - x| \wedge u \in U\} \subseteq \{n \in \mathbb{N} : |rx_n - rx| \wedge u \in U\}$ for each zero neighborhood U. It means that $rx_n \xrightarrow{st-u_\tau} rx$ holds for each $|r| \leq 1$.

Next, for $|r| > 1$, it follows from the third property of $\mathcal{N}$ that there is another $V \in \mathcal{N}$ such that $\{V + V + \cdots + V\}_k \subseteq U$, where k is the smallest integer greater or equal r. Then, by the inequality $|rx_n - rx| \wedge u = (|r||x_n - x|) \wedge u \leq (|k||x_n - x|) \wedge u = |kx_n - kx| \wedge u \in \{V + V + \cdots + V\}_k \subseteq U$, we can get that $|rx_n - rx| \wedge u \in U$ for each $u \in E_+$ and for all $n \in \mathbb{N}$ because U is a solid subset. Therefore, we get $\mu\big(\{n \in \mathbb{N} : |rx_n - rx| \wedge u \in U\}\big) = 1$ for each $u \in E_+$, i.e., $rx_n \xrightarrow{st-u_\tau} rx$.

Now, we show $x_n + y_n \xrightarrow{st-u_\tau} x + y$. Take U an arbitrary zero neighborhood. By applying the properties of $\mathcal{N}$, there exists another $V \in \mathcal{N}$ with $V + V \subseteq U$. Since $x_n \xrightarrow{st-u_\tau} x$ and $y_n \xrightarrow{st-u_\tau} y$, we have $\mu\big(\{n \in \mathbb{N} : |x_n - x| \wedge u \in V\}\big) = 1$ and $\mu\big(\{n \in \mathbb{N} : |y_n - y| \wedge u \in V\}\big) = 1$ for all $u \in E_+$. By the formula $|(x_n + y_n) - (x + y)| \wedge u \leq |(x_n - x)| \wedge u + |(y_n + y)| \wedge u \in V + V \subseteq U$ for every $u \in E_+$. So, we get $\mu\big(\{n \in \mathbb{N} : |(x_n + y_n) - (x + y)| \wedge u \in U\}\big) = 1$. That is, $x_n + y_n \xrightarrow{st-u_\tau} x + y$.

(iii) For arbitrary zero neighborhood U, it follows from $\{k \in \mathbb{N} : |x_{n_k} - x| \wedge u \notin U\} \subseteq \{n \in \mathbb{N} : |x_n - x| \wedge u \notin U\}$ for all $u \in E_+$ and $x_n \xrightarrow{st-u_\tau} x$ that $\mu\big(\{k \in \mathbb{N} : |x_{n_k} - x| \wedge u \notin U\}\big) = 0$ for every $u \in E_+$ i.e., $x_{n_k} \xrightarrow{st-u_\tau} x$ holds.

(iv) By applying the formula $|x| = x \vee (-x)$, and by using (i) and Theorem 2.1, one can get the desired result.

(v) Let U be a zero neighborhood. Take $K_n = \{n \in \mathbb{N} : |x_n - x| \wedge u \in U\}$ and $L_n = \{n \in \mathbb{N} : |x_n - y| \wedge u \in U\}$ for $u \in E_+$. Since $x_n \xrightarrow{st-u_\tau} x$ and $x_n \xrightarrow{st-u_\tau} y$, we have $\mu(K_n) = \mu(L_n) = 1$. On another hand, by applying the properties of $\mathcal{N}$, there is another $V \in \mathcal{N}$ such that $V + V \subseteq U$. Consider the inequality

$|x-y|\wedge u \leq |x-x_n|\wedge u + |x_n - y|\wedge u \in V+V \subseteq U$ for all $u \in E_+$ and for each $n \in \mathbb{N}$. Thus, for arbitrary zero neighborhood U, we have $|x-y|\wedge u \in U$ for every $u \in E_+$. But, since E is a Hausdorff space, the intersection of all zero neighborhood is the singleton 0. That is, $|x-y|\wedge u = 0$ for all $u \in E_+$, and so, we get $x = y$. □

By using the st-u_τ-convergence, we can define the st-u_τ-closed.

Definition 2.3 *Let (E,τ) be a locally solid vector lattice and A be a subset of E. Then, A is called st-u_τ-closed subset in E if, for any sequence (a_n) in A which is st-u_τ-convergent to $a \in E$, it holds $a \in A$.*

Proposition 2.4 *The positive cone $E_+ = \{x \in E : 0 \leq x\}$ of a locally solid Riesz space (E,τ) with Hausdorff topology is st-u_τ-closed.*

Proof Assume (x_n) is a sequence in E_+ such that it st-u_τ-converges $x \in E$. By applying Theorem 2.1, $x_n = x_n^+ = x_n \vee 0 \xrightarrow{st-u_\tau} x \vee 0 = x^+ \geq 0$. Thus, following from Theorem 2.2(v), we get $x = x^+ \in E_+$. □

Proposition 2.5 *Let $x_n \xrightarrow{st-u_\tau} x$ and $y_n \xrightarrow{st-u_\tau} y$ in a Hausdorff locally solid Riesz space (E,τ). If $x_n \geq y_n$ for all $n \in \mathbb{N}$ then $x \geq y$.*

Proof Suppose that $y_n \leq x_n$ holds for all $n \in \mathbb{N}$. Then we can get $0 \leq x_n - y_n \in E_+$ for each $n \in \mathbb{N}$. By using (i) and applying Proposition 2.4, we have $x_n - y_n \xrightarrow{st-u_\tau} x - y \in E_+$ because of $(x_n - y_n) \in E_+$. Thus, we get $x - y \geq 0$, i.e., $x \geq y$. □

The following proposition is an st-u_τ-version of Lemma 2.5(ii) in [5] and Theorem 2.8. in [4].

Proposition 2.6 *Any monotone st-u_τ-convergent sequence in a Hausdorff locally solid vector lattice order converges to its st-u_τ-limit.*

Proof It is enough to show that if a sequence $(x_n)\uparrow$ in a locally solid vector lattice (E,τ) and $x_n \xrightarrow{st-u_\tau} x$ then $x_n \uparrow x$. Let's fix arbitrary $n \in \mathbb{N}$. Then $x_m - x_n \in E_+$ for every $m \geq n$. By using Proposition 2.4 and Theorem 2.2(ii), we get $x_m - x_n \xrightarrow{st-u_\tau} x - x_n \in E_+$ as $m \to \infty$. Therefore, $x \geq x_n$ for any n. Thus, since n is arbitrary, x is an upper bound of (x_n). Assume y is another upper bound of (x_n), i.e., $y \geq x_n$ for all $n \in \mathbb{N}$. Then, again by Proposition 2.4, we have $y - x_n \xrightarrow{st-u_\tau} y - x \in E_+$, or $y \geq x$. Thus, we get the desired result $x_n \uparrow x$. □

In Remark 1.4, we show that the statistically τ-convergence implies statistically unbounded τ-convergence. For the converse, we give the following theorem.

Theorem 2.7 *Let (x_n) be a monotone sequence in a Hausdorff locally solid Riesz space (E,τ). Then the statistically unbounded τ-convergence of (x_n) implies the statistically τ-convergence of it.*

Proof Without loss of generality, we may assume that (x_n) positive and increasing sequence in E_+. It follows from Proposition 2.6 that $x_n \uparrow x$ for some $x \in E$ because of $x_n \xrightarrow{st-u_\tau} x$. Hence, we have $0 \leq x - x_n \leq x$ for all $n \in \mathbb{N}$. On the other hand, for each $u \in E_+$ and for any arbitrary zero neighborhood U, we have

$$\mu\big(\{n \in \mathbb{N} : |x_n - x| \wedge u \in U\}\big) = 1.$$

In particular, for $u = x$, we obtain that

$$\{n \in \mathbb{N} : |x_n - x| \wedge u \in U\} = \{n \in \mathbb{N} : (x - x_n) \in U\}. \tag{2}$$

Thus, we get $\mu\big(\{n \in \mathbb{N} : (x - x_n) \in U\}\big) = 1$ from (2), i.e., (x_n) statistically τ-converges to x. □

Recall that a band B in a vector lattice E is called a projection band whenever it satisfies $E = B \oplus B^d$, where $B^d = \{x \in E : |x| \wedge |b| = 0, \text{ for all } b \in B\}$ is disjoint complement set of B.

Proposition 2.8 *Let (E, τ) be a locally solid Riesz space and B be a projection band of E. If $x_n \xrightarrow{st-u_\tau} x$ in E then $P_B(x_n) \xrightarrow{st-u_\tau} P_B(x)$ in both E and B, where P_B is the corresponding order projection of B.*

Proof Let U be an arbitrary zero neighborhood. Then we have $\mu\big(\{n \in \mathbb{N} : (|x_n - x| \wedge u) \in U\}\big) = 1$ because of $x_n \xrightarrow{st-u_\tau} x$. It is known that every order projection is an order continuous lattice homomorphism (see [2, p.94]). Thus, by using Theorem 1.44. in [2], P_B is a lattice homomorphism and $0 \leq P_B \leq I$. Now, by applying Theorem 2.14. in [2], we can get $|P_B(x_n) - P_B(x)| \wedge u = P_B(|x_n - x|) \wedge u \leq |x_n - x| \wedge u$ for every $u \in E_+$. Then it follows easily from $\mu\big(\{n \in \mathbb{N} : (|P_B(x_n) - P_B(x)| \wedge u) \in U\}\big) = 1$ that $P_B(x_n) \xrightarrow{st-u_\tau} P_B(x)$ in E and B. □

We continue with several basic notions in locally solid vector lattice concerning the st-u_τ-convergence, which are motivated by their analogies from vector lattice theory.

Definition 2.9 *Let (E, τ) be a locally solid vector lattice. Then*

(1) a sequence (x_n) in E is said to be st-u_τ-Cauchy if the sequence $(x_m - x_n)_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ st-u_τ-converges to zero, i.e., for each zero neighborhood U, $\mu\big(\{n \in \mathbb{N} : (|x_m - x_n| \wedge u) \notin U\}\big) = 0$ for all $u \in E_+$;

(2) E is called order st-u_τ-continuous if $x_n \xrightarrow{o} 0$ implies $x_n \xrightarrow{st-u_\tau} 0$;

(3) E is called st-u_τ-complete if every st-u_τ-Cauchy sequence in E is st-u_τ-convergent.

The next proposition follows from the basic definitions and results, so its proof is omitted.

Proposition 2.10 *If a sequence (x_n) in a locally solid Riesz space (E, τ) is statistically u_τ-convergent then it is statistically u_τ-Cauchy.*

Theorem 2.11 *Let (E, τ) be a locally solid vector lattice and (x_n) be a sequence in E. Then E is order st-u_τ-continuous iff $x_n \downarrow 0$ implies $x_n \xrightarrow{st-u_\tau} 0$ in E.*

Proof The implication is obvious, so we show the converse direction. Suppose $x_n \downarrow 0$ implies $x_n \xrightarrow{st-u_\tau} 0$ in E. Take $x_n \xrightarrow{o} 0$ in E. We show $x_n \xrightarrow{st-u_\tau} 0$. By the order convergence of (x_n), there is another sequence $(z_n) \downarrow 0$ in E such that, for each n, there exists $n_k \in \mathbb{N}$ so that $|x_n| \leq z_n$ for all $n \geq n_k$. Thus, we have $z_n \xrightarrow{st-u_\tau} 0$ because of $z_n \downarrow 0$. Therefore, for arbitrary zero neighborhood U, we have $\mu\big(\{n \in \mathbb{N} : z_n \wedge u \in U\}\big) = 1$ for all $u \in E_+$. Since U is solid and $|x_n| \leq z_n$ for all $n \geq n_k$, we have $|x_n| \wedge u \in U$ for every $n \geq n_k$ and $u \in E_+$. As a result, $\mu\big(\{n \in \mathbb{N} : (|x_n| \wedge u) \in U\}\big) = 1$ for arbitrary U and for all $u \in E_+$. □

In the case of st-u_τ-complete locally solid vector lattice, we have the following result.

Theorem 2.12 *For an st-u_τ-complete Hausdorff locally solid vector lattice (E,τ), the following statements are equivalent:*

(i) *E is order st-u_τ-continuous;*

(ii) *if $0 \le x_n \uparrow \le x$ holds in E then (x_n) is an st-u_τ-Cauchy sequence;*

(iii) *$x_n \downarrow 0$ in E implies $x_n \xrightarrow{st-u_\tau} 0$.*

Proof $(i) \Rightarrow (ii)$: Let (x_n) be a positive increasing and bounded sequence in E_+. Then, by using Lemma 4.8. in [2], we have another sequence (y_k) in E such that $(y_k - x_n)_{(k,n)\in\mathbb{N}\times\mathbb{N}} \downarrow 0$. Thus, it follows from the assumption that we have $(y_k - x_n)_{(k,n)\in\mathbb{N}\times\mathbb{N}} \xrightarrow{st-u_\tau} 0$. Then, for any zero neighborhood U, we have $\mu\big(\{n \in \mathbb{N} : (|y_k - x_n| \wedge u) \in U\}\big) = 1$ for all $u \in E_+$ and for every $k, n \in \mathbb{N}$. By properties of $\mathcal{N}$, there is another zero neighborhood V such that $V + V \subseteq U$. So, by using the inequality $|x_n - x_m| \wedge u \le |x_n - y_k| \wedge u + |y_k - x_m| \wedge u \in V + V \subseteq U$, we have $|x_n - x_m| \wedge u \in U$. As a result, we get $\mu\big(\{n \in \mathbb{N} : (|x_n - x_m| \wedge u) \in U\}\big) = 1$ for all $u \in E_+$. It means that (x_n) is an st-u_τ-Cauchy sequence.

$(ii) \Rightarrow (iii)$: Assume that $x_n \downarrow 0$ is a sequence in E. Let's fix an arbitrary index n_0. Then, we have $x_n \le x_{n_0}$ whenever $n \ge n_0$. So, we get $0 \le (x_{n_0} - x_n)_{n \ge n_0} \uparrow \le x_{n_0}$. Thus, we can apply the condition (ii), and so the sequence $(x_{n_0} - x_n)_{n \ge n_0}$ is st-u_τ-Cauchy, i.e., $(x_n - x_{n'})_{(n,n')\in\mathbb{N}\times\mathbb{N}} \xrightarrow{st-u_\tau} 0$ as $n_0 \le n, n' \to \infty$. Now, it follows from the st-u_τ-completeness of E that there exists an element $x \in E$ such that $x_n \xrightarrow{st-u_\tau} x$ as $n_0 \le n \to \infty$. By Proposition 2.6, we have $x_n \downarrow x$, and so it is clear $x = 0$. As a result, we get $x_n \xrightarrow{st-u_\tau} 0$.

$(iii) \Rightarrow (i)$: It is just the implication of Theorem 2.11. □

Theorem 2.13 *Let (E,τ) be an st-u_τ-continuous and st-u_τ-complete Hausdorff locally solid vector lattice. Then E is order complete.*

Proof Let (x_n) be a positive, increasing and bounded sequence by a vector $e \in E_+$. Then by applying Theorem 2.12(ii), we see that (x_n) is an st-u_τ-Cauchy sequence because of st-u_τ-continuity of E. Then there is $x \in E$ such that $x_n \xrightarrow{st-u_\tau} x$ since E is st-u_τ-complete. It follows from Proposition 2.6 that $x_n \uparrow x$, and so E is order complete. □

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