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New multiple solutions for a Schrödinger–Poisson system involving concave-convex nonlinearities

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Abstract: In this paper, we study the following critical growth Schrödinger–Poisson system with concave-convex nonlinearities term

$$\begin{cases} -\Delta u + u + \eta\varphi u = \lambda f(x)u^{q-1} + u^5, & \text{in } \mathbb{R}^3, \\ -\Delta\varphi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{0.1}$$

where $1 < q < 2$, $\eta \in \mathbb{R}$, $\lambda > 0$ is a real parameter and $f \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ is a nonzero nonnegative function. Using the variational method, we obtain that there exists a positive constant $\lambda_* > 0$ such that for all $\lambda \in (0, \lambda_*)$, the system has at least two positive solutions.

Key words: Schrödinger–Poisson system, critical exponent, concave-convex nonlinearities

1. Introduction and main results

In this paper, we are interested in the existence of multiple positive solutions to the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + \eta\varphi u = \lambda f(x)u^{q-1} + u^5, & \text{in } \mathbb{R}^3, \\ -\Delta\varphi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $1 < q < 2$, $\eta \in \mathbb{R}$, $\lambda > 0$ is a real parameter and $f \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ is a nonzero nonnegative function. The first Schrödinger equation coupled with a Poisson equation means that the potential is determined by the charge of the wave function. The general term $f(x)u^{q-1}$ models the interaction between particles. The nonlocal term $\eta\varphi u$ concerns the interaction with the electric field. For detailed mathematical and physical interpretation, we refer readers to [2, 8, 9] and the references therein.

In recent years, the following form of Schrödinger–Poisson system with critical growth

$$\begin{cases} -\triangle u + V(x)u + K(x)\varphi u = Q(x)u^5 + f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle\varphi = K(x)u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

has been investigated extensively. For previous related results, please refer to [1, 4, 12, 15, 17–19, 23]. Further, the Schrödinger–Poisson system with concave-convex nonlinearities has attracted much attention. For example,

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2010 *AMS Mathematics Subject Classification:* 35J20; 35J60 on bounded domain, Guo and Liu in [12] were concerned about the following system

$$\begin{cases} -\Delta u + \omega u + \lambda\varphi u = \mu f(u) + u^5, & \text{in } \Omega, \\ -\Delta\varphi = u^2, & \text{in } \Omega, \\ \varphi = u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $|f(u)| \leq c_1 + c_2|u|^{q-1}$ with $1 < q < 2$ and infinitely many negative energy solutions were established for every $\mu > 0$ small enough and $\lambda > 0$. In [15], Lei and Suo established two positive solutions to the system

$$\begin{cases} -\Delta u + \lambda\varphi u = \lambda u^{q-1} + u^5, & \text{in } \Omega, \\ -\Delta\varphi = u^2, & \text{in } \Omega, \\ \varphi = u = 0, & \text{on } \partial\Omega, \end{cases}$$

for $\lambda > 0$ enough small when $1 < q < 2$.

On unbounded domains, Sun et al. in [21] considered the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda\varphi u = K(x)|u|^{q-2}u + f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\varphi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter, $1 < q < 2$ and $f(x,u)$ is linearly bounded in u at infinity. They established the existence and multiplicity of solutions (when λ is enough small) under suitable assumptions on V, K, f.

Recently, Li and Tang [18] proved the existence of $\lambda^* > 0$ such that the system

$$\begin{cases} -\Delta u - \eta l(x)\varphi u = \lambda f(x)|u|^{q-2}u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta\varphi = l(x)u^2, & \text{in } \mathbb{R}^3 \end{cases}$$

with $\eta = 1$, $1 < q < 2$, $f \in L^{\frac{6}{6-q}}(\mathbb{R}^3), f \geq 0, f \not\equiv 0$, $l \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3), l \geq 0, l \not\equiv 0$ has two positive solutions for $\lambda \in (0, \lambda^*)$. However, the authors did not consider the cases of $\eta = -1$ or on the whole space $H^1(\mathbb{R}^3)$. Indeed, it is very difficult to estimate the critical value level in the cases when $\eta = -1$ or on the space $H^1(\mathbb{R}^3)$ if we use the extremal function

$$\Psi(x) = \frac{3^{\frac{1}{2}}}{(1+|x|^2)^{\frac{1}{2}}}, \quad x \in \mathbb{R}^3.$$

Observing these results, it is natural to ask if system (1.1) has multiple positive solutions since system (1.1) has a concave and convex nonlinearity. In this paper, we study the existence of multiple positive solutions of system (1.1) in the case of $\eta \in \mathbb{R}$ through variational method. We have the following results.

Theorem 1.1 *Assume $\eta \in \mathbb{R}$, $1 < q < 2$ and $f \in L^{\frac{6}{6-q}}(\mathbb{R}^3), f \geq 0, f \not\equiv 0$. Then there exists a positive constant $\lambda_* > 0$ such that for all $\lambda \in (0, \lambda_*)$, system (1.1) has at least two positive solutions.*

Remark 1.2 *On one hand, as we shall see, Theorem 1.1 extends the result in [15] to unbounded domains. Moreover, we get rid of the restriction of the coefficient of the nonlocal term. On the other hand, our result extends the result in [18] to the whole space $H^1(\mathbb{R}^3)$ and two positive solutions are still obtained. Besides, assume $l \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), l \geq 0, l \not\equiv 0$ in $\mathbb{R}^3$, we have the following result.*

Corollary 1.3 *Assume* $1 < q < 2$ *and* $f \in L^{\frac{6}{6-q}}(\mathbb{R}^3), f \geq 0, f \not\equiv 0$. *Then there exists a positive constant* $\lambda_* > 0$ *such that for all* $\lambda \in (0, \lambda_*)$, *system*

$$\begin{cases} -\Delta u + l(x)\varphi u = \lambda f(x)|u|^{q-2}u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \varphi = l(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

has at least two positive solutions.

Throughout this paper, we make use of the following notation:

- The space in $H^1(\mathbb{R}^3)$ is equipped with the norm $\|u\| = (\int_{\mathbb{R}^3}(|\nabla u|^2 + |u|^2)dx)^{\frac{1}{2}}$, the norm in $L^p(\mathbb{R}^3)$ is denoted by $|u|_p = (\int_{\mathbb{R}^3}|u|^p dx)^{\frac{1}{p}}$;
- We denote by B_r (respectively, ∂B_r) the closed ball (respectively, the sphere) of center zero and radius r, i.e. $B_r = \{u \in H^1(\mathbb{R}^3) : \|u\| \leq r\}$, $\partial B_r = \{u \in H^1(\mathbb{R}^3) : \|u\| = r\}$;
- $C, C_0, C_1, C_2, ...$ denote various positive constants, which may vary from line to line;
- For each $p \in [2, 6)$, by the Sobolev embeddings, we denote

$$S_p = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|^2}{|u|_p^2}; \quad S := \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3}|\nabla u|^2 dx}{|u|_6^2}.$$

2. Proof of Theorem 1.1

With the help of the Lax–Milgram theorem, for every $u \in H^1(\mathbb{R}^3)$, the second equation of system (1.1) has a unique solution $\varphi_u \in H^1(\mathbb{R}^3)$. We substitute φ_u to the first equation of system (1.1), then system (1.1) transforms into the following equation

$$-\Delta u + u + \eta\varphi_u u = \lambda f(x)|u|^{q-2}u + |u|^4 u, \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

The energy functional corresponding to equation (2.1) is given by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\eta}{4}\int_{\mathbb{R}^3}\varphi_u|u|^2 dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} f(x)|u|^q dx - \int_{\mathbb{R}^3}|u|^6 dx.$$

So for all $u, v \in H^1(\mathbb{R}^3)$, it holds

$$\int_{\mathbb{R}^3}(\nabla u\nabla v + uv)dx + \eta\int_{\mathbb{R}^3}\varphi_u uv dx - \int_{\mathbb{R}^3}|u|^4 uv dx - \lambda\int_{\mathbb{R}^3} f(x)|u|^{q-2}uv dx = 0.$$

Before proving our Theorem 1.1, we need the following lemma (see [8, 10, 11]).

Lemma 2.1 *For every* $u \in H^1(\mathbb{R}^3)$, *there exists a unique* $\varphi_u \in D^{1,2}(\mathbb{R}^3)$ *solution of*

$$-\Delta\varphi = u^2, \text{in } \mathbb{R}^3$$

and the following results hold:

(1) $\|\varphi_u\|^2 = \int_{\mathbb{R}^3} \varphi_u u^2 dx$.

(2) $\varphi_u \geq 0$. *Moreover,* $\varphi_u > 0$ *when* $u \neq 0$.

(3) *For each* $t \neq 0$, $\varphi_{tu} = t^2 \varphi_u$. *For every* $u, v \in H^1(\mathbb{R}^3)$, *it holds*

$$\int_{\mathbb{R}^3} \varphi_u v^2 dx = \int_{\mathbb{R}^3} \varphi_v u^2 dx.$$

(4)

$$\int_{\mathbb{R}^3} \varphi_u u^2 dx = \int_{\mathbb{R}^3} |\nabla \varphi_u|^2 dx \leq C|u|_{12/5}^4.$$

(5) *Assume that* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$, *then* $\varphi_{u_n} \to \varphi_u$ *in* $H^1(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3} \varphi_{u_n} u_n v dx \to \int_{\mathbb{R}^3} \varphi_u u v dx$ *for every* $v \in H^1(\mathbb{R}^3)$.

(6) *Set* $\mathcal{F}(u) = \int_{\mathbb{R}^3} \varphi_u u^2 dx$, *then* $\mathcal{F} : H^1(\mathbb{R}^3) \to \mathbb{R}$ *is* C^1 *and*

$$\langle \mathcal{F}'(u), v \rangle = 4 \int_{\mathbb{R}^3} \varphi_u u v dx \quad \forall v \in H^1(\mathbb{R}^3).$$

Lemma 2.2 *There exist* $\rho, \Lambda_0 > 0$, *such that for each* $\lambda \in (0, \Lambda_0)$, *then it holds*

$$d \triangleq \inf_{u \in B_\rho(0)} I_\lambda(u) < 0 \quad \text{and} \quad I_\lambda|_{u \in \partial B_\rho(0)} > 0. \tag{2.2}$$

Proof By the Sobolev and Hölder inequalities, we obtain

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{\eta}{4}\int_{\mathbb{R}^3} \varphi_u |u|^2 dx - \frac{\lambda}{q}\int_{\mathbb{R}^3} f(x)|u|^q dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{6S_6^3}\|u\|^6 - \frac{\lambda}{q} S_6^{-\frac{q}{2}} |f|_{\frac{6}{6-q}} \|u\|^q \\ &= \|u\|^q \left\{ \frac{1}{2}\|u\|^{2-q} - \frac{1}{6S_6^3}\|u\|^{6-q} - \frac{\lambda}{q} S_6^{-\frac{q}{2}} |f|_{\frac{6}{6-q}} \right\}. \end{aligned}$$

Set $g(t) = \frac{1}{2}t^{2-q} - \frac{1}{6S_6^3}t^{6-q}$. We see that there exists a constant $\rho = \left(\frac{3S_6^3(2-q)}{6-q}\right)^{\frac{1}{4-q}}$ such that $\max_{t>0} g(t) = g(\rho) > 0$. Let $\Lambda_0 = \frac{qS_6^{\frac{q}{2}}}{2|f|_{\frac{6}{6-q}}} g(\rho)$. Consequently, $I_\lambda|_{\|u\|=\rho} \geq \frac{g(\rho)}{2}\rho^q$ for any $\lambda \in (0, \Lambda_0)$. Moreover, for $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, it holds

$$\begin{aligned} \lim_{t \to 0^+} \frac{I_\lambda(tu)}{t^q} &= -\frac{\lambda}{q}\int_{\mathbb{R}^3} f(x)|u|^q dx \\ &< 0. \end{aligned}$$

Thus, there exists u small enough such that $I_\lambda(u) \triangleq d < 0$. The proof is complete. □

Theorem 2.3 *Suppose* $0 < \lambda < \Lambda_0$ *(*Λ_0 *defined in Lemma 2.2). Then system* (1.1) *has a positive solution* $(u_1, \varphi_{u_1}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *satisfying* $I_\lambda(u_1) < 0$.

Proof First, we claim that there exists $u_1 \in B_R(0)$ such that $I_\lambda(u_1) = d < 0$.

Indeed, by (2.2), we can deduce that

$$\frac{1}{2}\|u\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx \geq \rho, \quad \text{for } u \in \partial B_R(0),$$

and

$$\frac{1}{2}\|u\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx \geq 0, \quad \text{for } u \in B_R(0). \tag{2.3}$$

In view of the definition of d, there exists a bounded minimizing sequence $\{u_n\} \subset B_R(0)$ such that $\lim_{n\to\infty} I_\lambda(u_n) = d < 0$. Up to a subsequence, there exists $u_\lambda \in H^1(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u_1, \quad \text{weakly in } H^1(\mathbb{R}^3), \quad u_n(x) \to u_1(x), \quad \text{a.e. in } \mathbb{R}^3.$$

From [18], it holds that

$$\int_{\mathbb{R}^3} f(x)|u_n|^q dx = \int_{\mathbb{R}^3} f(x)|u_1|^q dx + o(1).$$

Set $w_n = u_n - u_1$. By Brézis–Lieb's Lemma (see [6]), one has

$$\begin{cases} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_{\mathbb{R}^3} |\nabla w_n|^2 dx + \int_{\mathbb{R}^3} |\nabla u_1|^2 dx + o(1), \\ \int_{\mathbb{R}^3} u_n^6 dx = \int_{\mathbb{R}^3} w_n^6 dx + \int_{\mathbb{R}^3} u_1^6 dx + o(1). \end{cases}$$

If $u_1 = 0$, then $w_n = u_n$, which follows that $w_n \in B_R(0)$. If $u_1 \neq 0$, we also get $w_n \in B_R(0)$ for n large sufficiently. Hence, from (2.3) one has

$$\frac{1}{2}\|w_n\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx \geq 0. \tag{2.4}$$

Therefore, by Lemma 2.1, it follows from (2.4) that

$$\begin{aligned} d &= I_\lambda(u_n) + o(1) \\ &= I_\lambda(u_1) + \frac{1}{2}\|w_n\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} |w_n|^6 dx + o(1) \\ &\geq I_\lambda(u_1) + o(1). \end{aligned}$$

As $n \to \infty$, it holds that $d \geq I_\lambda(u_1)$. Since $B_R(0)$ is closed and convex; thus, $u_1 \in B_R(0)$. Hence, we obtain $I_\lambda(u_1) = d < 0$ and $u_1 \not\equiv 0$. It follows that u_1 is a local minimizer of I_λ. Then for any $\psi \in H^1(\mathbb{R}^3), \psi \geq 0$, setting $t > 0$ small enough such that $u_1 + t\psi \in B_R(0)$, one obtains

$$\begin{aligned} 0 &\leq I_\lambda(u_1 + t\psi) - I_\lambda(u_1) \\ &= \frac{1}{2}\|u_1 + t\psi\|^2 + \frac{\eta}{4}\int_{\mathbb{R}^3} \varphi_{u_1+t\psi}|u_1 + t\psi|^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u_1 + t\psi|^6 dx \\ &\quad -\frac{\lambda}{q}\int_{\mathbb{R}^3} f(x)|u_1 + t\psi|^q dx - \frac{1}{2}\|u_1\|^2 \\ &\quad -\frac{\eta}{4}\int_{\mathbb{R}^3} \varphi_{u_1}|u_\lambda|^2 dx + \frac{1}{6}\int_{\mathbb{R}^3} |u_1|^6 dx + \frac{\lambda}{q}\int_{\mathbb{R}^3} f(x)|u_1|^q dx. \end{aligned} \tag{2.5}$$

By Lemma 2.1, we have

$$\begin{aligned}
& \lim_{t\to 0^+}\frac{1}{t}\int_{\mathbb{R}^3}[\varphi_{u_1+t\psi}|u_1+t\psi|^2-\varphi_{u_1}|u_1|^2]dx \\
= & \lim_{t\to 0^+}\frac{1}{t}\int_{\mathbb{R}^3}[\varphi_{u_1+t\psi}|u_1+t\psi|^2-\varphi_{u_1}|u_1+t\psi|^2]dx \\
& +\lim_{t\to 0^+}\frac{1}{t}\int_{\mathbb{R}^3}[\varphi_{u_1}(u_1+t\psi)^2-\varphi_{u_1}|u_1|^2]dx \\
= & \lim_{t\to 0^+}\frac{1}{t}\int_{\mathbb{R}^3}[\varphi_{u_1+t\psi}(u_1+t\psi)^2-\varphi_{u_1+t\psi}u_1^2]dx \\
& +\lim_{t\to 0^+}\frac{1}{t}\int_{\mathbb{R}^3}[\varphi_{u_1}(u_1+t\psi)^2-\varphi_{u_1}u_1^2]dx \\
= & 4\int_{\mathbb{R}^3}\varphi_{u_1}u_1\psi dx.
\end{aligned}$$

Dividing by $t>0$ and passing to the limit as $t\to 0^+$ in (2.5), we get

$$\int_{\mathbb{R}^3}(\nabla u_1\nabla\psi+u_1\psi)dx+\eta\int_{\mathbb{R}^3}\varphi_{u_1}u_1\psi dx-\int_{\mathbb{R}^3}u_1^5\psi dx-\lambda\int_{\mathbb{R}^3}f(x)u_1^{q-2}\psi dx\geq 0.$$

By the arbitrariness of ψ, the above inequality also holds for $-\psi$. Hence, (u_1,φ_{u_1}) is a nonzero solution of system (1.1). Moreover, similar to the discussion of Theorem 3.3 in [14], we can also obtain that (u_1,φ_{u_1}) is a positive solution of system (1.1) with $I_\lambda(u_1)<0$. This completes the proof of Theorem 2.3. □

Lemma 2.4 *The functional I_λ satisfies the $(PS)_c$ condition provided $c<\frac{1}{3}S^{\frac{3}{2}}-D\lambda^{\frac{2}{2-q}}$, where $D=\left(\frac{(4-q)}{4q}|f|_{\frac{6}{6-q}}S_6^{-\frac{q}{2}}\right)^{\frac{2}{2-q}}(2q)^{\frac{q}{2-q}}$.*

Proof Let $\{v_n\}\subset H^1(\mathbb{R}^3)$ be a $(PS)_c$ sequence for I_λ, i.e.

$$I_\lambda(v_n)\to c,\quad I'_\lambda(v_n)\to 0,\qquad \text{as } n\to\infty. \tag{2.6}$$

Consequently,

$$\begin{aligned}
1+c+o(\|v_n\|) \geq\ & I_\lambda(v_n)-\frac{1}{6}\langle I'_\lambda(v_n),v_n\rangle \\
=\ & \frac{1}{3}\|v_n\|^2+\frac{1}{12}\int_{\mathbb{R}^3}\varphi_{v_n}|v_n|^2dx-\lambda\left(\frac{1}{q}-\frac{1}{6}\right)\int_{\mathbb{R}^3}f(x)|v_n|^q dx \\
\geq\ & \frac{1}{3}\|v_n\|^2-\lambda\left(\frac{1}{q}-\frac{1}{6}\right)S_6^{-\frac{q}{2}}|f|_{\frac{6}{6-q}}\|v_n\|^q,
\end{aligned}$$

which implies that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Note that $I_\lambda(v_n)=I_\lambda(|v_n|)$. Thus, there exist a nonnegative subsequence, still denoted by itself, and $v_*\in H^1(\mathbb{R}^3)$ such that $v_n\rightharpoonup v_*$ weakly in $H^1(\mathbb{R}^3)$ as $n\to\infty$. Set $w_n=v_n-v_*$, then $\int_{\mathbb{R}^3}|\nabla w_n|^2dx\to 0$. Otherwise, there exists a subsequence (still denoted by itself) such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla w_n|^2dx=l>0.$$

From (2.6), as $n\to\infty$, for every $\varphi\in H^1(\mathbb{R}^3)$, it follows

$$\int_{\mathbb{R}^3}(\nabla v_*\nabla\varphi+v_*\varphi)dx+\eta\int_{\mathbb{R}^3}\varphi_{v_*}v_*\varphi dx-\int_{\mathbb{R}^3}v_*^5\varphi dx-\lambda\int_{\mathbb{R}^3}f(x)v_*^{q-1}\varphi dx=0. \tag{2.7}$$

Take the test function $\varphi = v_*$ in (2.7), then it holds that

$$\|v_*\|^2 + \eta \int_{\mathbb{R}^3} \varphi_{v_*} v_*^2 dx - \int_{\mathbb{R}^3} v_*^6 dx - \lambda \int_{\mathbb{R}^3} f(x) v_*^{1-\gamma} dx = 0. \tag{2.8}$$

Putting $\varphi = v_n$ in (2.6) and using the Brézis–Lieb's lemma, we obtain

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^3} |\nabla w_n|^2 dx + \|v_*\|^2 + \eta \int_{\mathbb{R}^3} \varphi_{v_*} v_*^2 dx \\ &\quad - \int_{\mathbb{R}^3} w_n^6 dx - \int_{\mathbb{R}^3} v_*^6 dx - \lambda \int_{\mathbb{R}^3} f(x) v_*^q dx. \end{aligned} \tag{2.9}$$

It follows from (2.8) and (2.9) such that

$$\int_{\mathbb{R}^3} |\nabla w_n|^2 dx - \int_{\mathbb{R}^3} w_n^6 dx = o(1). \tag{2.10}$$

Note that $|w_n|_6^2 \leq S^{-1} \int_{\mathbb{R}^3} |\nabla w_n|^2 dx$. Then we have

$$l \geq S^{\frac{3}{2}}.$$

On the one hand, from (2.8), by the Young inequality, it holds

$$\begin{aligned} I_\lambda(v_*) &= \frac{1}{2}\|v_*\|^2 + \frac{\eta}{4} \int_{\mathbb{R}^3} \varphi_{v_*} v_*^2 dx - \frac{1}{6} \int_{\mathbb{R}^3} v_*^6 dx - \frac{\lambda}{q} \int_{\mathbb{R}^3} f(x) v_*^q dx \\ &= \frac{1}{4}\|v_*\|^2 + \frac{1}{12} \int_{\mathbb{R}^3} v_*^6 dx - \lambda \left(\frac{1}{q} - \frac{1}{4}\right) \int_{\mathbb{R}^3} f(x) v_*^q dx \\ &\geq \frac{1}{4}\|v_*\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{4}\right) |f|_{\frac{6}{6-q}} S_6^{-\frac{q}{2}} \|v_*\|^q \\ &\geq -D\lambda^{\frac{2}{2-q}}, \end{aligned}$$

where $D = \left(\frac{(4-q)}{4q}|f|_{\frac{6}{6-q}} S_6^{-\frac{q}{2}}\right)^{\frac{2}{2-q}} (2q)^{\frac{q}{2-q}}$. On the other hand, by (2.6) and (2.10), it follows

$$\begin{aligned} I_\lambda(v_*) &= I_\lambda(v_n) - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w_n|^2 dx + \frac{1}{6} \int_{\mathbb{R}^3} |w_n|^6 dx + o(1) \\ &= I_\lambda(v_n) - \frac{1}{3} \int_{\mathbb{R}^3} |\nabla w_n|^2 dx + o(1) \\ &\leq c - \frac{1}{3} l \\ &< \frac{1}{3} S^{\frac{3}{2}} - D\lambda^{\frac{2}{2-q}} - \frac{1}{3} S^{\frac{3}{2}} \\ &= -D\lambda^{\frac{2}{2-q}}. \end{aligned}$$

This is a contradiction. Therefore, $l = 0$ implies that $\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \to \int_{\mathbb{R}^3} |\nabla v_*|^2 dx$ in $D^{1,2}(\mathbb{R}^3)$. Note that

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^3} v_n^6 dx - \int_{\mathbb{R}^3} v_*^6 dx \\ &= \int_{\mathbb{R}^3} w_n^6 dx + o(1) \\ &= \int_{\mathbb{R}^3} |\nabla w_n|^2 dx + o(1) \\ &\to 0, \end{aligned}$$

which suggests that $\int_{\mathbb{R}^3} v_n^6 dx \to \int_{\mathbb{R}^3} v_*^6 dx$ as $n \to \infty$. The proof is complete. □

We know that the extremal function

$$U(x) = \frac{3^{\frac{1}{4}}}{(1+|x|^2)^{\frac{1}{2}}}, \quad x \in \mathbb{R}^3$$

solves

$$-\Delta u = u^5 \quad \text{in } \mathbb{R}^3 \backslash \{0\}$$

and $|\nabla U|_2^2 = |U|_6^6 = S^{\frac{3}{2}}$. We choose a function $\zeta \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leq \zeta(x) \leq 1$ in Ω. $\zeta(x) = 1$ near $x = 0$ and it is radially symmetric. We define

$$u_\varepsilon(x) = \varepsilon^{\frac{-1}{2}} \zeta(x) U\left(\frac{x}{\varepsilon}\right) = \frac{3^{\frac{1}{4}} \varepsilon^{\frac{1}{2}} \eta(x)}{(\varepsilon^2 + |x|^2)^{\frac{1}{2}}}.$$

Besides, since (u_1, φ_{u_1}) is a positive solution of system (1.1), we can see that there exist $m, M > 0$ such that $m \leq u_1 \leq M$ for each $x \in \text{supp}\zeta$. □

Lemma 2.5 *Under the assumptions of Theorem 1.1, it holds*

$$\sup_{t \geq 0} I_\lambda(u_1 + tu_\varepsilon) < \frac{1}{3} S^{\frac{3}{2}} - D\lambda^{\frac{2}{2-q}}. \tag{2.11}$$

Proof It is known (see [7]) that

$$\begin{cases} |u_\varepsilon|_6^6 = |U|_6^6 + O(\varepsilon^3) = S^{\frac{3}{2}} + O(\varepsilon^3); \\ \|u_\varepsilon\|^2 = |\nabla U|_2^2 + O(\varepsilon) = S^{\frac{3}{2}} + O(\varepsilon). \end{cases}$$

According to the definition of u_ε, it follows

$$\begin{aligned} \int_{\mathbb{R}^3} u_\varepsilon^p dx &= C\varepsilon^{\frac{p}{2}} \int_0^R \frac{r^2}{(\varepsilon^2 + r^2)^{p/2}} dx \\ &\leq C\varepsilon^{\frac{p}{2}} \int_0^R \frac{1}{r^{p-2}} dx \\ &= O(\varepsilon^{\frac{p}{2}}), \end{aligned} \tag{2.12}$$

where the last equality holds provided $p < 3$. For $1 < q < 2$, one has

$$(a+b)^q \geq a^q + qa^{q-1}b, \quad \text{for } a, b \geq 0.$$

It is obvious that the following inequality

$$(a+b)^6 \geq a^6 + b^6 + 6a^5 b + 6ab^5,$$

holds for each $a, b \geq 0$.

Using the inequalities above, for all $t \geq 0$, we obtain

$$
\begin{aligned}
& I_\lambda(u_1+tu_\varepsilon) \\
= \; & I_\lambda(u_1) + \frac{t^2}{2}\|u_\varepsilon\|^2 + t\int_{\mathbb{R}^3}\Big[(\nabla u_1, \nabla u_\varepsilon) + \eta\varphi_{u_1}u_1u_\varepsilon - u_1^5u_\varepsilon - \lambda f(x)u_1^{q-1}u_\varepsilon\Big]dx \\
& + \frac{\eta}{4}\int_{\mathbb{R}^3}[\varphi_{u_1+tu_\varepsilon}(u_1+tu_\varepsilon)^2 - \varphi_{u_1}u_1^2 - 4\varphi_{u_1}u_1(tu_\varepsilon)]dx \\
& - \frac{1}{6}\int_{\mathbb{R}^3}[|u_1+tu_\varepsilon|^6 - u_1^6 - 6u_1^5tu_\varepsilon]dx - \frac{\lambda}{q}\int_{\mathbb{R}^3}f(x)[(u_1+tu_\varepsilon)^q - u_1^q - qu_\lambda^{q-1}tu_\varepsilon]dx \\
\leq \; & \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^6}{6}\int_{\mathbb{R}^3}|u_\varepsilon|^6dx - t^5\int_{\mathbb{R}^3}u_1|u_\varepsilon|^5dx + g_\varepsilon(t) \\
\leq \; & \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^6}{6}\int_{\mathbb{R}^3}|u_\varepsilon|^6dx - mt^5\int_{\mathbb{R}^3}|u_\varepsilon|^5dx + g_\varepsilon(t),
\end{aligned}
$$

where

$$
g_\varepsilon(t) = \frac{\eta}{4}\int_{\mathbb{R}^3}[\varphi_{u_1+tu_\varepsilon}(u_1+tu_\varepsilon)^2 - \varphi_{u_1}u_1^2 - 4\varphi_{u_1}u_1(tu_\varepsilon)]dx.
$$

By Lemma 2.1 (3), we have

$$
\begin{aligned}
& \int_{\mathbb{R}^3}[\varphi_{u_1+tu_\varepsilon}(u_1+tu_\varepsilon)^2 - \varphi_{u_1}u_1^2 - 4\varphi_{u_1}u_1(tu_\varepsilon)]dx \\
= \; & \int_{\mathbb{R}^3}[\varphi_{u_1+tu_\varepsilon}(u_1^2 + 2tu_1u_\varepsilon + t^2u_\varepsilon^2) - \varphi_{u_1}u_1^2 - 4\varphi_{u_1}u_1(tu_\varepsilon)]dx \\
= \; & \int_{\mathbb{R}^3}[\varphi_{u_1+tu_\varepsilon}u_1^2 + 2t\varphi_{u_1+tu_\varepsilon}u_1u_\varepsilon + t^2\varphi_{u_1+tu_\varepsilon}u_\varepsilon^2 - \varphi_{u_1}u_1^2 - 4\varphi_{u_1}u_1(tu_\varepsilon)]dx \\
= \; & \int_{\mathbb{R}^3}[\varphi_{u_1}(u_1+tu_\varepsilon)^2 + 2t\varphi_{u_1+tu_\varepsilon}u_1u_\varepsilon + t^2\varphi_{u_\varepsilon}(u_1+tu_\varepsilon)^2 - \varphi_{u_1}u_1^2 - 4\varphi_{u_1}u_1(tu_\varepsilon)]dx \\
= \; & \int_{\mathbb{R}^3}[\varphi_{u_1}(u_1^2 + 2tu_1u_\varepsilon + t^2u_\varepsilon^2) + t^2\varphi_{u_\varepsilon}(u_1^2 + 2tu_1u_\varepsilon + t^2u_\varepsilon^2) - \varphi_{u_1}u_1^2 - 4\varphi_{u_1}u_1(tu_\varepsilon)]dx \\
& + \int_{\mathbb{R}^3}\frac{2tu_1u_\varepsilon}{4\pi}\int_{\mathbb{R}^3}\frac{[u_1(y)+tu_\varepsilon(y)]^2}{|x-y|}dydx \\
= \; & \int_{\mathbb{R}^3}[t^2\varphi_{u_1}u_\varepsilon^2 + t^2\varphi_{u_1}u_\varepsilon^2 + 2t^3\varphi_{u_\varepsilon}u_1u_\varepsilon + t^4\varphi_{u_\varepsilon}u_\varepsilon^2 - 2\varphi_{u_1}u_1(tu_\varepsilon)]dx \\
& + \int_{\mathbb{R}^3}\frac{2tu_1u_\varepsilon}{4\pi}\int_{\mathbb{R}^3}\frac{u_1^2(y) + 2tu_1(y)u_\varepsilon(y) + t^2u_\varepsilon^2(y)}{|x-y|}dydx \\
= \; & \int_{\mathbb{R}^3}[2t^2\varphi_{u_1}u_\varepsilon^2 + 2t^3\varphi_{u_\varepsilon}u_1u_\varepsilon + t^4\varphi_{u_\varepsilon}u_\varepsilon^2 - 2\varphi_{u_1}u_1(tu_\varepsilon)]dx \\
& + \int_{\mathbb{R}^3}2tu_1u_\varepsilon[\varphi_{u_1} + t^2\varphi_{u_\varepsilon}]dx + \frac{t^2}{\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u_1(x)u_\varepsilon(x)u_1(y)u_\varepsilon(y)}{|x-y|}dxdy \\
= \; & \int_{\mathbb{R}^3}[2t^2\varphi_{u_1}u_\varepsilon^2 + 4t^3\varphi_{u_\varepsilon}u_1u_\varepsilon + t^4\varphi_{u_\varepsilon}u_\varepsilon^2]dx + \frac{t^2}{\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u_1(x)u_\varepsilon(x)u_1(y)u_\varepsilon(y)}{|x-y|}dxdy.
\end{aligned}
$$

It follows from Hölder's inequality and (2.12) such that

$$
\int_{\mathbb{R}^3}\varphi_{u_1}u_\varepsilon^2dx \leq |\varphi_{u_1}|_6|u_\varepsilon|_{12/5}^2 \leq C|u_\varepsilon|_{12/5}^2 \leq C\varepsilon;
$$

$$
\int_{\mathbb{R}^3}\varphi_{u_\varepsilon}u_1u_\varepsilon dx \leq |\varphi_{u_\varepsilon}|_6|u_1|_{12/5}|u_\varepsilon|_{12/5} \leq C|u_\varepsilon|_{12/5}^3 \leq C\varepsilon^{\frac{3}{2}};
$$

$$
\int_{\mathbb{R}^3}\varphi_{u_\varepsilon}u_\varepsilon^2dx \leq |\varphi_{u_\varepsilon}|_6|u_\varepsilon|_{12/5}^2 \leq C|u_\varepsilon|_{12/5}^4 \leq C\varepsilon^2.
$$

According to [20], we have

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u_1(x)u_\varepsilon(x)u_1(y)u_\varepsilon(y)}{|x-y|}dxdy\le C\varepsilon.$$

Therefore, one obtains

$$g_\varepsilon(t)\le Ct^2\varepsilon+Ct^3\varepsilon^{\frac{3}{2}}+Ct^4\varepsilon^2.$$

Set

$$h_\varepsilon(t)=\frac{t^2}{2}\|u_\varepsilon\|^2-\frac{t^6}{6}\int_{\mathbb{R}^3}|u_\varepsilon|^6dx-mt^5\int_{\mathbb{R}^3}|u_\varepsilon|^5dx+Ct^2\varepsilon+Ct^3\varepsilon^{\frac{3}{2}}+Ct^4\varepsilon^2.$$

Now, we prove that there exist $t_\varepsilon>0$ and positive constants t_1,t_2 independent of ε,λ, such that $\sup_{t\ge0}h_\varepsilon(t)=h_\varepsilon(t_\varepsilon)$ and

$$0<t_1\le t_\varepsilon\le t_2<\infty. \tag{2.13}$$

Indeed, we see that $\lim_{t\to+\infty}h_\varepsilon(t)=-\infty$ and $h_\varepsilon(0)=0$. Then there exists $t_\varepsilon>0$ such that

$$h_\varepsilon(t_\varepsilon)=\sup_{t\ge0}h_\varepsilon(t),\quad\text{and } h'_\varepsilon(t)|_{t=t_\varepsilon}=0.$$

It is similar to the paper [16] that (2.13) holds. Note that $\int_{\mathbb{R}^3}|u_\varepsilon|^5dx=C\varepsilon^{\frac{1}{2}}+O(\varepsilon^{\frac{5}{2}})$. Then it follows

$$\begin{aligned}\sup_{t\ge0}h_\varepsilon(t)&\le\sup_{t\ge0}\left\{\frac{t^2}{2}S^{\frac{3}{2}}-\frac{t^6}{6}S^{\frac{3}{2}}\right\}+C_1\varepsilon-C_2\varepsilon^{\frac{1}{2}}\\&\le\frac{1}{3}S^{\frac{3}{2}}+C_1\varepsilon-C_2\varepsilon^{\frac{1}{2}},\end{aligned}$$

where $C_1,C_2>0$ (independent of ε,λ). Let $\varepsilon=\lambda^{\frac{2}{2-q}}$, $0<\lambda<\Lambda_1:=\left(\frac{C_2}{C_1+D}\right)^{2-q}$. Then it holds that

$$\begin{aligned}C_1\varepsilon-C_2\varepsilon^{\frac{1}{2}}&=C_1\lambda^{\frac{2}{2-q}}-C_2\lambda^{\frac{1}{2-q}}\\&=\lambda^{\frac{2}{2-q}}(C_1-C_2\lambda^{\frac{-1}{2-q}})\\&<-D\lambda^{\frac{2}{2-q}},\end{aligned}$$

which implies that $\sup_{t\ge0}h_\varepsilon(t)<\frac{1}{3}S^{\frac{3}{2}}-D\lambda^{\frac{2}{2-q}}$. From the above information, we can deduce that (2.11) holds true when $\lambda<\Lambda_1$. The proof is complete. □

Theorem 2.6 *Under the assumptions of Theorem 1.1, the system (1.1) admits another positive solution* (u_2,φ_{u_2}) *with* $I_\lambda(u_2)>0$.

Proof Let $\lambda_*=\min\left\{\Lambda_0,\Lambda_1,(\frac{S^{\frac{3}{2}}}{3D})^{\frac{2-q}{2}}\right\}$. By Lemma 2.5, we can choose a sufficiently large $T_0>0$ such that $I_\lambda(u_1+T_0u_\varepsilon)<0$, with the fact that $I_\lambda(u_1)<0$. Then we apply the mountain-pass lemma [5] to obtain that there is a sequence $\{u_n\}\subset H^1(\mathbb{R}^3)$ such that

$$I_\lambda(u_n)\to c>0\quad\text{and}\quad I'_\lambda(u_n)\to0,$$

where

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_\lambda(\gamma(t)),$$

and

$$\Gamma = \left\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = u_1, \gamma(1) = u_1 + T_0 u_\varepsilon\right\}.$$

From Lemma 2.4, $\{u_n\}$ has a convergent subsequence (still denoted by $\{u_n\}$) and there exists $u_2 \in H^1(\mathbb{R}^3)$ such that $u_n \to u_2$ in $H^1(\mathbb{R}^3)$. Moreover, we can obtain (u_2, φ_{u_2}) is a nonnegative weak solution of system (1.1) and

$$I_\lambda(u_2) = \lim_{n\to\infty} I_\lambda(u_n) = c > 0.$$

Therefore, we infer that $u_2 \not\equiv 0$. It is similar to Theorem 2.3 that $u_2 > 0$ in $\mathbb{R}^3$. The proof is complete. □

3. Proof of Corollary 1.3

Proof From Lemma 2.5, when estimating the critical value level, we know that the nonlocal term does not work. Consequently, similar to the proof of Theorem 2.3 and 2.6, we can prove that system (1.2) has two positive solutions. The proof is complete. □

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