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Higher-order character Dedekind sum

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Abstract: In this paper, we are interested in higher-order character Dedekind sum

$$\sum_{v=0}^{ck-1} \chi_1(v)\, \mathcal{B}_{p,\chi_2}\left(a\frac{v+z}{c}+x\right) \mathcal{B}_q\left(b\frac{v+z}{ck}+y\right),\ a,b,c\in\mathbb{N} \text{ and } x,y,z\in\mathbb{R},$$

where χ_1 and χ_2 are primitive characters of modulus k, $\mathcal{B}_p(x)$ and $\mathcal{B}_{p,\chi_2}(x)$ are Bernoulli and generalized Bernoulli functions, respectively. We employ the Fourier series technique to demonstrate reciprocity formulas for this sum. Derived formulas are analogues of Mikolás' reciprocity formula. Moreover, we offer Petersson–Knopp type identities for this sum.

Key words: Dedekind sum, Bernoulli polynomials, Fourier series

1. Introduction

Let $\mathcal{B}_n(x)$ denote the nth Bernoulli function defined by

$$\mathcal{B}_n(x) = \begin{cases} B_n(x-[x]), & \text{if } n\neq 1 \text{ or } x\notin\mathbb{Z}, \\ 0, & \text{if } n=1 \text{ and } x\in\mathbb{Z}, \end{cases}$$

where $[x]$ denotes the largest integer $\leq x$ and $B_n(x)$ is the nth Bernoulli polynomial defined by (see [3, p. 264])

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

The classical Dedekind sum $s(a,c)$, arising in the theory of Dedekind eta-function, is defined by

$$s(a,c) = \sum_{v=0}^{c-1} \mathcal{B}_1\left(\frac{av}{c}\right) \mathcal{B}_1\left(\frac{v}{c}\right),\ a,c\in\mathbb{Z},\ c>0.$$

This sum was generalized by various authors. One of the generalizations, due to Hall et al [25], is the higher-order Dedekind sum (or generalized Dedekind–Rademacher sum)

$$s_{p,q}\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} = \sum_{v=0}^{c-1} \mathcal{B}_p\left(a\frac{v+z}{c}-x\right) \mathcal{B}_q\left(b\frac{v+z}{c}-y\right). \tag{1.1}$$

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This sum contains generalized Dedekind sums previously defined by Carlitz [17] (see also [8, 40])

$$s_p(a,c:x,y)=\sum_{v=0}^{c-1}\left(\left(\frac{v+y}{c}\right)\right)\mathcal{B}_p\left(a\frac{v+y}{c}+x\right), \tag{1.2}$$

Rademacher [37]

$$s(a,c:x,y)=\sum_{v=0}^{c-1}\left(\left(a\frac{v+y}{c}+x\right)\right)\left(\left(\frac{v+y}{c}\right)\right), \tag{1.3}$$

Mikolás [32] (see also [16])

$$s_{p,q}(a,b;c)=\sum_{v=0}^{c-1}\mathcal{B}_p\left(\frac{av}{c}\right)\mathcal{B}_q\left(\frac{bv}{c}\right) \tag{1.4}$$

and Apostol [2]

$$s_p(a,c)=\sum_{v=0}^{c-1}\left(\left(\frac{v}{c}\right)\right)\mathcal{B}_p\left(\frac{av}{c}\right). \tag{1.5}$$

Another generalization of $s(a,c)$ is character Dedekind sum introduced by Berndt [7]: Let χ be a nonprinciple primitive character modulo k. The generalized Bernoulli functions $\mathcal{B}_{n,\chi}(x)$ are functions with period k and can be defined by [9, Theorem 3.1]

$$\mathcal{B}_{n,\chi}(x)=k^{n-1}\sum_{v=1}^{k-1}\overline{\chi}(v)\mathcal{B}_n\left(\frac{v+x}{k}\right), \tag{1.6}$$

for all real x and $n\geq 0$. Berndt's character Dedekind sum is [7]

$$s(a,c;\chi)=\sum_{v=0}^{ck-1}\chi(v)\,\mathcal{B}_{1,\chi}\left(\frac{av}{c}\right)\mathcal{B}_1\left(\frac{v}{ck}\right), \tag{1.7}$$

which appears in the transformation formulas of a generalized Eisenstein series. Generalizations of $s(a,c;\chi)$ in the sense of (1.2), (1.3) and (1.5) have also been studied in [7, 19, 20].

For further generalizations of $s(a,c)$, the reader may consult to [5, 6, 10–12, 14, 18, 21, 22, 26, 28, 31, 33, 39, 42].

One of the most important properties of the Dedekind sum is its reciprocity formula: it plays a key role in proving a bias phenomena [1], distribution properties [27] and unboundedness [36] of the sum.

It should be mentioned that Hall and Wilson [24] classified all linear relations (reciprocity formulas) for the sums $s_{p,q}(a,b;c)$ and $s_{p,q}(a,1;c)$, and it emerged that Mikolás' relations form a complete set [32, Eq. (5.5)] (see also [24, Eq. (8)]). Moreover, reciprocity formulas of Hall et al [25], Bayad and Raouj [5] and Beck and Chavez [6] are in terms of the generating functions.

The reciprocity formula for the sum $s(a,c)$, due to Richard Dedekind, is

$$s(a,c)+s(c,a)=-\frac{1}{4}+\frac{1}{12}\left(\frac{a}{c}+\frac{c}{a}+\frac{1}{ac}\right), \tag{1.8}$$

where $(a,c)=1$ (for several proofs see [38]), whereas the reciprocity formula for the sum $s(a,c;\chi)$ is [7, Theorem 4]

$$s\left(c,a;\chi\right)+s\left(a,c;\overline{\chi}\right)=B_{1,\chi}B_{1,\overline{\chi}},$$

where $(a,c)=1$ and either a or $c\equiv 0\,(\mathrm{mod}\,k)$.

Another basic identity for Dedekind sum is the Petersson–Knopp identity [29]:

$$\sum_{d|n}\sum_{r=1}^{d}s\left(\frac{n}{d}a+rc,dc\right)=\sigma\left(n\right)s\left(a,c\right),\tag{1.9}$$

where $\sigma\left(n\right)$ is the sum of the positive divisors of n. Elementary proofs of this identity have been offered by several authors [23, 34]. Analogues of (1.9) have also been given for certain generalized Dedekind sums, for example, [4, 30, 35, 42].

In this paper, we are interested in character extension of (1.1), i.e., higher-order character Dedekind sum

$$s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right)=\sum_{v=0}^{ck-1}\chi_1\left(v\right)\mathcal{B}_{p,\chi_2}\left(a\frac{v+z}{c}+x\right)\mathcal{B}_q\left(b\frac{v+z}{ck}+y\right).$$

It is worth noting that the sums

$$s_{q,p+1-q}\left(a,1,c;\chi_1,\chi_2;x,0,z\right),\ \ 0\le q\le p+1,$$

can be viewed as coefficients appearing in Berndt's transformation formula [10, Theorem 1] although not explicitly stated. So the sum $s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right)$ can be viewed as a natural generalization of the character Dedekind sum $s(a,c;\chi)$.

The reciprocity formulas in this concept are proved by employing various techniques and theories such as transformation formulas, residue theory, generating functions, summation formulas and arithmetic methods.

In this study, we employ the Fourier series technique to demonstrate reciprocity formulas for the sum $s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right)$. Derived formulas are analogues of Mikolás' relation [32, Eq. (5.5)] (see also [24, Eq. (8)]) and are presented in Section 2. Moreover, in Section 3, we offer Petersson–Knopp type identities for the sum $s_{p,q}(a,b,c;\chi_1,\chi_2;0,0,0)$.

2. Reciprocity formulas

In this section, we prove two reciprocity formulas for higher-order character Dedekind sum. Let

$$s^*_{p,q}\left(a,c,b;\chi_2,\chi_1;x,z,y\right)=\sum_{v=0}^{b-1}\mathcal{B}_{p,\chi_2}\left(ak\frac{v+y}{b}+x\right)\mathcal{B}_{q,\chi_1}\left(ck\frac{v+y}{b}+z\right).$$

The first reciprocity formula is given by the following theorem.

Theorem 2.1 *Let a, b and c be pairwise coprime positive integers and $a \equiv 0 \,(\operatorname{mod} k)$. Let χ_1 and χ_2 be nonprincipal primitive characters of modulus k. Then, for $x, y, z \in \mathbb{R}$*

$$\begin{aligned}
&s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right)\\
&= q\left(b/ck\right)^{q-1}\sum_{j=0}^{p-1}\binom{p}{j}\frac{\left(a/c\right)^{j}}{q+j}s^{*}_{q+j,p-j}\left(c,-a,b;\overline{\chi}_1,\chi_2;z,x,y\right)\\
&+pk^{q-p}\left(a/c\right)^{p-1}\overline{\chi}_1\left(b\right)\chi_1\left(-c\right)\\
&\quad\times\sum_{l=0}^{q-1}\binom{q}{l}\frac{\left(bk/c\right)^{l}}{p+l}\left(-1\right)^{q-l}s_{q-l,p+l}\left(b,c,a;\overline{\chi}_2,\overline{\chi}_1;-ky,z/k,x\right)+E_{p,q},
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
E_{p,q} &= E_{p,q}\left(x,y,z;\chi_1,\chi_2\right)\\
&= \begin{cases} -\dfrac{1}{4}\chi_1(-z_1)\overline{\chi}_2(-x_1), & \text{if } p=q=1 \text{ and}\\ & (x,y,z)\in(-ak,-b,ck)\,\mathbb{R}+\mathbb{Z}^3,\\ 0, & \text{otherwise,}\end{cases}
\end{aligned} \tag{2.2}$$

here $(x_1,y_1,z_1) = (x+akR, y+bR, z-ckR) \in \mathbb{Z}^3$ for some $R \in \mathbb{R}$.

Proof We first recall the Fourier series representations of the Bernoulli functions $\mathcal{B}_n(x)$ and $\mathcal{B}_{n,\chi}(x)$:

$$\mathcal{B}_n\left(x\right) = A_n \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{e^{2\pi imx}}{m^n},\ \text{(see [3, p. 267])} \tag{2.3}$$

$$\mathcal{B}_{n,\chi}\left(x\right) = A_n\left(\chi\right) \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{\chi\left(m\right)}{m^n}e^{2\pi imx/k},\ \text{([9, p. 421])} \tag{2.4}$$

for all real x and $n \geq 1$. Here $A_n = -n!/(2\pi i)^n$, $A_n(\chi) = -G(\overline{\chi})\,n!/k(2\pi i/k)^n$ and $G(\chi) = G(1,\chi)$ is the Gauss sum defined by

$$G(z,\chi) = \sum_{h=0}^{k-1}\chi(h)e^{2\pi ihz/k}.$$

$G(z,\chi)$ satisfies [3, p. 168] (χ primitive)

$$G(n,\chi) = \overline{\chi}(n)G(\chi),\ n\in\mathbb{Z}. \tag{2.5}$$

We want to perform (2.3) and (2.4) on the right-hand side of

$$s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right) = \sum_{\lambda=0}^{ck-1}\chi_1\left(\lambda\right)\mathcal{B}_{p,\chi_2}\left(a\frac{\lambda+z}{c}+x\right)\mathcal{B}_q\left(b\frac{\lambda+z}{ck}+y\right).$$

Note that the left-hand sides of (2.3) and (2.4) are in the form

$$\lim_{\delta\to 0^+}\frac{f\left(X+\delta\right)+f\left(X-\delta\right)}{2} = f\left(X\right)$$

(the functions $\mathcal{B}_n(X)$ and $\mathcal{B}_{n,\chi}(X)$ have discontinuity at $X \in \mathbb{Z}$ when $n=1$, with

$$\begin{aligned} &\lim\nolimits_{\delta\to 0^+} \mathcal{B}_1(X \mp \delta) = \pm 1/2, \text{ for } X \in \mathbb{Z}, \\ &\lim\nolimits_{\delta\to 0^+} \mathcal{B}_{1,\chi}(X \mp \delta) = \mathcal{B}_{1,\chi}(X) \pm \frac{\overline{\chi}(-X)}{2}, \text{ for } X \in \mathbb{Z}, \end{aligned} \tag{2.6}$$

and otherwise are continuous). It is clear that

$$\begin{aligned} &\mathcal{B}_{p,\chi}(X)\mathcal{B}_q(Y) \\ &= \lim_{\delta\to 0^+} \frac{1}{2}\left\{\mathcal{B}_{p,\chi}(X+\delta\alpha) + \mathcal{B}_{p,\chi}(X-\delta\alpha)\right\} \frac{1}{2}\left\{\mathcal{B}_q(Y+\delta\beta) + \mathcal{B}_q(Y-\delta\beta)\right\} \\ &= \lim_{\delta\to 0^+} \frac{1}{2}\left\{\mathcal{B}_{p,\chi}(X+\delta\alpha)\mathcal{B}_q(Y+\delta\beta) + \mathcal{B}_{p,\chi}(X-\delta\alpha)\mathcal{B}_q(Y-\delta\beta)\right\} \\ &\quad + \lim_{\delta\to 0^+} \frac{1}{4}\left\{\mathcal{B}_{p,\chi}(X-\delta\alpha) - \mathcal{B}_{p,\chi}(X+\delta\alpha)\right\}\left\{\mathcal{B}_q(Y+\delta\beta) - \mathcal{B}_q(Y-\delta\beta)\right\}, \end{aligned} \tag{2.7}$$

where $0 \le \alpha, \beta \in \mathbb{R}$. Then, (2.6) and (2.7) entail that

$$\begin{aligned} &\lim_{\delta\to 0^+} \frac{1}{2}\left\{\mathcal{B}_{p,\chi}(X+\delta\alpha)\mathcal{B}_q(Y+\delta\beta) + \mathcal{B}_{p,\chi}(X-\delta\alpha)\mathcal{B}_q(Y-\delta\beta)\right\} \\ &= \mathcal{B}_{p,\chi}(X)\mathcal{B}_q(Y) + C_{p,q}(X,Y;\chi), \end{aligned} \tag{2.8}$$

where

$$C_{p,q}(X,Y;\chi) = \begin{cases} \dfrac{\overline{\chi}(-X)}{4}, & \text{if } p=q=1 \text{ and } (X,Y) \in \mathbb{Z}^2, \\ 0, & \text{otherwise.} \end{cases}$$

- **The case** $p \neq 1$ **or** $q \neq 1$ **or** $(x,y,z) \notin (-ak,-b,ck)\mathbb{R} + \mathbb{Z}^3$.

$((a(\lambda+z)/c + x, b(\lambda+z)/ck + y) \in \mathbb{Z}^2 \Rightarrow x = -akR + x_1,\ y = -bR + y_1$ and $z = ckR + z_1$ for some $R \in \mathbb{R}$ and $x_1, y_1, z_1 \in \mathbb{Z}.)$

Setting $\lambda = v + jk,\ 0 \le v < k,\ 0 \le j < c$ we have

$$\begin{aligned} s_{p,q}(a,b,c;\chi_1,\chi_2;x,y,z) &= \sum_{\lambda=0}^{ck-1} \chi_1(\lambda)\mathcal{B}_{p,\chi_2}\left(a\frac{\lambda+z}{c} + x\right)\mathcal{B}_q\left(b\frac{\lambda+z}{ck} + y\right) \\ &= A_p(\chi_2) A_q \sum_{\lambda=0}^{ck-1} \chi_1(\lambda) \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} \frac{\chi_2(m)}{m^p r^q} e^{2\pi i\left((ma+rb)\frac{\lambda+z}{kc} + \frac{mx}{k} + ry\right)} \\ &= A_p(\chi_2) A_q \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} \frac{\chi_2(m)}{m^p r^q} e^{2\pi i\left(\frac{ma+rb}{kc}z + \frac{mx}{k} + ry\right)} \\ &\quad \times \sum_{v=0}^{k-1} \chi_1(v) e^{2\pi i(ma+rb)v/ck} \sum_{j=0}^{c-1} e^{2\pi i(ma+rb)j/c}. \end{aligned} \tag{2.9}$$

by (2.3) and (2.4). Since

$$\sum_{j=0}^{c-1} e^{2\pi i(ma+rb)j/c} = \begin{cases} c, & ma+rb \equiv 0 \,(\operatorname{mod} c)\,, \\ 0, & ma+rb \not\equiv 0 \,(\operatorname{mod} c)\,, \end{cases}$$

we have

$$\begin{aligned} & s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right) \\ & = cb^q A_p\left(\chi_2\right) A_q \sum_{ma\neq nc}{}' \frac{\chi_2\left(m\right) e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k}}{m^p\left(nc-ma\right)^q} \sum_{v=1}^{k-1} \chi_1\left(v\right) e^{2\pi i n v/k} \\ & = cb^q G\left(\chi_1\right) A_p\left(\chi_2\right) A_q \sum_{ma\neq nc}{}' \frac{\chi_2\left(m\right)\overline{\chi}_1\left(n\right)}{m^p\left(nc-ma\right)^q} e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k}, \end{aligned} \tag{2.10}$$

where we have used (2.5). Here and in the sequel, we write

$$\sum{}' \text{ for } \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} = \lim_{M\to\infty} \sum_{\substack{m=-M \\ m\neq 0}}^{M} \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N}.$$

We now utilize the partial fractions

$$\frac{1}{x^p\left(1-x\right)^q} = \sum_{j=0}^{p-1} \frac{\alpha_{p-j}}{x^{p-j}} + \sum_{l=0}^{q-1} \frac{\beta_{q-l}}{\left(1-x\right)^{q-l}}$$

in (2.10). Simple calculation gives $\alpha_{p-j} = \binom{q+j-1}{j}$, $0 \leq j < p$ and $\beta_{q-l} = \binom{p+l-1}{l}$, $0 \leq l < q$. Then,

$$\begin{aligned} & \sum_{ma\neq nc}{}' \frac{\chi_2\left(m\right)\overline{\chi}_1\left(n\right)}{m^p\left(nc-ma\right)^q} e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k} \\ & = a^p \sum_{ma\neq nc}{}' \frac{\chi_2\left(m\right)\overline{\chi}_1\left(n\right)}{\left(\frac{ma}{nc}\right)^p\left(1-\frac{ma}{nc}\right)^q\left(nc\right)^{p+q}} e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k} \\ & = a^p \sum_{ma\neq nc}{}' \frac{\chi_2\left(m\right)\overline{\chi}_1\left(n\right)}{\left(nc\right)^{p+q}} e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k} \left(\sum_{j=0}^{p-1} \frac{\alpha_{p-j}}{\left(\frac{ma}{nc}\right)^{p-j}} + \sum_{l=0}^{q-1} \frac{\beta_{q-l}}{\left(1-\frac{ma}{nc}\right)^{q-l}}\right) \\ & = a^p \sum_{j=0}^{p-1} \frac{\alpha_{p-j}}{a^{p-j}c^{q+j}} \sum_{ma\neq nc}{}' \frac{\chi_2\left(m\right)\overline{\chi}_1\left(n\right)}{m^{p-j}n^{q+j}} e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k} \\ & \quad + a^p \sum_{l=0}^{q-1} \frac{\beta_{q-l}}{c^{p+l}} \sum_{ma\neq nc}{}' \frac{\overline{\chi}_1\left(n\right)\chi_2\left(m\right)}{n^{p+l}\left(nc-ma\right)^{q-l}} e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k}. \end{aligned} \tag{2.11}$$

Using the Fourier series representation of $\mathcal{B}_{n,\chi}\left(x\right)$ we see that

$$\begin{aligned} & \sum_{ma\neq nc}{}' \frac{\chi_2\left(m\right)\overline{\chi}_1\left(n\right)}{m^{p-j}n^{q+j}} e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k} \\ & \quad = \frac{\mathcal{B}_{p-j,\chi_2}\left(x-\frac{aky}{b}\right)\mathcal{B}_{q+j,\overline{\chi}_1}\left(\frac{cky}{b}+z\right)}{A_{p-j}\left(\chi_2\right)A_{q+j}\left(\overline{\chi}_1\right)} - \sum_{ma=nc}{}' \frac{\chi_2\left(m\right)\overline{\chi}_1\left(n\right)}{m^{p-j}n^{q+j}} e^{2\pi i\left(nz+mx\right)/k}. \end{aligned} \tag{2.12}$$

Since $(a,c)=1$ and $ma=nc$, we have $n=ra$ and $m=rc$. Then

$$\sum_{ma=nc}{}' \frac{\chi_2(m)\overline{\chi}_1(n)}{m^{p-j}n^{q+j}}e^{2\pi i(nz+mx)/k}=\frac{\overline{\chi}_1(a)\chi_2(c)}{c^{p-j}a^{q+j}}\sum_{\substack{r=-\infty\\ r\neq 0}}^{\infty}\frac{\chi_2(r)\overline{\chi}_1(r)}{r^{p+q}}e^{2\pi i(az+cx)r/k}. \tag{2.13}$$

Notice that the contribution of (2.13) to (2.12) is zero since $\overline{\chi}_1(a)=0$ by the condition $a\equiv 0 \,(\mathrm{mod}\, k)$. Similar to (2.9), we have

$$\begin{aligned}
&s_{q,p}(b,c,a;\overline{\chi}_2,\overline{\chi}_1;y,z,x)\\
&=A_q(\overline{\chi}_1)A_p\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\sum_{\substack{r=-\infty\\ r\neq 0}}^{\infty}\frac{\overline{\chi}_1(n)}{n^q r^p}e^{2\pi i\left(\frac{nb+rc}{ak}x+\frac{ny}{k}+rz\right)}\sum_{v=0}^{k-1}\overline{\chi}_2(v)e^{2\pi i(nb+rc)v/ak}\sum_{j=0}^{a-1}e^{2\pi i(nb+rc)j/a}.
\end{aligned}$$

Using

$$\sum_{j=0}^{a-1}e^{2\pi i(nb+rc)j/a}=\begin{cases} a, & nb+rc\equiv 0\,(\mathrm{mod}\, a),\\ 0, & nb+rc\not\equiv 0\,(\mathrm{mod}\, a),\end{cases}$$

and then taking $nb=ma-rc$, with the use of $(bc,k)=1$ and $a\equiv 0\,(\mathrm{mod}\, k)$, we find that

$$\begin{aligned}
&s_{q,p}(b,c,a;\overline{\chi}_2,\overline{\chi}_1;y,z,x)\\
&=ab^q\chi_1(b)A_q(\overline{\chi}_1)A_p\sum_{ma\neq rc}{}'\frac{\overline{\chi}_1(ma-rc)}{r^p(ma-rc)^q}e^{2\pi i\left(mx+rkz+\frac{ma-rc}{b}y\right)/k}\sum_{v=0}^{k-1}\overline{\chi}_2(v)e^{2\pi imv/k}\\
&=ab^q\chi_1(b)\overline{\chi}_1(-c)G(\overline{\chi}_2)A_q(\overline{\chi}_1)A_p\sum_{ma\neq rc}{}'\frac{\overline{\chi}_1(r)\chi_2(m)}{r^p(ma-rc)^q}e^{2\pi i\left(mx+rkz+\frac{ma-rc}{b}y\right)/k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{ma\neq nc}{}'\frac{\overline{\chi}_1(n)\chi_2(m)}{n^{p+l}(nc-ma)^{q-l}}e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k}\\
&=\frac{\chi_1(-c)\overline{\chi}_1(b)(-1)^{q-l}}{ab^{q-l}G(\overline{\chi}_2)A_{q-l}(\overline{\chi}_1)A_{p+l}}s_{q-l,p+l}(b,c,a;\overline{\chi}_2,\overline{\chi}_1;-yk,z/k,x).
\end{aligned} \tag{2.14}$$

Combining (2.10), (2.11), (2.12), (2.13) and (2.14), after some simplifications, we arrive at

$$\begin{aligned}
&s_{p,q}(a,b,c;\chi_1,\chi_2;x,y,z)\\
&=q\frac{b^q}{(ck)^{q-1}}\sum_{j=0}^{p-1}\binom{p}{j}\frac{(a/c)^j}{q+j}\mathcal{B}_{q+j,\overline{\chi}_1}\left(ck\frac{y}{b}+z\right)\mathcal{B}_{p-j,\chi_2}\left(-ak\frac{y}{b}+x\right)\\
&\quad+p\overline{\chi}_1(b)\chi_1(-c)(a/c)^{p-1}k^{p-q}\\
&\quad\times\sum_{l=0}^{q-1}\binom{q}{l}\frac{(bk/c)^l}{p+l}(-1)^{q-l}s_{q-l,p+l}(b,c,a;\overline{\chi}_2,\overline{\chi}_1;-ky,z/k,x).
\end{aligned} \tag{2.15}$$

Taking $y \to y+h$ in (2.15) and summing over h from 0 to $b-1$ give

$$\begin{aligned} & b s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right) \\ & = \frac{b^q}{(ck)^{q-1}} \sum_{j=0}^{p-1} \binom{p}{j} \frac{q}{q+j} (a/c)^j\, s^*_{q+j,p-j}\left(c,-a,b;\overline{\chi}_1,\chi_2;x,z,y\right) \\ & \quad + bp\overline{\chi}_1(b)\,\chi_1(-c)\,(a/c)^{p-1} k^{p-q} \\ & \qquad \times \sum_{l=0}^{q-1} \binom{q}{l} \frac{(bk/c)^l}{p+l} (-1)^{q-l} s_{q-l,p+l}\left(b,c,a;\overline{\chi}_2,\overline{\chi}_1;-ky,z/k,x\right), \end{aligned} \tag{2.16}$$

which is (2.1) for $p \neq 1$ or $q \neq 1$ or $(x,y,z) \notin (-ak,-b,ck)\,\mathbb{R}+\mathbb{Z}^3$.

- **The case** $p=q=1$ **and** $(x,y,z) \in (-ak,-b,ck)\,\mathbb{R}+\mathbb{Z}^3$.

For $p=q=1$ and $(x,y,z) \notin (-ak,-b,ck)\,\mathbb{R}+\mathbb{Z}^3$, we have already proved that

$$\begin{aligned} & s_{1,1}\left(a,b,c;\chi_1,\chi_2;x,y,z\right) \\ & = s^*_{1,1}\left(c,-a,b;\overline{\chi}_1,\chi_2;z,x,y\right) - \overline{\chi}_1(b)\,\chi_1(-c)\, s_{1,1}\left(b,c,a;\overline{\chi}_2,\overline{\chi}_1;-ky,z/k,x\right). \end{aligned} \tag{2.17}$$

Without loss of generality, we substitute $x=-akR+x_1$, $y=-bR+y_1$ and $z=ckR+z_1 \mp ck\delta$ in (2.17) for some $R \in \mathbb{R}$ and $x_1,y_1,z_1 \in \mathbb{Z}$. Then adding gives

$$\begin{aligned} & \frac{1}{2}\sum_{\lambda=0}^{ck-1} \chi_1(\lambda) \left\{ \mathcal{B}_{1,\chi_2}\left(a\frac{\lambda+z_1}{c}+x_1+\delta ak\right) \mathcal{B}_1\left(b\frac{\lambda+z_1}{ck}+y_1+\delta b\right) \right. \\ & \quad \left. + \mathcal{B}_{1,\chi_2}\left(a\frac{\lambda+z_1}{c}+x_1-\delta ak\right) \mathcal{B}_1\left(b\frac{\lambda+z_1}{ck}+y_1-\delta b\right) \right\} \\ & = \frac{1}{2}\sum_{j=0}^{b-1} \mathcal{B}_{1,\chi_2}\left(-ak\frac{j+y_1}{b}+x_1\right) \\ & \quad \times \left\{ \mathcal{B}_{1,\overline{\chi}_1}\left(ck\frac{j+y_1}{b}+z_1+\delta ck\right) + \mathcal{B}_{1,\overline{\chi}_1}\left(ck\frac{j+y_1}{b}+z_1-\delta ck\right) \right\} \\ & - \overline{\chi}_1(b)\,\chi_1(-c)\,\frac{1}{2}\sum_{v=0}^{ak-1} \overline{\chi}_2(v)\, \mathcal{B}_{1,\overline{\chi}_1}\left(b\frac{v+x_1}{ak}-ky_1\right) \\ & \quad \times \left\{ \mathcal{B}_1\left(c\frac{v+x_1}{a}+\frac{z_1}{k}+c\delta\right) + \mathcal{B}_1\left(c\frac{v+x_1}{a}+\frac{z_1}{k}-c\delta\right) \right\}. \end{aligned}$$

Since $z_1 \in \mathbb{Z}$, there exists $\lambda_{z_1} \in \{0,1,\ldots,ck-1\}$ such that $\lambda_{z_1}+z_1 \equiv 0 \,(\operatorname{mod} ck)$. Therefore, by letting $\delta \to 0^+$

it follows from (2.8) and $\mathcal{B}_1\left(b\frac{\lambda_{z_1}+z_1}{ck}+y_1\right)=\mathcal{B}_1(0)=0$ that

$$\sum_{\substack{\lambda=0\\ \lambda\not\equiv -z_1 (\operatorname{mod} ck)}}^{ck-1}\chi_1(\lambda)\mathcal{B}_{1,\chi_2}\left(a\frac{\lambda+z_1}{c}+x_1\right)\mathcal{B}_1\left(b\frac{\lambda+z_1}{ck}+y_1\right)+\frac{1}{4}\chi_1(-z_1)\overline{\chi}_2(-x_1)$$

$$=s_{1,1}(a,b,c;\chi_1,\chi_2;x_1,y_1,z_1)+\frac{1}{4}\chi_1(-z_1)\overline{\chi}_2(-x_1)$$

$$=s^*_{1,1}(c,-a,b;\overline{\chi}_1,\chi_2;x_1,z_1,y_1)-\overline{\chi}_1(b)\chi_1(-c)s_{1,1}(b,c,a;\overline{\chi}_2,\overline{\chi}_1;-ky_1,z_1/k,x_1),$$

which is equivalent to (2.1) for $p=q=1$ and $(x,y,z)\in(-ak,-b,ck)\mathbb{R}+\mathbb{Z}^3$ because of

$$a\frac{\lambda+z}{c}+x=a\frac{\lambda+z_1}{c}+x_1 \text{ and } b\frac{\lambda+z}{ck}+y=b\frac{\lambda+z_1}{ck}+y_1$$

for $x=-akR+x_1,\ y=-bR+y_1,\ z=ckR+z_1,\ R\in\mathbb{R}$ and $x_1,y_1,z_1\in\mathbb{Z}$. □

Note that for $q=1$, (2.1) reduces to

$$\chi_1(b)s_{p,1}(a,b,c;\chi_1,\chi_2;x,y,z)+(a/ck)^{p-1}\chi_1(-c)s_{1,p}(b,c,a;\overline{\chi}_2,\overline{\chi}_1;-ky,z/k,x)$$

$$=\frac{\chi_1(b)}{p+1}\sum_{j=1}^{p}\binom{p+1}{j}\left(\frac{a}{c}\right)^{j-1}s^*_{j,p+1-j}(c,-a,b;\overline{\chi}_1,\chi_2;z,x,y)+\chi_1(b)E_{p,1}.$$

Observing that $s^*_{j,p+1-j}(c,-a,b;\overline{\chi}_1,\chi_2;z,x,y)=\mathcal{B}_{j,\overline{\chi}_1}(z)\mathcal{B}_{p-j,\chi_2}(x)$ for $b=1$ and $y=0$, this formula will be simpler, for example,

$$s_{p,1}(a,c;\chi_1,\chi_2)+(a/ck)^{p-1}\chi_1(-c)\chi_2(c)s_{1,p}\left(c^{-1},a;\overline{\chi}_2,\overline{\chi}_1\right)$$

$$=\frac{1}{p+1}\sum_{j=1}^{p}\binom{p+1}{j}(a/c)^{j-1}\mathcal{B}_{j,\overline{\chi}_1}\mathcal{B}_{p-j,\chi_2},$$

where $s_{m,n}(a,c;\chi_1,\chi_2)=s_{m,n}(a,1,c;\chi_1,\chi_2;0,0,0)$ and $cc^{-1}\equiv 1\,(\operatorname{mod} ak)$.

Although the sum $s_{p,q}(a,b,c;\chi_1,\chi_2;x,y,z)$ is a generalization of the sums defined in [7, 19, 20], Theorem 2.1 does not include the corresponding reciprocity formulas. However, the following theorem covers the reciprocity formulas derived previously.

Theorem 2.2 *Let a, b and c be pairwise coprime positive integers. Let χ_1 and χ_2 be nonprincipal primitive characters of modulus k. Then, for $x,y,z\in\mathbb{R}$ and $p,q\in\mathbb{N}$*

$$s_{p,q}(a,b,c;\chi_1,\chi_2;x,y,z)$$

$$=q(b/ck)^{q-1}\sum_{j=0}^{p-1}\binom{p}{j}\frac{(a/c)^j}{q+j}s^*_{p-j,q+j}(-a,c,b;\chi_2,\overline{\chi}_1;x,z,y)$$

$$+p(a/c)^{p-1}\sum_{l=0}^{q}\binom{q}{l}\frac{(b/kc)^l}{p+l}(-1)^{q-l}s_{p+l,q-l}(c,b,a;\overline{\chi}_2,\overline{\chi}_1;z,-y,x)+E_{p,q}, \tag{2.18}$$

where $E_{p,q}$ is given by (2.2),

Remark 2.3 *The reciprocity formulas given by Dağlı and Can [20, Theorem 1], Cenkci et al. [19, Theorem 5.4], Berndt [7, Theorem 7] are special cases of Theorem 2.2 for* $x = z = 0$, *for* $\chi_1 = \chi_2$, *for* $p = 1$, $\chi_1 = \chi_2$ *and in all cases* $q = b = 1$ *and* $y = 0$, *respectively.*

In particular,

$$\begin{aligned} & s_{1,1}\left(a,b,c;\chi_1,\chi_2;x,y,z\right) + s_{1,1}\left(c,b,a;\overline{\chi}_2,\overline{\chi}_1;z,-y,x\right) \\ & \quad - s^{*}_{1,1}\left(-a,c,b;\chi_2,\overline{\chi}_1;x,z,y\right) = \frac{b}{2kc} s_{2,0}\left(c,b,a;\overline{\chi}_2,\overline{\chi}_1;z,-y,x\right) + E_{1,1}. \end{aligned} \tag{2.19}$$

In the case $x = y = z = 0$, (2.19) was first proved by Berndt [12, Theorem 7.2] for arbitrary periodic sequences $\{a_n\}$ and $\{c_n\}$, $(-\infty < n < \infty)$ of period k (instead of $\{\chi_1(n)\}$ and $\{\chi_2(n)\}$).

Moreover, for $p = 1$ and $x = y = z = 0$, (2.18) reduces to

$$\begin{aligned} & s_{1,q}\left(a,b,c;\chi_1,\chi_2\right) - \left(b/ck\right)^{q-1} s^{*}_{1,q}\left(-a,c,b;\chi_2,\overline{\chi}_1\right) \\ & = \frac{1}{q+1}\sum_{l=1}^{q+1}\binom{q+1}{l}\left(b/kc\right)^{l-1}\left(-1\right)^{q+1-l} s_{l,q+1-l}\left(c,b,a;\overline{\chi}_2,\overline{\chi}_1\right), \end{aligned}$$

where $s_{p,q}\left(a,b,c;\chi_1,\chi_2\right) = s_{p,q}\left(a,b,c;\chi_1,\chi_2;0,0,0\right)$. For $b = 1$, this is analogue to the formulas including character Hardy–Berndt sums given in [13, p. 15 and p. 17].

Proof [Proof of Theorem 2.2]We consider the following three cases: **I)** $p \geq 2,\ q \geq 2$, **II)** $p \geq 2,\ q = 1$ and **III)** $p = 1,\ q \geq 1$.

I) The case $p \geq 2,\ q \geq 2$.

We already have from (2.10)

$$\begin{aligned} & s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right) \\ & = cb^{q} G\left(\chi_1\right) A_p\left(\chi_2\right) A_q \sum_{\substack{m=-\infty \\ m\neq 0 \\ ma\neq nc}}^{\infty}\ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\chi_2\left(m\right)\overline{\chi}_1\left(n\right)}{m^{p}\left(nc-ma\right)^{q}} e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k} \\ & = cb^{q} G\left(\chi_1\right) A_p\left(\chi_2\right) A_q \sum_{\substack{n=-\infty \\ n\neq 0 \\ nc\neq ma}}^{\infty}\ \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{\chi_2\left(m\right)\overline{\chi}_1\left(n\right)}{m^{p}\left(nc-ma\right)^{q}} e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k}, \end{aligned} \tag{2.20}$$

where the implied interchange of order of summation is justified by absolute convergence. We now combine

(2.20), (2.11), (2.12) and (2.13) to obtain

$$\begin{aligned}
&\frac{s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right)}{cb^qG\left(\chi_1\right)A_p\left(\chi_2\right)A_q} \\
&= a^p\sum_{j=0}^{p-1}\frac{\alpha_{p-j}}{a^{p-j}c^{q+j}}\frac{\mathcal{B}_{p-j,\chi_2}\left(x-\frac{aky}{b}\right)\mathcal{B}_{q+j,\overline{\chi}_1}\left(\frac{cky}{b}+z\right)}{A_{p-j}\left(\chi_2\right)A_{q+j}\left(\overline{\chi}_1\right)} \\
&- a^p\sum_{j=0}^{p-1}\frac{\alpha_{p-j}}{a^{p-j}c^{q+j}}\frac{\overline{\chi}_1\left(a\right)\chi_2\left(c\right)}{c^{p-j}a^{q+j}}\sum_{\substack{r=-\infty\\ r\neq 0}}^{\infty}\frac{\chi_2\left(r\right)\overline{\chi}_1\left(r\right)}{r^{p+q}}e^{2\pi i(az+cx)r/k} \\
&+ a^p\sum_{l=0}^{q-1}\frac{\beta_{q-l}}{c^{p+l}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\sum_{\substack{m=-\infty\\ m\neq 0\\ nc\neq ma}}^{\infty}\frac{\overline{\chi}_1\left(n\right)\chi_2\left(m\right)}{n^{p+l}\left(nc-ma\right)^{q-l}}e^{2\pi i\left(nz+mx+\frac{nc-ma}{b}yk\right)/k}.
\end{aligned}\tag{2.21}$$

Taking $a\leftrightarrow c,\ x\leftrightarrow z$ and $\overline{\chi}_1\leftrightarrow\chi_2$ in (2.10) gives

$$\begin{aligned}
&s_{p,q}\left(c,b,a;\overline{\chi}_2,\overline{\chi}_1;z,y,x\right) \\
&= ab^qG\left(\overline{\chi}_2\right)A_p\left(\overline{\chi}_1\right)A_q\sum_{\substack{m=-\infty\\ m\neq 0\\ ma\neq nc}}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\overline{\chi}_1\left(m\right)\chi_2\left(n\right)}{m^p\left(na-mc\right)^q}e^{2\pi i\left(nx+mz+\frac{na-mc}{b}yk\right)/k}.
\end{aligned}\tag{2.22}$$

Moreover, similar to (2.10), we have

$$\begin{aligned}
&s_{p+q,0}\left(c,b,a;\overline{\chi}_2,\overline{\chi}_1;z,-y,x\right) \\
&= \overline{\chi}_1\left(a\right)\chi_2\left(c\right)\frac{G\left(\overline{\chi}_2\right)A_{p+q}\left(\overline{\chi}_1\right)}{a^{p+q-1}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\chi_2\left(n\right)\overline{\chi}_1\left(n\right)}{n^{p+q}}e^{2\pi i(az+cx)n/k}.
\end{aligned}\tag{2.23}$$

Therefore, combining (2.21), (2.22), (2.23) and using that

$$\sum_{j=0}^{p-1}\alpha_{p-j}=\sum_{j=0}^{p-1}\binom{q+j-1}{j}=\binom{q+p-1}{p-1},$$

after simplifications, we find that

$$\begin{aligned}
&s_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right) \\
&= \frac{b^q}{(ck)^{q-1}}\sum_{j=0}^{p-1}\binom{p}{j}\frac{q}{q+j}\left(a/c\right)^j\mathcal{B}_{p-j,\chi_2}\left(-ak\frac{y}{b}+x\right)\mathcal{B}_{q+j,\overline{\chi}_1}\left(ck\frac{y}{b}+z\right) \\
&+ \left(\frac{a}{c}\right)^{p-1}\sum_{l=0}^{q}\binom{q}{l}\frac{p}{p+l}\left(\frac{b}{ck}\right)^l(-1)^{q-l}s_{p+l,q-l}\left(c,b,a;\overline{\chi}_2,\overline{\chi}_1;z,-y,x\right).
\end{aligned}$$

As in (2.16), taking $y \to y+h$ and then summing over h from 0 to $b-1$ give

$$\begin{aligned} & bs_{p,q}\left(a,b,c;\chi_1,\chi_2;x,y,z\right) \\ & = b\frac{b^{q-1}}{(ck)^{q-1}}\sum_{j=0}^{p-1}\binom{p}{j}\frac{q}{q+j}\left(a/c\right)^{j} s^{*}_{p-j,q+j}\left(-a,c,b;\chi_2,\overline{\chi}_1;x,z,y\right) \\ & + b\left(\frac{a}{c}\right)^{p-1}\sum_{l=0}^{q}\binom{q}{l}\frac{p}{p+l}\left(\frac{b}{ck}\right)^{l}(-1)^{q-l} s_{p+l,q-l}\left(c,b,a;\overline{\chi}_2,\overline{\chi}_1;z,-y,x\right), \end{aligned} \tag{2.24}$$

which is (2.18) for $p \geq 2,\ q \geq 2$ and $x,y,z \in \mathbb{R}$.

II) The case $p \geq 2,\ q = 1$.

In this case we take advantage of the differential properties of the Bernoulli functions

$$\frac{d}{dx}\mathcal{B}_n\left(x\right) = n\mathcal{B}_{n-1}\left(x\right) \text{ and } \frac{d}{dx}\mathcal{B}_{n,\chi}\left(x\right) = n\mathcal{B}_{n-1,\chi}\left(x\right),\ n \geq 1,\ x \notin \mathbb{Z}.$$

We use (2.24) for $p \geq 2,\ q = 2:$

$$\begin{aligned} & \sum_{v=0}^{ck-1}\chi_1\left(v\right)\mathcal{B}_{p,\chi_2}\left(a\frac{v+z}{c}+x\right)\mathcal{B}_2\left(b\frac{v+z}{ck}+y\right) \\ & = 2\frac{b}{ck}\sum_{j=0}^{p-1}\binom{p}{j}\frac{(a/c)^{j}}{2+j}\sum_{v=0}^{b-1}\mathcal{B}_{p-j,\chi_2}\left(-ak\frac{v+y}{b}+x\right)\mathcal{B}_{2+j,\overline{\chi}_1}\left(ck\frac{v+y}{b}+z\right) \\ & \quad + \left(\frac{a}{c}\right)^{p-1}\sum_{l=0}^{2}\binom{2}{l}\frac{p}{p+l}\left(\frac{b}{ck}\right)^{l}(-1)^{l} \\ & \quad\quad \times \sum_{v=0}^{ak-1}\overline{\chi}_2\left(v\right)\mathcal{B}_{p+l,\overline{\chi}_1}\left(c\frac{v+x}{a}+z\right)\mathcal{B}_{2-l}\left(b\frac{v+x}{ak}-y\right). \end{aligned} \tag{2.25}$$

Let $bz+cky \notin \mathbb{Z}$ and $bx-aky \notin \mathbb{Z}$. Differentiating both sides of (2.25) with respect to y gives

$$\begin{aligned} & s_{p,1}\left(a,b,c;\chi_1,\chi_2;x,y,z\right) \\ & = \sum_{j=0}^{p-1}\binom{p}{j}\frac{1}{1+j}\left(a/c\right)^{j} s^{*}_{p-j,1+j}\left(-a,c,b;\chi_2,\overline{\chi}_1;x,z,y\right) \\ & + \left(\frac{a}{c}\right)^{p-1}\sum_{l=0}^{1}\binom{1}{l}\frac{p}{p+l}\left(\frac{b}{ck}\right)^{l}(-1)^{1-l} s_{p+l,1-l}\left(c,b,a;\overline{\chi}_2,\overline{\chi}_1;z,-y,x\right), \end{aligned} \tag{2.26}$$

which is (2.18) for $p \geq 2,\ q = 1$ and $x,y,z \in \mathbb{R}$ with $bz+cky \notin \mathbb{Z}$ and $bx-aky \notin \mathbb{Z}$.

If $bz+cky \in \mathbb{Z}$ or $bx-aky \in \mathbb{Z}$, then we take $y \to y \mp \delta$ in (2.25) such that $bz+ck\left(y \mp \delta\right) \notin \mathbb{Z}$ and $bx-ak\left(y \mp \delta\right) \notin \mathbb{Z}$. Then (2.26) holds for $bz+ck\left(y \mp \delta\right) \notin \mathbb{Z}$ and $bx-ak\left(y \mp \delta\right) \notin \mathbb{Z}$. Adding (as in the case $p=q=1$ and $(x,y,z) \in (-ak,-b,ck)\mathbb{R}+\mathbb{Z}^3$ in the proof of Theorem 2.1) and then letting $\delta \to 0^{+}$, with the use of (2.6), give (2.18) for $p \geq 2,\ q = 1$ and for all $x,y,z \in \mathbb{R}$.

III) The case $p=1,\ q\geq 1$.

If $bx-aky\notin\mathbb{Z}$, the proof is similar to the previous case, where now we set $p=2,\ q\geq 1$ in (2.24). Differentiating both sides of the resulting equation with respect to x gives (2.18) for $p=1,\ q\geq 1$ and $x,y,z\in\mathbb{R}$ with $bx-aky\notin\mathbb{Z}$.

If $bx-aky\in\mathbb{Z}$, then we take $x\to x\mp\delta$ so that (2.18) holds for $p=1,\ q\geq 1$ and $x,y,z\in\mathbb{R}$ such that $b(x\mp\delta)-aky\notin\mathbb{Z}$. Adding the resulting equations gives

$$\begin{aligned}
&\sum_{\lambda=0}^{ck-1}\chi_1(\lambda)\mathcal{B}_q\left(b\frac{\lambda+z}{ck}+y\right)\\
&\quad\times\frac{1}{2}\left\{\mathcal{B}_{1,\chi_2}\left(a\frac{\lambda+z}{c}+x+\delta\right)+\mathcal{B}_{1,\chi_2}\left(a\frac{\lambda+z}{c}+x-\delta\right)\right\}\\
&=(b/ck)^{q-1}\sum_{v=0}^{b-1}\mathcal{B}_{q,\overline{\chi}_1}\left(ck\frac{v+y}{b}+z\right)\\
&\quad\times\frac{1}{2}\left\{\mathcal{B}_{1,\chi_2}\left(-ak\frac{v+y}{b}+x+\delta\right)+\mathcal{B}_{1,\chi_2}\left(-ak\frac{v+y}{b}+x-\delta\right)\right\}\\
&\quad+\sum_{l=0}^{q}\binom{q}{l}\frac{(b/ck)^l}{1+l}(-1)^{q-l}\sum_{v=0}^{ak-1}\overline{\chi}_2(v)\\
&\quad\times\frac{1}{2}\left\{\mathcal{B}_{1+l,\overline{\chi}_1}\left(c\frac{v+x}{a}+z+\frac{c\delta}{a}\right)\mathcal{B}_{q-l}\left(b\frac{v+x}{ak}-y+\frac{b\delta}{ak}\right)\right.\\
&\quad\left.+\mathcal{B}_{1+l,\overline{\chi}_1}\left(c\frac{v+x}{a}+z-\frac{c\delta}{a}\right)\mathcal{B}_{q-l}\left(b\frac{v+x}{ak}-y-\frac{b\delta}{ak}\right)\right\}.
\end{aligned}$$

Now assume that $x=-akR+x_1,\ y=-bR+y_1$ for some $R\in\mathbb{R}$ and $x_1,y_1\in\mathbb{Z}$, so $bx-aky\in\mathbb{Z}$. Let $z=ckR+z_1$. Then, (2.18) follows from (2.8) by letting $\delta\to 0^+$ for the cases ($q\neq 1$ or $z_1\notin\mathbb{Z}$) and ($q=1$ and $z_1\in\mathbb{Z}$), separately. □

3. Petersson–Knopp type identities

For the sum $s_{p,q}(a,b,c;\chi_1,\chi_2)$ the following Petersson–Knopp type identities hold.

Theorem 3.1 *Let χ_1 and χ_2 be nonprincipal primitive characters of modulus k. Then, for $(n,k)=1,\ n\in\mathbb{N}$*

$$\begin{aligned}
&\sum_{d|n}\overline{\chi}_1(d)\overline{\chi}_2(d)d^{p+q-2}\sum_{r=1}^{d}\sum_{j=1}^{d}s_{p,q}\left(\frac{n}{d}a+rck,\frac{n}{d}b+jck,dc;\chi_1,\chi_2\right)\\
&=n\overline{\chi}_1(n)\sigma_{p+q-1}(n;\chi_1,\overline{\chi}_2)s_{p,q}(a,b,c;\chi_1,\chi_2),
\end{aligned}\tag{3.1}$$

where $\sigma_m(n;\chi_1,\chi_2)=\sum_{d|n}d^m\chi_1(d)\chi_2(d)$.

In addition, if $(n,b)=1,$ *then*

$$\sum_{d|n}\overline{\chi}_1(d)\,\overline{\chi}_2(d)\,d^{p-1}\sum_{r=1}^{d}s_{p,q}\left(\frac{n}{d}a+rck,b,dc;\chi_1,\chi_2\right)$$
$$=n^{1-q}\overline{\chi}_1(n)\,\sigma_{p+q-1}\left(n;\chi_1,\overline{\chi}_2\right)s_{p,q}\left(a,b,c;\chi_1,\chi_2\right). \tag{3.2}$$

Note that for $\chi_1=\overline{\chi}_2=\chi,$ (3.1) reduces to

$$\sum_{d|n}d^{p+q-2}\sum_{r=1}^{d}\sum_{j=1}^{d}s_{p,q}\left(\frac{n}{d}a+rck,\frac{n}{d}b+jck,dc;\chi,\overline{\chi}\right)$$
$$=n\overline{\chi}(n)\,\sigma_{p+q-1}\left(n;\chi,\chi\right)s_{p,q}\left(a,b,c;\chi,\overline{\chi}\right).$$

To prove Theorem 3.1, we need the following two lemmas.

Lemma 3.2 *([42, Lemma 3]) If* $v\in\mathbb{Z}$ *and* $d\in\mathbb{N},$ *then*

$$\sum_{j=1}^{d}\mathcal{B}_q\left(x+\frac{jv}{d}\right)=(v,d)^q\,d^{1-q}\mathcal{B}_q\left(\frac{dx}{(v,d)}\right). \tag{3.3}$$

Lemma 3.3 *Let* χ *be a nonprincipal primitive character modulo* k*. Then, for* $(d,k)=1,$

$$\sum_{r=1}^{d}\mathcal{B}_{p,\chi}\left(x+vk\frac{r}{d}\right)=(d,v)^p\,d^{1-p}\chi\left(\frac{d}{(d,v)}\right)\mathcal{B}_{p,\chi}\left(\frac{xd}{(d,v)}\right). \tag{3.4}$$

Proof Let $(d,v)=l$ and set $d=ld_1,\ v=lv_1$. From (1.6)

$$\sum_{r=1}^{d}\mathcal{B}_{p,\chi}\left(x+vk\frac{r}{d}\right)=k^{p-1}\sum_{j=0}^{k-1}\overline{\chi}(j)\sum_{r=1}^{d}\mathcal{B}_p\left(\frac{x+j}{k}+\frac{vr}{d}\right)$$
$$=lk^{p-1}\sum_{j=0}^{k-1}\overline{\chi}(j)\sum_{r=1}^{d_1}\mathcal{B}_p\left(\frac{x+j}{k}+\frac{v_1r}{d_1}\right)$$
$$=lk^{p-1}\sum_{j=0}^{k-1}\overline{\chi}(j)\sum_{r=1}^{d_1}\mathcal{B}_p\left(\frac{x+j}{k}+\frac{r}{d_1}\right).$$

From the well-known multiplication formula of $\mathcal{B}_n(x)$

$$\mathcal{B}_n(mx)=m^{n-1}\sum_{r=0}^{m-1}\mathcal{B}_n\left(x+\frac{r}{m}\right) \tag{3.5}$$

we have

$$\sum_{r=1}^{d}\mathcal{B}_{p,\chi}\left(x+vk\frac{r}{d}\right)=l\left(d_1\right)^{1-p}k^{p-1}\sum_{j=0}^{k-1}\overline{\chi}\left(j\right)\mathcal{B}_p\left(\frac{xd_1+jd_1}{k}\right)$$

$$\overset{(d_1,k)=1}{=} l^p d^{1-p}\chi\left(\frac{d}{l}\right)k^{p-1}\sum_{j=0}^{k-1}\overline{\chi}\left(j\right)\mathcal{B}_p\left(\frac{xd_1+j}{k}\right)$$

$$=l^p d^{1-p}\chi\left(\frac{d}{l}\right)\mathcal{B}_{p,\chi}\left(\frac{xd}{l}\right)$$

by (1.6). □

Note that in the case $v=1$, (3.4) occurs in [15, Eq. (3.13)].

Proof [Proof of Theorem 3.1]We prove (3.1), the proof of (3.2) is similar, so we omit it. From (3.3) and (3.4),

$$\sum_{d|n}\overline{\chi}_1\left(d\right)\overline{\chi}_2\left(d\right)d^{p+q-2}\sum_{r=1}^{d}\sum_{j=1}^{d}s_{p,q}\left(\frac{n}{d}a+rck,\frac{n}{d}b+jck,dc;\chi_1,\chi_2\right)$$

$$=\sum_{d|n}\overline{\chi}_1\left(d\right)\overline{\chi}_2\left(d\right)d^{p+q-2}\sum_{v=1}^{dck}\chi_1\left(v\right)\sum_{r=1}^{d}\mathcal{B}_{p,\chi_2}\left(\frac{nav}{d^2c}+\frac{rvk}{d}\right)\sum_{j=1}^{d}\mathcal{B}_q\left(\frac{nbv}{d^2ck}+\frac{jv}{d}\right)$$

$$=\sum_{d|n}\overline{\chi}_1\left(d\right)\overline{\chi}_2\left(d\right)\sum_{v=1}^{dck}\chi_1\left(v\right)\left(d,v\right)^{p+q}\chi_2\left(d/\left(d,v\right)\right)$$

$$\times\mathcal{B}_{p,\chi_2}\left(\frac{nav}{\left(d,v\right)dc}\right)\mathcal{B}_q\left(\frac{nbv}{\left(d,v\right)dck}\right)$$

$$\overset{l=(d,v)}{=}\sum_{d|n}\overline{\chi}_1\left(d\right)\sum_{l|d}\overline{\chi}_2\left(l\right)l^{p+q}\sum_{\substack{v=1\\(v,d)=l}}^{dck}\chi_1\left(v\right)\mathcal{B}_{p,\chi_2}\left(\frac{nav}{ldc}\right)\mathcal{B}_q\left(\frac{nbv}{ldck}\right)$$

$$=\sum_{d|n}\overline{\chi}_1\left(d\right)\sum_{l|d}\overline{\chi}_2\left(l\right)l^{p+q}\sum_{\substack{v=1\\\left(v,\frac{d}{l}\right)=1}}^{dck/l}\chi_1\left(l\right)\chi_1\left(v\right)\mathcal{B}_{p,\chi_2}\left(\frac{nav}{dc}\right)\mathcal{B}_q\left(\frac{nbv}{dck}\right). \tag{3.6}$$

We need the following two properties of the Mobius μ-function:

$$\sum_{d|n}\mu\left(d\right)=\begin{cases}1, & \text{if } n=1,\\ 0, & \text{otherwise,}\end{cases} \tag{3.7}$$

and

$$\sum_{\substack{v=1\\(v,d)=1}}^{ad}f\left(v\right)=\sum_{t|d}\mu\left(t\right)\sum_{v=1}^{ad/t}f\left(tv\right). \tag{3.8}$$

Utilizing (3.7) and (3.8), the right-hand side of (3.6) becomes

$$\sum_{d|n}\overline{\chi}_1(d)\sum_{l|d}\chi_1(l)\overline{\chi}_2(l)\,l^{p+q}\sum_{t|\frac{d}{l}}\mu(t)\sum_{v=1}^{dck/lt}\chi_1(vt)\,\mathcal{B}_{p,\chi_2}\left(\frac{vna/l}{dc/lt}\right)\mathcal{B}_q\left(\frac{nbv/l}{dck/lt}\right)$$

$$=\sum_{d|n}\sum_{l|d}\sum_{t|\frac{d}{l}}\overline{\chi}_1(d)\,\chi_1(lt)\,\overline{\chi}_2(l)\,l^{p+q}\mu(t)\,s_{p,q}\left(\frac{na}{l},\frac{nb}{l},\frac{dc}{lt};\chi_1,\chi_2\right)$$

$$=\sum_{tlh|n}\overline{\chi}_1(h)\,\overline{\chi}_2(l)\,l^{p+q}\mu(t)\,s_{p,q}\left(\frac{na}{l},\frac{nb}{l},hc;\chi_1,\chi_2\right)$$

$$=\sum_{lh|n}\overline{\chi}_1(h)\,\overline{\chi}_2(l)\,l^{p+q}s_{p,q}\left(\frac{na}{l},\frac{nb}{l},hc;\chi_1,\chi_2\right)\sum_{t|\frac{n}{lh}}\mu(t)$$

$$=\sum_{lh=n}\overline{\chi}_1(h)\,\overline{\chi}_2(l)\,l^{p+q}s_{p,q}(ha,hb,hc;\chi_1,\chi_2)\,.$$

Here, using that

$$s_{p,q}(ha,hb,hc;\chi_1,\chi_2)=hs_{p,q}(a,b,c;\chi_1,\chi_2)\,,\ h\in\mathbb{N}$$

which follows from the definition of $s_{p,q}(a,b,c;\chi_1,\chi_2)$, gives

$$\sum_{d|n}d^{p+q-2}\overline{\chi}_1(d)\,\overline{\chi}_2(d)\sum_{r=1}^{d}\sum_{j=1}^{d}s_{p,q}\left(\frac{n}{d}a+rck,\frac{n}{d}b+jck,dc;\chi_1,\chi_2\right)$$

$$=\sum_{lh=n}\overline{\chi}_1(h)\,\overline{\chi}_2(l)\,l^{p+q}hs_{p,q}(a,b,c;\chi_1,\chi_2)$$

$$=n\sum_{l|n}\overline{\chi}_1\left(\frac{n}{l}\right)\overline{\chi}_2(l)\,l^{p+q-1}s_{p,q}(a,b,c;\chi_1,\chi_2)$$

$$=n\overline{\chi}_1(n)\,\sigma_{p+q-1}(n;\chi_1,\overline{\chi}_2)\,s_{p,q}(a,b,c;\chi_1,\chi_2)\,.$$

This proves (3.1). □

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References

[1] Alkan E, Xiong M, Zaharescu A. A bias phenomenon on the behavior of Dedekind sums. Mathematical Research Letters 2008; (15): 1039-1052.

[2] Apostol TM. Generalized Dedekind sums and transformation formulae of certain Lambert series. Duke Mathematical Journal 1950; (17): 147-157.

[3] Apostol TM. Introduction to Analytic Number Theory. New York, NY, USA: Springer-Verlag, 1976.

[4] Apostol TM, Vu TH. Identities for sums of Dedekind type. Journal of Number Theory 1982; (14): 391-396.

[5] Bayad A, Raouj A. Reciprocity formulae for multiple Dedekind–Rademacher sums. Comptes Rendus de l'Académie des Sciences - Series I - Mathematics 2011; (349): 131-136.

[6] Beck M, Chavez A. Bernoulli–Dedekind sums. Acta Arithmetica 2011; (149): 65-82.

[7] Berndt BC. Character transformation formulae similar to those for the Dedekind Eta–function. In: 'Analytic Number Theory', Proceedings of Symposia in Pure Mathematics XXIV. Providence, RI, USA: American Mathematical Society, 1973, pp. 9-30.

[8] Berndt BC. Generalized Dedekind eta-functions and generalized Dedekind sums. Transactions of the American Mathematical Society 1973; (178): 495-508.

[9] Berndt BC. Character analogues of Poisson and Euler–MacLaurin summation formulas with applications. Journal of Number Theory 1975; (7): 413-445.

[10] Berndt BC. On Eisenstein series with characters and the values of Dirichlet L-functions. Acta Arithmetica 1975; (28): 299-320.

[11] Berndt BC. Generalized Eisenstein series and modified Dedekind sums. Journal für die reine und angewandte Mathematik 1975; (272): 182-193.

[12] Berndt BC. Reciprocity theorems for Dedekind sums and generalizations. Advances in Mathematics 1977; (23): 285-316.

[13] Boztaş MÇ, Can M. Transformation formulas of a character analogue of $\log \theta_2(z)$. The Ramanujan Journal 2019; (48): 323-349.

[14] Can M, Cenkci M, Kurt V, Simsek Y. Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions. Advanced Studies in Contemporary Mathematics 2009; (18): 135-160.

[15] Can M, Kurt V. Character analogues of certain Hardy–Berndt sums. International Journal of Number Theory 2014; (10): 737-762.

[16] Carlitz L. Some theorems on generalized Dedekind sums. Pacific Journal of Mathematics 1953; (3): 513-522.

[17] Carlitz L. Generalized Dedekind sums. Mathematische Zeitschrift 1964; (85): 83-90.

[18] Cenkci M. On p-adic character Dedekind sums. Palestine Journal of Mathematics 2015; (4): 502-507.

[19] Cenkci M, Can M, Kurt V. Degenerate and character Dedekind sums. Journal of Number Theory 2007; (124): 346-363.

[20] Dağlı MC, Can M. On reciprocity formula of character Dedekind sums and the integral of products of Bernoulli polynomials. Journal of Number Theory 2015; (156): 105-124.

[21] Dağlı MC, Can M. Periodic analogues of Dedekind sums and transformation formulas of Eisenstein series. The Ramanujan Journal 2017; (44): 301-341.

[22] Dağlı MC, Can M. On generalized Eisenstein series and Ramanujan's formula for periodic zeta-functions. Monatshefte für Mathematik 2017; (184): 77-103.

[23] Goldberg LA. An elementary proof of Knopp's theorem on Dedekind sums. Journal of Number Theory 1980; (12): 541-542.

[24] Hall RR, Wilson JC., On reciprocity formulae for inhomogeneous and homogeneous Dedekind sums. Mathematical Proceedings of the Cambridge Philosophical Society 1993; (114): 9-24.

[25] Hall RR, Wilson JC, Zagier D. Reciprocity formulae for general Dedekind–Rademacher sums. Acta Arithmetica 1995; (73): 389-396.

[26] Hamahata Y. Dedekind sums with a parameter in finite fields. Finite Fields and Their Applications 2014; (28): 57-66.

[27] Hickerson D. Continued fractions and density results for Dedekind sums. Journal für die reine und angewandte Mathematik 1977; (290): 113-116.

[28] Kim MS, Son JW. On generalized Dedekind sums involving quasi-periodic Euler functions. Journal of Number Theory 2014; (144): 267-280.

[29] Knopp MI. Hecke operators and an identity for Dedekind sums. Journal of Number Theory 1980; (12): 2-9.

[30] Liu H. Some generalizations of Knopp's identity. Bulletin of the Brazilian Mathematical Society, New Series 2007; (38): 179-188.

[31] Meyer JL. Character analogues of Dedekind sums and transformations of analytic Eisenstein series. Pacific Journal of Mathematics 2000; (194): 137-164.

[32] Mikolás M. On certain sums generating the Dedekind sums and their reciprocity laws. Pacific Journal of Mathematics 1957; (7): 1167-1178.

[33] Nagasaka Y, Ota K, Sekine C. Generalizations of Dedekind sums and their reciprocity laws. Acta Arithmetica 2003; (106): 355-378.

[34] Parson LA. Dedekind sums and Hecke operators. Mathematical Proceedings of the Cambridge Philosophical Society 1980; (88): 11-14.

[35] Parson LA, Rosen K. Hecke operators and Lambert series. Mathematica Scandinavica 1981; (49): 5-14.

[36] Rademacher H. Zur theorie der Dedekindschen summen. Mathematische Zeitschrift 1956; (63): 445-463.

[37] Rademacher H. Some remarks on certain generalized Dedekind sums. Acta Arithmetica 1964; (9): 97-105.

[38] Rademacher H, Grosswald E. Dedekind sums. Carus Math Monographs 16, Mathematics Association America 1972.

[39] Rosen KH, Snyder WM. p-adic Dedekind sums. Journal für die reine und angewandte Mathematik 1985; (361): 23-26.

[40] Takács L. On generalized Dedekind sums. Journal of Number Theory 1979; (11): 264-272.

[41] Zagier D. Higher dimensional Dedekind sums. Mathematische Annalen 1973; (202): 149-172

[42] Zheng Z. The Petersson–Knopp identity for the homogeneous Dedekind sums. Journal of Number Theory 1996; (57): 223-230.