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The Fekete-Szegö inequality for subclasses of analytic functions related to modified Sigmoid functions

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Abstract: In this paper, the authors investigate the initial coefficient bounds for a new generalized subclass of analytic functions related to Sigmoid functions. Also, the relevant connections with the famous classical Fekete–Szegö inequality for these classes are discussed.

Key words: Analytic functions, modified Hadamard product, Sigmoid function, Fekete-Szegö inequality

1. Introduction and preliminaries

Special functions is a branch of mathematics which is of utmost importance for scientists and engineers who are concerned with actual mathematical calculations. It has applications in specific problems of physics, engineering, and computer science.

The theory of special functions has been developed by C. F. Gauss, C. G. J. Jacobi, F. Klein, and many others in 19th century. However, in the twentieth century, the theory of special functions has been overshadowed by other fields such as real and functional analysis, topology, algebra and differential equations. Special functions play an important role in geometric function theory. An example of special function is an activation function. An activation function acts as a squashing function which is the output of a neuron in a neural network taking certain values (usually 0 and 1, or -1 and 1). There are three types of activation functions, namely threshold function, piecewise-linear function, and Sigmoid function.

The most popular activation function is the Sigmoid function. There are different methods to evaluate this function, such as truncating series expansion, looking-up tables, or piecewise approximation.

The Sigmoid function of the form

$$g(z) = \frac{1}{1+e^{-z}} \tag{1.1}$$

is differentiable and has the following properties.

- It outputs real numbers between 0 and 1.
- It maps from a very large input domain to a small range of outputs.

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- It never loses information because it is a one-to-one function.
- It increases monotonically.

These properties enable us to use Sigmoid function in univalent function theory.

Let A denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in U) \tag{1.2}$$

normalized by the conditions $f(0) = f'(0) - 1 = 0$ which are defined in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Also, let S be subclass of A consisting of functions the form (1.2) which are univalent in U.

We briefly recall the following results and definitions needed our investigation.

Lemma 1.1 *[8] If a function $p \in P$ is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \quad (z \in U),$$

then $|p_k| \leq 2,\ k \in \mathbb{N}$ where P is the family of all analytic functions in U for which $p(0) = 1$ and $\Re\{p(z)\} > 0$.

Lemma 1.2 *[6] Let g be a Sigmoid function defined in (1.1) and*

$$\varphi(z) = 2g(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m,$$

then $\varphi(z) \in P,\ |z| < 1$ where $\varphi(z)$ is a modified Sigmoid function.

Lemma 1.3 *[6] Let*

$$\varphi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m,$$

then $|\varphi_{n,m}(z)| < 2$.

Lemma 1.4 *[6] Let $\varphi(z) \in P$ and be starlike, then f is a normalized univalent function of the form (1.2). Setting $m = 1$, Fadipe et al. [6] remarked that*

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

where $c_n = \frac{(-1)^{n+1}}{2n!}$, then $|c_n| \leq 2,\ n = 1, 2, 3, ...$ and the result is sharp for each n.

Definition 1.5 *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. The modified Hadamard product of two functions f and g which belong to A is defined by*

$$F(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

Definition 1.6 *Let $f \in A$. Then the qth Hankel determinant of f is defined for $q \geq 1$ and $n \geq 1$ by*

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Thus, the second Hankel determinant

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

C. Ramachandran and K. Dhanalakshmi [9] studied the following various classes of analytic and univalent functions defined by Sigmoid function.

For two analytic functions f and g, the function f is subordinate to g, written as follows:

$$f(z) \prec g(z)$$

if there exists an analytic function w, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1.7 *[9] Let $b \in \mathbb{C} \backslash \{0\}$ and the class $M_\lambda(b, \varphi_{n,m})$ denote the subclass of A consisting of functions f of the form (1.2), and satisfying the following subordination condition*

$$1 + \frac{1}{b}\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} - 1\right\} \prec \varphi_{n,m}(z) \tag{1.4}$$

for $0 \leq \lambda \leq 1$ and $\varphi_{n,m}$ is a simple logistic Sigmoid activation function.

Definition 1.8 *[9] Let $b \in \mathbb{C} \backslash \{0\}$ and the class $\Im_\lambda(b, \varphi_{n,m})$ denote the subclass of A consisting of functions f of the form (1.2), and satisfying the following subordination condition*

$$1 + \frac{1}{b}\left\{\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} - 1\right\} \prec \varphi_{n,m}(z) \tag{1.5}$$

for $0 \leq \lambda \leq 1$ and $\varphi_{n,m}$ is a simple logistic Sigmoid activation function.

Definition 1.9 *Let $\gamma \in \mathbb{C} \backslash \{0\}$ and the class $M_{\lambda(*)}(\gamma, \varphi_{n,m})$ denote the subclass of A consisting of functions $F(z)$ of the form (1.3), and satisfying the following subordination condition*

$$1 + \frac{1}{\gamma}\left\{\frac{z(f*g)'(z) + \lambda z^2 (f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z (f*g)'(z)} - 1\right\} \prec \varphi_{n,m}(z) \tag{1.6}$$

for $0 \leq \lambda \leq 1$ and $\varphi_{n,m}$ is a simple logistic Sigmoid activation function.

Definition 1.10 *Let $\gamma \in \mathbb{C} \backslash \{0\}$ and the class $\Im_{\lambda(*)}(\gamma, \varphi_{n,m})$ denote the subclass of A consisting of functions $F(z)$ of the form (1.3), and satisfying the following subordination condition*

$$1 + \frac{1}{\gamma}\left\{\frac{z(f*g)'(z)}{(f*g)(z)} + \lambda \frac{z^2(f*g)''(z)}{(f*g)(z)} - 1\right\} \prec \varphi_{n,m}(z) \tag{1.7}$$

for $0 \leq \lambda \leq 1$ and $\varphi_{n,m}$ is a simple logistic Sigmoid activation function.

As can be seen from the definitions of the classes $M_{\lambda(*)}(\gamma, \varphi_{n,m})$ and $\Im_{\lambda(*)}(\gamma, \varphi_{n,m})$, for $\lambda = 0$ the equality $M_{\lambda(*)}(\gamma, \varphi_{n,m}) = \Im_{\lambda(*)}(\gamma, \varphi_{n,m})$ holds.

2. Some coefficient estimates for the classes of $M_{\lambda(*)}(\gamma, \varphi_{n,m})$ and $\Im_{\lambda(*)}(\gamma, \varphi_{n,m})$

At first, we will find the estimates on the coefficients a_2b_2, a_3b_3, a_4b_4, and a_5b_5 for functions in the classes $M_{\lambda(*)}(\gamma, \varphi_{n,m})$ and $\Im_{\lambda(*)}(\gamma, \varphi_{n,m})$.

Theorem 2.1 *Let*

$$\varphi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m$$

where $\varphi_{n,m}(z) \in A$ is a modified logistic Sigmoid activation function and $\varphi'_{n,m}(0) > 0$. If $F(z) = (f*g)(z)$ given by (1.3) belongs to the class $M_{\lambda(*)}(\gamma, \varphi_{n,m})$ then,

$$a_2b_2 = \frac{\gamma}{2(1+\lambda)}, \tag{2.1}$$

$$a_3b_3 = \frac{\gamma^2}{8(1+2\lambda)}, \tag{2.2}$$

$$a_4b_4 = \frac{\left(3\gamma^2 - 2\right)\gamma}{144(1+3\lambda)}, \tag{2.3}$$

$$a_5b_5 = \frac{\left(3\gamma^2 - 8\right)\gamma^2}{1152(1+4\lambda)}. \tag{2.4}$$

Proof Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. Then, we can write the following equalities:

$$(f*g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \Rightarrow (f*g)'(z) = 1 + \sum_{k=2}^{\infty} k a_k b_k z^{k-1} \Rightarrow (f*g)''(z) = \sum_{k=2}^{\infty} k(k-1) a_k b_k z^{k-2},$$

$$z(f*g)'(z) = z + \sum_{k=2}^{\infty} k a_k b_k z^k,$$

$$z^2(f*g)''(z) = \sum_{k=2}^{\infty} k(k-1) a_k b_k z^k.$$

Thus, we obtain

$$z(f*g)'(z) + \lambda z^2(f*g)''(z) = z + \sum_{k=2}^{\infty} k\{1 + \lambda(k-1)\} a_k b_k z^k$$

and

$$(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)=z+\sum_{k=2}^{\infty}\{1+\lambda(k-1)\}a_kb_kz^k.$$

If $F\in M_{\lambda(*)}(\gamma,\varphi_{n,m})$, then we have

$$\frac{1}{\gamma}\left\{\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-1\right\}=\varphi_{n,m}(z)-1, \tag{2.5}$$

where $\varphi_{n,m}(z)$ is a modified Sigmoid function given by

$$\varphi_{n,m}(z)=1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{17}{40320}z^7+.... \tag{2.6}$$

In view of (2.5) and (2.6), expanding in series forms we have

$$\frac{1}{\gamma}\sum_{k=2}^{\infty}(k-1)\{1+\lambda(k-1)\}a_kb_kz^k=\left\{z+\sum_{k=2}^{\infty}\{1+\lambda(k-1)\}a_kb_kz^k\right\}\left\{\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{17}{40320}z^7+...\right\}$$

$$\begin{aligned}&\Rightarrow\quad \frac{1}{\gamma}\left\{(1+\lambda)a_2b_2z^2+2(1+2\lambda)a_3b_3z^3+3(1+3\lambda)a_4b_4z^4+4(1+4\lambda)a_5b_5z^5+...\right\} \qquad (2.7)\\ &=\quad \left\{z+(1+\lambda)a_2b_2z^2+(1+2\lambda)a_3b_3z^3+(1+3\lambda)a_4b_4z^4+(1+4\lambda)a_5b_5z^5+...\right\}\\ &\quad\times\left\{\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{17}{40320}z^7+...\right\}.\end{aligned}$$

Comparing the coefficients of z^2, z^3, z^4, and z^5 in (2.7), we obtain

$$\begin{aligned}a_2b_2&=\frac{\gamma}{2(1+\lambda)},\\ a_3b_3&=\frac{\gamma^2}{8(1+2\lambda)},\\ a_4b_4&=\frac{(3\gamma^2-2)\gamma}{144(1+3\lambda)},\\ a_5b_5&=\frac{(3\gamma^2-8)\gamma^2}{1152(1+4\lambda)}.\end{aligned}$$

□

Corollary 2.2 *For coefficient* a_2b_2,

$$|a_2b_2|=\frac{|\gamma|}{2(1+\lambda)}$$

is written and since $\varphi(\lambda)=\frac{1}{1+\lambda}$, $\varphi'(\lambda)<0$ *in the interval* $0\le\lambda\le1$ *and* $\varphi(\lambda)$ *is decreasing, it will be*

$$\frac{|\gamma|}{4}\le|a_2b_2|\le\frac{|\gamma|}{2} \tag{2.8}$$

for $\frac{1}{2} \leq \frac{1}{1+\lambda} \leq 1$.

Similarly, since the coefficients a_2b_2, a_3b_3, a_4b_4, *and* a_5b_5 *depend on* λ *and are decreasing with respect to* λ, *the following inequalities can be written easily:*

$$\frac{|\gamma|^2}{24} \leq |a_3b_3| \leq \frac{|\gamma|^2}{8}, \tag{2.9}$$

$$\frac{\left|3\gamma^3 - 2\gamma\right|}{576} \leq |a_4b_4| \leq \frac{\left|3\gamma^3 - 2\gamma\right|}{144}, \tag{2.10}$$

$$\frac{\left|3\gamma^4 - 8\gamma^2\right|}{5760} \leq |a_5b_5| \leq \frac{\left|3\gamma^4 - 8\gamma^2\right|}{1152}. \tag{2.11}$$

Theorem 2.3 *Let*

$$\varphi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m$$

where $\varphi_{n,m}(z) \in A$ *is a modified logistic Sigmoid activation function and* $\varphi'_{n,m}(0) > 0$. *If* $F(z) = (f * g)(z)$ *given by (1.3) belongs to the class* $\Im_{\lambda(*)}(\gamma, \varphi_{n,m})$ *then,*

$$a_2b_2 = \frac{\gamma}{2(1+2\lambda)}, \tag{2.12}$$

$$a_3b_3 = \frac{\gamma^2}{8(1+2\lambda)(1+3\lambda)}, \tag{2.13}$$

$$a_4b_4 = \frac{\left\{3\gamma^2 - 2(1+2\lambda)(1+3\lambda)\right\}\gamma}{144(1+2\lambda)(1+3\lambda)(1+4\lambda)}, \tag{2.14}$$

$$a_5b_5 = \frac{\left\{3\gamma^2 - 2(1+3\lambda)(14\lambda+4)\right\}\gamma^2}{1152(1+2\lambda)(1+3\lambda)(1+4\lambda)}. \tag{2.15}$$

Proof If $F(z) = (f * g)(z) \in \Im_{\lambda(*)}(\gamma, \varphi_{n,m})$, then we have

$$1 + \frac{1}{\gamma}\left\{ \frac{z(f*g)'(z)}{(f*g)(z)} + \lambda \frac{z^2(f*g)''(z)}{(f*g)(z)} - 1 \right\} = \varphi_{n,m}(z) \tag{2.16}$$

where $\varphi_{n,m}(z)$ is a modified Sigmoid function given by

$$\varphi_{n,m}(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + ... \tag{2.17}$$

In view of (2.16) and (2.17), expanding in series forms we have

$$1 + \frac{1}{\gamma}\left\{ \frac{z(f*g)'(z)}{(f*g)(z)} + \lambda \frac{z^2(f*g)''(z)}{(f*g)(z)} - 1 \right\} = \varphi_{n,m}(z)$$

$$\Rightarrow \frac{1}{\gamma}\left\{ \frac{\sum_{k=2}^{\infty}(k-1)(1+\lambda k)a_kb_kz^k}{z + \sum_{k=2}^{\infty}a_kb_kz^k} \right\} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + ...$$

$$\Rightarrow \frac{1}{\gamma}\left\{(1+2\lambda)a_2b_2z^2+2(1+3\lambda)a_3b_3z^3+3(1+4\lambda)a_4b_4z^4+4(1+5\lambda)a_5b_5z^5+...\right\}$$
$$= \left\{z+a_2b_2z^2+a_3b_3z^3+a_4b_4z^4+a_5b_5z^5+...\right\}\left\{\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{17}{40320}z^7+...\right\}$$

$$\Rightarrow \frac{1}{\gamma}(1+2\lambda)a_2b_2z^2+\frac{2}{\gamma}(1+3\lambda)a_3b_3z^3+\frac{3}{\gamma}(1+4\lambda)a_4b_4z^4+\frac{4}{\gamma}(1+5\lambda)a_5b_5z^5+... \tag{2.18}$$
$$= \frac{1}{2}z^2+\frac{1}{2}a_2b_2z^3+\left\{-\frac{1}{24}+\frac{a_3b_3}{2}\right\}z^4+\left\{-\frac{a_2b_2}{24}+\frac{a_4b_4}{2}\right\}z^5+...$$

Thus, comparing the coefficients of z^2, z^3, z^4, and z^5 in (2.18), we obtain

$$a_2b_2=\frac{\gamma}{2(1+2\lambda)},$$

$$a_3b_3=\frac{\gamma^2}{8(1+2\lambda)(1+3\lambda)},$$

$$a_4b_4=\frac{\left\{3\gamma^2-2(1+2\lambda)(1+3\lambda)\right\}\gamma}{144(1+2\lambda)(1+3\lambda)(1+4\lambda)},$$

$$a_5b_5=\frac{\left\{3\gamma^2-4(1+3\lambda)(7\lambda+2)\right\}\gamma^2}{1152(1+2\lambda)(1+3\lambda)(1+4\lambda)(1+5\lambda)}.$$

□

Corollary 2.4 *Since the coefficients a_2b_2, a_3b_3, a_4b_4, and a_5b_5 depend on λ and are decreasing with respect to λ, the following inequalities can be written as the result of Theorem 2.1.*

$$\frac{|\gamma|}{6} \leq |a_2b_2| \leq \frac{|\gamma|}{2}, \tag{2.19}$$

$$\frac{|\gamma|^2}{96} \leq |a_3b_3| \leq \frac{|\gamma|^2}{8}, \tag{2.20}$$

$$\frac{|\gamma^3-8\gamma|}{2880} \leq |a_4b_4| \leq \frac{|3\gamma^3-2\gamma|}{144}, \tag{2.21}$$

$$\frac{|\gamma^4-48\gamma^2|}{138240} \leq |a_5b_5| \leq \frac{|3\gamma^4-8\gamma^2|}{1152}. \tag{2.22}$$

3. Some results connected with the Fekete–Szegö inequality and Hankel Determinant

The Fekete-Szegö problem may be considered one of the most important results about univalent functions, which is related to coefficients a_n of a function's Taylor series and was introduced by Fekete and Szegö [1].

We state it as:

If $f \in S$ given by (1.2), then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 3-4\mu & , \quad \mu \leq 0 \\ 1+2\exp\left(\frac{2\mu}{\mu-1}\right) & , \quad 0 \leq \mu \leq 1 \\ 4\mu-3 & , \quad \mu \geq 1 \end{cases}.$$

The problem of maximizing the absolute value of functional $a_3-\mu a_2^2$ is called the Fekete–Szegö problem. This result is sharp and is studied thoroughly by many researchers. The equality holds true for the Koebe function. In 1969, Keogh and Merkes [2] obtained the sharp upper bound of the Fekete–Szegö functional $\left|a_3-\mu a_2^2\right|$ for some subclasses of univalent function.

Recently, Murugusundarmoorthy and Janani [3], Olantunji et al. [5], Olantunji [4], and Orhan and Çağlar [7] have studied Sigmoid function for various classes of analytic and univalent functions.

In this section, we first prove the following Fekete–Szegö result for the function in the classes $M_{\lambda(*)}(\gamma,\varphi_{n,m})$ and $\Im_{\lambda(*)}(\gamma,\varphi_{n,m})$ with the values of a_2b_2 and a_3b_3.

Theorem 3.1 *If $F\in A$ given by (1.3) is in the class $M_{\lambda(*)}(\gamma,\varphi_{n,m})$, then*

$$\left|a_3b_3-\mu(a_2b_2)^2\right|\leq\frac{|\gamma|^2}{8}(1+2|\mu|). \tag{3.1}$$

Proof If the values of a_2b_2 and a_3b_3 determined by (2.1) and (2.2) are written instead of $a_3b_3-\mu(a_2b_2)^2$, the absolute value of both sides of the equation is taken and triangle inequality is applied, we get

$$a_3b_3-\mu(a_2b_2)^2=\frac{\gamma^2}{8(1+2\lambda)}-\mu\left(\frac{\gamma}{2(1+\lambda)}\right)^2$$

$$\left|a_3b_3-\mu(a_2b_2)^2\right|\leq\frac{|\gamma|^2}{8(1+2\lambda)}+|\mu|\frac{|\gamma|^2}{4(1+\lambda)^2}.$$

Here $\zeta_1(\lambda)=\frac{1}{1+2\lambda}$ and $\zeta_2(\lambda)=\frac{1}{(1+\lambda)^2}$ are taken and these functions depending on λ are considered to be decreasing in the interval $0\leq\lambda\leq1$, since $\max\limits_{0\leq\lambda\leq1}\left\{\frac{1}{1+2\lambda}\right\}=1$ and $\max\limits_{0\leq\lambda\leq1}\left\{\frac{1}{(1+\lambda)^2}\right\}=1$,

$$\left|a_3b_3-\mu(a_2b_2)^2\right|\leq\frac{|\gamma|^2}{8}+|\mu|\frac{|\gamma|^2}{4},$$

then we obtain

$$\left|a_3b_3-\mu(a_2b_2)^2\right|\leq\frac{|\gamma|^2}{8}(1+2|\mu|)=\begin{cases}\frac{|\gamma|^2}{8}(1+2|\mu|); & \mu\geq0\\ \frac{|\gamma|^2}{8}(1-2|\mu|); & \mu<0\end{cases}.$$

□

Theorem 3.2 *If $F\in A$ given by (1.3) is in the class $\Im_{\lambda(*)}(\gamma,\varphi_{n,m})$, then*

$$\left|a_3b_3-\mu(a_2b_2)^2\right|\leq\frac{|\gamma|^2}{8}(1+2|\mu|). \tag{3.2}$$

Proof If the values of a_2b_2 and a_3b_3 determined by (2.12) and (2.13) are written instead of $a_3b_3-\mu(a_2b_2)^2$, the absolute value of both sides of the equation is taken and triangle inequality is applied, we have

$$a_3b_3-\mu(a_2b_2)^2=\frac{\gamma^2}{8(1+2\lambda)(1+3\lambda)}-\mu\left(\frac{\gamma}{2(1+2\lambda)}\right)^2$$

$$\Rightarrow \left|a_3b_3 - \mu\left(a_2b_2\right)^2\right| \leq \frac{|\gamma|^2}{8\left(1+2\lambda\right)\left(1+3\lambda\right)} + |\mu|\frac{|\gamma|^2}{4\left(1+2\lambda\right)^2}.$$

Here $\zeta_3(\lambda) = \frac{1}{(1+2\lambda)(1+3\lambda)}$ and $\zeta_4(\lambda) = \frac{1}{(1+2\lambda)^2}$ are taken and these functions depending on λ are considered to be decreasing in the interval $0 \leq \lambda \leq 1$, since $\max_{0\leq\lambda\leq1}\left\{\frac{1}{(1+2\lambda)(1+3\lambda)}\right\} = 1$ and $\max_{0\leq\lambda\leq1}\left\{\frac{1}{(1+2\lambda)^2}\right\} = 1$,

$$\left|a_3b_3 - \mu\left(a_2b_2\right)^2\right| \leq \frac{|\gamma|^2}{8} + |\mu|\frac{|\gamma|^2}{4},$$

then we obtain

$$\left|a_3b_3 - \mu\left(a_2b_2\right)^2\right| \leq \frac{|\gamma|^2}{8}\left(1+2|\mu|\right) = \begin{cases} \frac{|\gamma|^2}{8}\left(1+2|\mu|\right); & \mu \geq 0 \\ \frac{|\gamma|^2}{8}\left(1-2|\mu|\right); & \mu < 0 \end{cases}.$$

□

In what follows, we will give some results related to by Hankel determinant for the functions belonging to classes $M_{\lambda(*)}\left(\gamma,\varphi_{n,m}\right)$ and $\Im_{\lambda(*)}\left(\gamma,\varphi_{n,m}\right)$.

Theorem 3.3 *If $F \in A$ given by (1.3) is in the class $M_{\lambda(*)}\left(\gamma,\varphi_{n,m}\right)$, then*

$$\left|\left(a_2b_2\right)\left(a_4b_4\right) - \left(a_3b_3\right)^2\right| \leq \frac{|\gamma|^2}{576}\left\{2\left|\left(3\gamma^2-2\right)\right| + 9|\gamma|^2\right\}. \tag{3.3}$$

Proof From (2.1), (2.2), and (2.3), we get

$$\begin{aligned} H_2(2) &= \begin{vmatrix} a_2b_2 & a_3b_3 \\ a_3b_3 & a_4b_4 \end{vmatrix} = \left(a_2b_2\right)\left(a_4b_4\right) - \left(a_3b_3\right)^2 = \left\{\frac{\gamma}{2\left(1+\lambda\right)}\right\}\left\{\frac{\left(3\gamma^2-2\right)\gamma}{144\left(1+3\lambda\right)}\right\} - \left\{\frac{\gamma^2}{8\left(1+2\lambda\right)}\right\}^2 \\ &= \left\{\frac{\left(3\gamma^2-2\right)\gamma^2}{288\left(1+\lambda\right)\left(1+3\lambda\right)}\right\}^2 - \left\{\frac{\gamma^4}{64\left(1+2\lambda\right)^2}\right\} \end{aligned}$$

and thus

$$|H_2(2)| = \left|\left(a_2b_2\right)\left(a_4b_4\right) - \left(a_3b_3\right)^2\right| \leq \frac{\left|3\gamma^4-2\gamma^2\right|}{288\left(1+\lambda\right)\left(1+3\lambda\right)} + \frac{|\gamma|^4}{64\left(1+2\lambda\right)^2}.$$

Here $\zeta_3(\lambda) = \frac{1}{(1+\lambda)(1+3\lambda)}$ and $\zeta_4(\lambda) = \frac{1}{(1+2\lambda)^2}$ are taken and these functions depending on λ are considered to be decreasing in the interval $0 \leq \lambda \leq 1$, since $\max_{0\leq\lambda\leq1}\left\{\frac{1}{(1+\lambda)(1+3\lambda)}\right\} = 1$ and $\max_{0\leq\lambda\leq1}\left\{\frac{1}{(1+2\lambda)^2}\right\} = 1$, then we obtain

$$\left|\left(a_2b_2\right)\left(a_4b_4\right) - \left(a_3b_3\right)^2\right| \leq \frac{|\gamma|^2}{576}\left\{2\left|\left(3\gamma^2-2\right)\right| + 9|\gamma|^2\right\}.$$

□

Theorem 3.4 *If $F \in A$ given by (1.3) is in the class $M_{\lambda(*)}\left(\gamma,\varphi_{n,m}\right)$, then*

$$\left|\left(a_2b_2\right)\left(a_4b_4\right) - \mu\left(a_3b_3\right)^2\right| \leq \frac{|\gamma|^2}{576}\left\{2\left|\left(3\gamma^2-2\right)\right| + 9|\gamma|^2 + 72|\lambda-1|\right\}. \tag{3.4}$$

Proof Since

$$(a_2b_2)(a_4b_4) - \mu (a_3b_3)^2 = (a_2b_2)(a_4b_4) - (\mu - 1)(a_3b_3)^2 - (a_3b_3)^2$$

can be written, the absolute value of both sides of this equation is taken and triangle inequality is applied, then we obtain

$$\begin{aligned}\left|(a_2b_2)(a_4b_4) - \mu (a_3b_3)^2\right| &= \left|(a_2b_2)(a_4b_4) - (a_3b_3)^2 - (\mu - 1)(a_3b_3)^2\right| \\ &\leq \left|(a_2b_2)(a_4b_4) - (a_3b_3)^2\right| + |(\mu - 1)| \left|(a_3b_3)\right|^2 .\end{aligned}$$

If (3.3) and (2.9) are used in this inequality, we get

$$\left|(a_2b_2)(a_4b_4) - \mu (a_3b_3)^2\right| \leq \frac{|\gamma|^2}{576}\left\{2\left|\left(3\gamma^2 - 2\right)\right| + 9|\gamma|^2 + 72|\lambda - 1|\right\}.$$

□

Remark 3.5 *Setting $\lambda = \mu = 1$ in Theorem 3.4, we obtain Theorem 3.3.*

Theorem 3.6 *If $F \in A$ given by (1.3) is in the class $\Im_{\lambda(*)}(\gamma, \varphi_{n,m})$, then*

$$\left|(a_2b_2)(a_4b_4) - (a_3b_3)^2\right| \leq \frac{|\gamma|^2}{288}\left\{15|\gamma|^2 + 4\right\}. \tag{3.5}$$

Proof From (2.12), (2.13), and (2.14), we get

$$\begin{aligned}\begin{vmatrix} a_2b_2 & a_3b_3 \\ a_3b_3 & a_4b_4 \end{vmatrix} &= (a_2b_2)(a_4b_4) - (a_3b_3)^2 \\ &= \left\{\frac{\gamma}{2(1+2\lambda)}\right\}\left\{\frac{\left[3\gamma^2 - 2(1+2\lambda)(1+3\lambda)\right]\gamma}{144(1+2\lambda)(1+3\lambda)(1+4\lambda)}\right\} - \left\{\frac{\gamma^2}{8(1+2\lambda)(1+3\lambda)}\right\}^2 \\ &= \left\{\frac{\left[3\gamma^2 - 2(1+2\lambda)(1+3\lambda)\right]\gamma^2}{288(1+2\lambda)^2(1+3\lambda)(1+4\lambda)}\right\} - \left\{\frac{\gamma^2}{8(1+2\lambda)(1+3\lambda)}\right\}^2\end{aligned}$$

and thus

$$\begin{aligned}|H_2(2)| &= \left|(a_2b_2)(a_4b_4) - (a_3b_3)^2\right| \\ &\leq \frac{|\gamma|^2}{288}\left|\frac{3\gamma^2}{(1+2\lambda)^2(1+3\lambda)(1+4\lambda)} - \frac{2}{(1+2\lambda)(1+4\lambda)}\right| + \left|\frac{\gamma^4}{(8(1+2\lambda)(1+3\lambda))^2}\right| \\ &\leq \frac{|\gamma|^2}{288}\left\{\frac{3|\gamma|^2}{(1+2\lambda)^2(1+3\lambda)(1+4\lambda)} + \frac{2}{(1+2\lambda)(1+4\lambda)}\right\} + \frac{|\gamma|^4}{(8(1+2\lambda)(1+3\lambda))^2} \\ &\leq \frac{|\gamma|^2}{288}\left\{3|\gamma|^2 + 2\right\} + \frac{|\gamma|^4}{64} = \frac{|\gamma|^2}{288}\left\{15|\gamma|^2 + 4\right\}.\end{aligned}$$

□

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